Sharp thresholds and percolation in the plane

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Abstract

Recently, it was shown in [4] that the critical probability for random Voronoi percolation in the plane is 1/2. As a by-product of the method, a short proof of the Harris-Kesten Theorem was given in [5]. The aim of this paper is to show that the techniques used in these papers can be applied to many other planar percolation models, both to obtain short proofs of known results, and to prove new ones.

1 Introduction

In [5] a short proof was given of the fundamental result of Harris [13] and Kesten [15] that the critical probability $p_H = p_H(\mathbb{Z}^2, \text{bond})$ for bond percolation in the planar square lattice $\mathbb{Z}^2$ is equal to $1/2$, where p_H is the critical probability for the occurrence of percolation (see below), and $\mathbb{Z}^2$ is the graph with vertex set $\mathbb{Z}^2$ in which vertices are adjacent if and only if they are at Euclidean distance 1. The methods used in [5] were developed in [4] to prove the new result that the critical probability for percolation in random plane Voronoi tilings is also $1/2$. Here we show that the same methods easily give exponential decay of the volume below the critical probability. Furthermore, while the arguments in [5] are written specifically for bond percolation in $\mathbb{Z}^2$, they can also be applied in many other planar contexts. We illustrate this by considering two well-known examples, site percolation in the square and triangular lattices. We also consider a new bond-percolation model in the square lattice, where the states of the edges are not independent, showing that an analogue of the Harris-Kesten result holds in this context. It is very likely that the methods of [4] and [5] can be applied to many other percolation models.

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In the rest of this introduction we shall recall some of the fundamental concepts of percolation theory. Then, in Section 2, we present the basic tools we shall use to prove our results. In Section 3 we show that the method of [5] easily extends to prove an exponential decay result of Kesten [16]. In Section 4 we apply our method to give short proofs of well-known results for site percolation in the square and triangular lattices. Finally, in Section 5 we consider two percolation models that do not correspond to (independent) site percolation on any lattice, proving results that we believe to be new.

A *bond percolation measure* on an infinite graph G is a probability measure on the space of assignments of a *state*, namely *open* or *closed*, to each edge $e \in E(G)$ of G (with the usual σ-field of measurable events). Similarly, a *site percolation measure* on G is a probability measure on assignments of states to vertices. Here G will usually be a planar lattice; in particular, we consider the square lattice $\mathbb{Z}^2$ and the triangular lattice $L_\triangle$.

Given a lattice L, when discussing bond percolation on L we consider the measure $\mathbb{P}_p^{L,\mathrm{bond}}$ in which the states of the edges are independent, and each edge is open with probability p. Similarly, when discussing site percolation on L we consider the measure $\mathbb{P}_p^{L,\mathrm{site}}$ in which the vertices are open independently with probability p. When there is no danger of confusion, we write $\mathbb{P}_p$ for either of these measures.

An *open cluster* is a maximal connected subgraph of L all of whose edges (vertices) are open. We write C_v for the open cluster containing a given vertex $v \in L$. Thus a vertex w lies in C_v if and only if w can be reached from v by an open path, i.e., a path in L all of whose edges (vertices) are open. In the case of site percolation, if v is closed then $C_v = \emptyset$.

Writing $|C_v|$ for the number of vertices of C_v, let

$$\theta(p) = \mathbb{P}_p\left(|C_0| = \infty\right),$$

where $0 = (0,0)$ is the origin. We shall always take 0 to be a vertex of L. By Kolmogorov's 0-1 law, percolation occurs if and only if $\theta(p) > 0$. More precisely, if $\theta(p) > 0$ then with probability 1 there is an infinite open cluster somewhere in L, while if $\theta(p) = 0$ then with probability 1 there is no such cluster. As $\theta(p)$ is increasing in p, there is a critical probability p_H such that $\theta(p) > 0$ if and only if $p > p_H$. This critical probability depends on the lattice L and type of percolation under consideration. To emphasize this dependence we may write $p_H(L, \mathrm{bond})$ or $p_H(L, \mathrm{site})$. Here, following Welsh (see [21]) the H is in honour of Hammersley; Broadbent and Hammersley introduced the basic concepts of percolation in a 1957 paper [7], where they posed the problem of determining p_H in a variety of contexts. Hammersley [10, 11, 12] proved general upper and lower bounds which imply, for example, that $0.35 < p_H(\mathbb{Z}^2, \mathrm{bond}) < 0.65$.

Writing $\mathbb{E}_p$ for the expectation corresponding to $\mathbb{P}_p$, let

$$\chi(p) = \mathbb{E}_p|C_0|$$

be the expected size of the open cluster of the origin. It is immediate that $\chi(p)$

is increasing in p, so there is a second critical probability,

$$p_T = \inf\{p : \chi(p) = \infty\},$$

with the T in honour of Temperley. As $\theta(p) > 0$ implies $\chi(p) = \infty$, we have $p_T \leq p_H$.

For many years it was believed that $p_T = p_H = 1/2$ for bond percolation in $\mathbb{Z}^2$; this conjecture seems not be have been made explicitly, but, supported by various results and numerical evidence, this belief gradually arose. In 1978, Russo [20] and Seymour and Welsh [21] made significant progress. In particular, they proved independently that $p_T + p_H = 1$. It was only in 1980, twenty years after Harris' proof of the inequality $p_H \geq 1/2$, that Kesten [15] proved that $p_T = p_H = 1/2$. Since then, Menshikov [19] and Aizenman and Barsky [1] (see Grimmett [9]) have shown that $p_T = p_H$ in great generality, in particular, for site percolation in any lattice graph; see Section 4.1 for a formal definition. Note that bond percolation in a lattice graph L corresponds to site percolation in the line-graph of L, which can be realized as a lattice graph, so results for site percolation in general lattices apply for bond percolation as well.

Below the critical probability, much stronger results are known than $\chi(p) < \infty$. In particular, Kesten [16] showed in 1981 that for site percolation in a lattice, when $p < p_T$ there is exponential decay of the size of $|C_0|$. (See also Aizenman and Newman [2] and Grimmett [9].) In the light of the $p_T = p_H$ results mentioned above, Kesten's result implies that there is a single critical probability p_H, with percolation above p_H and exponential decay of the open cluster of the original below p_H. Here we shall show that the method of [5] easily gives exponential decay for $p < p_H$, in various contexts, implying that $p_T = p_H$.

An important property of bond percolation in $\mathbb{Z}^2$ is the 'self-duality' of $\mathbb{Z}^2$. This property is key to the results of Harris and Kesten. In the context of bond percolation, the appropriate notion of duality is the standard one for plane graphs: the dual G^* of a graph G drawn in the plane has a vertex for each face of G, and an edge e^* for each edge e of G. The edge e^* joins the two vertices of G^* corresponding to the faces of G in whose boundary e lies. Taking $G = \mathbb{Z}^2$, there is a vertex v of G^* for each square $[a, a+1] \times [b, b+1]$, $a, b \in \mathbb{Z}$, which we may take to be the point $v = (a + 1/2, b + 1/2)$. It is easy to see that G^* is isomorphic to G; see Figure 1. This self-duality is the 'reason why' $p_H(\mathbb{Z}^2, \text{bond}) = 1/2$, but this trivial observation, made, of course, when the question first arose, is very far from giving a proof of the Harris-Kesten result.

2 Preliminaries

As in [5], the proofs here will be mostly self-contained. The main result we shall use is a sharp-threshold result of Friedgut and Kalai [8], a simple consequence of a result of Kahn, Kalai and Linial [14] concerning the influences of coordinates in a product space. (See also [6].)

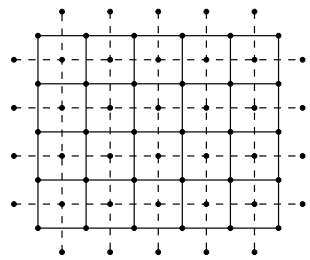

Figure 1: Portions of the lattice $L = \mathbb{Z}^2$ (solid lines) and the isomorphic dual lattice L^* (dashed lines).

Let X be a fixed ground set with N elements, and let X_p be a random subset of X obtained by selecting each $x \in X$ independently with probability p. For a family $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of X, let $\mathbb{P}_p^X(\mathcal{A})$ be the probability that $X_p \in \mathcal{A}$. In this context, $\mathcal{A}$ is *increasing* if $A \in \mathcal{A}$ and $A \subset B \subset X$ imply $B \in \mathcal{A}$. Also, $\mathcal{A}$ is *symmetric* if there is a permutation group acting transitively on X which fixes $\mathcal{A}$. In other words, $\mathcal{A}$ is a union of orbits of the induced action on $\mathcal{P}(X)$. In our notation the result of Friedgut and Kalai [8] we shall need is as follows.

Theorem 1. *There is an absolute constant c_1 such that if $|X| = N$, $\mathcal{A} \subset \mathcal{P}(X)$ is symmetric and increasing, and $\mathbb{P}_p^X(\mathcal{A}) > \varepsilon$, then $\mathbb{P}_q^X(\mathcal{A}) > 1 - \varepsilon$ whenever $q - p \geq c_1 \log(1/(2\varepsilon))/\log N$.*

We shall also make frequent use of Harris' Lemma.

Lemma 2. *If $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(X)$ are increasing, then for any p we have*

$$\mathbb{P}_p^X(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p^X(\mathcal{A})\mathbb{P}_p^X(\mathcal{B}).$$

Taking complements, the lemma also applies to two *decreasing* events, where a decreasing event is the complement of an increasing one. In other contexts Lemma 2 is often known as Kleitman's Lemma [17]. The present context is exactly that of Harris' original paper [13]: X will be a set of edges or vertices in the lattice (according to whether we are considering site or bond percolation), and X_p will be the subset of X consisting of open edges/vertices. Thus an event is increasing if it is preserved by changing the states of one or more edges/vertices from closed to open, and Harris' Lemma states that increasing events are positively correlated.

In addition to the results above, we shall need two observations concerning k-dependent percolation. A bond percolation measure on a graph G is *k-dependent* if for every pair S, T of sets of edges of G at graph distance at least k, the states (being open or closed) of the edges in S are independent of the states of the edges in T. When $k = 1$, the separation condition is exactly that no edge of S shares a vertex with an edge of T. The definition of k-dependence for a site percolation measure on G is exactly the same, except that S and T run over all

sets of *vertices* at graph distance at least k. Here we shall consider dependent measures only on the lattice $\mathbb{Z}^2$.

These k-dependent measures arise very naturally in a variety of contexts (for example, static renormalization arguments), and have been considered by several authors; see Liggett, Schonmann and Stacey [18] and the references therein. In [18], a very general comparison result between k-dependent and product measures is proved: working on any fixed countable graph G of bounded degree (for example, $\mathbb{Z}^d$), for any $p < 1$ there is an $f(G, k, p) < 1$ such that any k-dependent measure in which each edge (vertex) is open with probability at least $f(G, k, p)$ dominates the product measure $\mathbb{P}_p$ in which edges (vertices) are open independently with probability p.

In particular, provided the individual edge probabilities are high enough, percolation occurs in $\mathbb{Z}^2$ under the assumption of 1- (or k-) dependence.

Lemma 3. *There is a $p_0 < 1$ such that in any 1-dependent bond percolation measure on $\mathbb{Z}^2$ satisfying the additional condition that each edge is open with probability at least p_0, the probability that $|C_0| = \infty$ is positive.*

In applications, the value of p_0 is frequently important. Currently, the best known bound is the result of Balister, Bollobás and Walters [3] that one can take $p_0 = 0.8639$. Here, the value of p_0 will be irrelevant: all we shall need is the essentially trivial Lemma 3. For completeness, we give a very simple proof that one can take $p_0 = 0.995$.

Indeed, suppose that the open cluster C_0 containing the origin is finite. Consider the edge-boundary B of the (unique) infinite component of $\mathbb{Z}^2 \setminus C_0$, noting that all edges in B are closed. Passing to the lattice L^* dual to $L = \mathbb{Z}^2$ as defined above, the edges of L^* corresponding to the edges of L in B form a

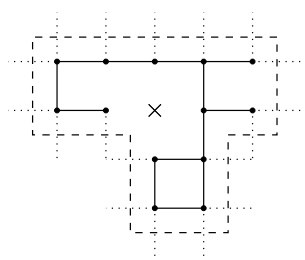

Figure 2: An open cluster C in $L = \mathbb{Z}^2$ (dots and solid lines), the edge boundary B of the infinite component of $L \setminus C$ (dotted lines), and the corresponding cycle S in L^* (dashed lines). The point marked with a cross is in a finite component of $L \setminus C$.

simple cycle S in L^* that surrounds the origin. Given the length $\ell \geq 4$ of S, there are crudely at most $\frac{\ell-2}{2}3^{\ell-2}$ possibilities for S (and hence B): S must cross the x axis at some x-coordinate between $\frac{1}{2}$ and $\frac{\ell-3}{2}$. Walking round S, at each stage there are at most three possibilities for the next edge, and at most

one choice that closes the cycle at the end. Passing back to $L = \mathbb{Z}^2$, the edges of L may be partitioned into four complete matchings, one of which must contain a set B' of at least $|B|/4 = \ell/4$ edges of B. Now the states of the edges in B' are independent of each other, and each $e \in B'$ is closed with probability at most $1 - p_0$. Putting everything together, we see that the probability that $|C_0|$ is finite, which is exactly the probability that some closed cycle in the dual surrounds the origin, is at most

$$\sum_{\ell \geq 4,\, \ell \text{ even}} \frac{\ell - 2}{2} 3^{\ell-2} (1 - p_0)^{\ell/4}.$$

This is strictly less than 1 if $p_0 = 0.995$.

Finally, a corresponding negative result is just as easy: we repeat the statement and proof from [4]. This time, it is easier to work with site percolation. Recall that in the site percolation context, C_0, the open cluster of the origin, is the set of vertices of $\mathbb{Z}^2$ joined to the origin by a path in $\mathbb{Z}^2$ every one of whose vertices is open.

Lemma 4. *Let k be a fixed positive integer, and let $\widetilde{\mathbb{P}}$ be a k-dependent site percolation measure on $\mathbb{Z}^2$ in which every vertex $v \in \mathbb{Z}^2$ is open with probability at most p. There is a constant $p_1 = p_1(k) > 0$ such that for every $p \leq p_1$ there is a $c(p, k) > 0$ for which*

$$\widetilde{\mathbb{P}}(|C_0| \geq n) \leq \exp(-c(p, k)n)$$

for all $n \geq 1$.

Proof. If $|C_0| \geq n$, then the subgraph of $\mathbb{Z}^2$ induced by the open vertices contains a tree T with n vertices, one of which is the origin. It is well known and easy to check that the number of such trees in $\mathbb{Z}^2$ grows exponentially, and is at most $(4e)^n$. Fix any such tree T. Then there is a subset S of at least $n/(2k^2 - 2k + 1)$ vertices of T such that any $a, b \in S$ are at graph distance at least k; indeed, one can find such a set by a greedy algorithm: whenever a vertex a is chosen, the number of other vertices it rules out is at most the number of other vertices of $\mathbb{Z}^2$ within graph distance $k - 1$ of a, namely $4\binom{k}{2} = 2k^2 - 2k$. The vertices of S are open independently, so the probability that every vertex of T is open is at most $p^{|S|}$. Hence,

$$\widetilde{\mathbb{P}}(|C_0| \geq n) \leq (4e)^n p^{n/(2k^2-2k+1)}.$$

Provided p is small enough that $r = 4ep^{1/(2k^2-2k+1)} < 1$, the conclusion follows, taking $c(p, k) = -\log r$. □

3 Bond percolation in $\mathbb{Z}^2$: exponential decay

In this section we consider bond percolation in $\mathbb{Z}^2$, writing $\mathbb{P}_p$ for the probability measure $\mathbb{P}_p^{\mathbb{Z}^2,\text{bond}}$ in which each edge of $\mathbb{Z}^2$ is open with probability p, independently of all other edges. In [5], a short proof was given of the Harris-Kesten

result that in this context $p_H = 1/2$, using Theorem 1 as the main ingredient. In fact the method also gives a simple proof that for $p < 1/2$ there is exponential decay of the 'volume' $|C_0|$ of the open cluster containing the origin. It follows that $\chi(p)$ is finite for $p < 1/2$, and hence that $p_T = p_H = 1/2$. The result below was first proved by Kesten [16] in 1981.

Theorem 5. *Let $p < 1/2$ be fixed. There is a constant $a = a(p) > 0$ such that $\mathbb{P}_p(|C_0| \geq n) \leq \exp(-an)$ for all $n \geq 0$.*

As in [5], we shall work with 'open crossings of rectangles'. We identify a rectangle $R = [x_0, x_1] \times [y_0, y_1]$, where $x_0 < x_1$ and $y_0 < y_1$ are integers, with an induced subgraph of $\mathbb{Z}^2$. This subgraph includes all vertices and edges in the interior and boundary of R. We write $H(R)$ for the event that there is a *horizontal open crossing* of R, i.e., a path from the left side of R to the right side consisting entirely of open edges of R. Similarly, we write $V(R)$ for the event that there is a vertical open crossing of R.

We shall deduce Theorem 5 from Lemma 8 of [5], reproduced below as Lemma 6. In fact, most of the work in [5] went into proving this lemma; the deduction of the Harris-Kesten Theorem was then easy.

Lemma 6. *Let $p > 1/2$, $c < 1$ and an integer $\rho > 1$ be fixed. There is a constant $n_0 = n_0(p, c, \rho)$ such that, if $n \geq n_0$ and R_n is a $4\rho n$ by $4n$ rectangle in $\mathbb{Z}^2$, then $\mathbb{P}_p(H(R_n)) \geq c$.*

Note that we take our rectangles to have integer coordinates. This is the only reason for taking ρ to be an integer. Of course, a corresponding result for general ρ follows immediately.

Proof of Theorem 5. Fix $p < 1/2$, let $p_1 > 0$ be a constant for which Lemma 4 holds with $k = 9$, and set $c = (1 - p_1)^{1/4}$. Let $s = 4n_0(1 - p, c, 3)$, where n_0 is the function in Lemma 6, and set $m = s + 4$.

We shall apply Lemma 6 to the lattice L^* dual to $L = \mathbb{Z}^2$, which is isomorphic to $\mathbb{Z}^2$. Defining the state of a dual edge e^* to be the state of e, each edge of L^* is closed with probability $1 - p > 1/2$, independently of all other edges. By Lemma 6, if R is a $3m$ by m rectangle in L^*, then the probability that R is crossed the long way by a path of closed dual edges is at least c.

Let S be an s by s square in $\mathbb{Z}^2$. We can arrange four $3m$ by m rectangles in the dual lattice to form an annulus A as in Figure 3, with the inside of the

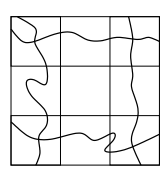

Figure 3: Four rectangles forming an annulus.

annulus surrounding S. Using Lemma 2, with probability at least $c^4 = 1 - p_1$,

each of the four rectangles is crossed the long way by a path of closed dual edges. If this happens, then there is a cycle of closed dual edges in A which surrounds S. (See Figure 3.) It follows that in the original lattice, no vertex in S is connected by an open path to a vertex outside A.

Returning to $\mathbb{Z}^2$, given an s by s square S in $\mathbb{Z}^2$, let $B(S)$ be the event that some vertex in S is connected by an open path to a vertex at L_∞-distance $2s > m + 4$ from S. We have shown that $\mathbb{P}_p(B(S)) \leq p_1$.

Let us define a site percolation measure $\widetilde{\mathbb{P}}$ on $\mathbb{Z}^2$ as follows: each $v = (x, y) \in \mathbb{Z}^2$ is open if $B(S)$ holds for the square $S_v = [sx, s(x + 1)] \times [sy, s(y + 1)]$. As $B(S_v)$ depends only on the states of edges within L_∞-distance $2s$ of S_v, $\widetilde{\mathbb{P}}$ is a 9-dependent measure. Furthermore, each $v \in \mathbb{Z}^2$ is open with $\widetilde{\mathbb{P}}$-probability at most p_1. Let C_0 be the open cluster of the origin in our original bond percolation, and let C_0' be the open cluster in the site percolation we have just defined. By Lemma 4 there is an $a > 0$ such that

$$\widetilde{\mathbb{P}}(|C_0'| \geq n) \leq \exp(-an)$$

for every n.

Recall that s is a large constant depending on p. If $|C_0| > (6s + 1)^2$, then every vertex w of C_0 is joined by an open path to some vertex at L_∞-distance $3s$ from w. If $w \in S_v$, then it follows that $B(S_v)$ holds. Thus, if $|C_0| > (6s+1)^2$, then $B(S_v)$ holds for every v such that S_v contains vertices of C_0. The set of such v forms an open cluster with respect to $\widetilde{\mathbb{P}}$, and is thus a subset of C_0'. Hence, as each S_v contains only $(s + 1)^2$ vertices, for $n \geq (6s + 1)^2$ we have

$$\mathbb{P}_p(|C_0| \geq n) \leq \widetilde{\mathbb{P}}(|C_0'| \geq n/(s + 1)^2) \leq \exp(-an/(s + 1)^2),$$

completing the proof of Theorem 5. □

4 Percolation in other lattices

The arguments given in [5] were specific to the case of bond percolation in $\mathbb{Z}^2$, because there our aim was to give the simplest possible proof that $p_H = 1/2$ in this case. However, parts of the proofs are applicable in many other contexts. In particular, the method used in Section 4 of [5] applies to any planar lattice, and can be extended to other contexts. The heart of the method is a simple application of Theorem 1; we present this in the setting of a general lattice as Lemma 8 in the next subsection.

In fact, the method of [5] was developed in [4] in a rather different, continuous, context, namely random Voronoi percolation; in [4] it is shown that the critical probability for random Voronoi percolation in the plane is 1/2. The arguments needed for the random Voronoi case are much more complicated than those for lattices; we shall not even outline them here.

In order to apply Lemma 1 to deduce results about critical probabilities, one needs an appropriate equivalent of the Russo-Seymour-Welsh Theorem, stating

essentially that if (very large) squares may be crossed with significant probability, then the same applies to rectangles with a fixed aspect ratio. As in [5], in many contexts simpler methods can be used to prove an essentially equivalent result. To illustrate this we give two examples, in Subsections 4.2 and 4.3. The first, site percolation in the square lattice, shows that knowing the critical probability is not necessary. The second, site percolation in the triangular lattice, shows that the square geometry is not necessary.

4.1 Sharp thresholds in lattices

In this subsection we consider percolation on lattices in $\mathbb{R}^d$. We say that L is a d-dimensional *lattice graph* or *lattice* if L is a connected locally finite graph with a countable vertex set $V = V(L) \subset \mathbb{R}^d$, such that there are d automorphisms α_i of L acting on V by translation through linearly independent vectors $\mathbf{v}_i \in \mathbb{R}^d$. We work throughout with site percolation on the graph L: for bond percolation we may realize the line-graph of L as a lattice L' and work with site percolation on L'. Note that in the 2-dimensional case, L need not be a planar graph.

A basic property of any lattice graph is that its vertex set V has a partition into finitely many classes V_j so that the automorphism group of the graph L acts transitively on each V_j.

We shall need the following slightly strengthened form of Theorem 1.

Lemma 7. *Let X be a finite ground set with $|X| = N$, and suppose that $\mathcal{A} \subset \mathcal{P}(X)$ is increasing. Suppose also that there is a group G acting on X so that every orbit of the action of G on X has size at least M, and so that $\mathcal{A}$ is a union of orbits of the induced action of G on $\mathcal{P}(X)$. There is an absolute constant c_1 such that if $\mathbb{P}_p^X(\mathcal{A}) > \varepsilon$, then $\mathbb{P}_q^X(\mathcal{A}) > 1 - \varepsilon$ whenever*

$$q - p \geq c_1 \frac{\log(1/(2\varepsilon))}{\log N} \frac{N}{M}.$$

Proof. The proof is the same as that of Theorem 1, i.e., of Theorem 2.1 of Friedgut and Kalai [8]. The only modification is that having found one variable with influence at least x, the conclusion is that the sum of the influences of all variables is at least Mx, rather than at least Nx. □

For notational convenience, we state the following result only in the 2-dimensional case. In d-dimensions corresponding results concerning paths from one face of a hypercuboid to the opposite face, or surfaces separating one face from the opposite face, can be proved in exactly the same way.

We work with the probability measure $\mathbb{P}_p = \mathbb{P}_p^{L,\text{site}}$ in which each vertex of L is open independently with probability p. An *open path* is a path in L all of whose vertices are open. If L is a 2-dimensional lattice and $R \subset \mathbb{R}^2$ is a rectangle, then we write $H(R) = H_L(R)$ for the event that R has a *horizontal open crossing*, i.e., that there is a path in L consisting of open vertices of R joining vertices v_1 and v_2, where v_1 is incident with an edge of L that meets the left-hand side of R, and v_2 with an edge that meets the right-hand side of R.

In fact, for the application below the precise definition of $H(R)$ (i.e., how we deal with vertices near the boundary of R) will not matter – the statement of our lemma will not be affected if the dimensions of the rectangles involved are altered by $O(1)$.

In this section, all our rectangles have a fixed orientation, which we take without loss of generality to be parallel to the coordinate axes. We also suppose that the origin is a lattice point. Note that $\mathbb{P}_p(H(R))$ may depend not just on the dimensions of R, but also on its position with respect to L; we do not assume that the corners of our rectangles are lattice points. In the case $L = \mathbb{Z}^2$, this assumption might be natural, but it would make no difference – the statement of the lemma is unaffected if we round the coordinates to integers.

Lemma 8. *Let L be a 2-dimensional lattice graph. Let $0 < p_1 < p_2 < 1$, $\varepsilon > 0$, and positive real numbers $x_1 > x_2$, $y_1 < y_2$ be fixed. There is an n_0 such that if $n \geq n_0$ and R is an $x_1 n$ by $y_1 n$ rectangle for which $\mathbb{P}_{p_1}(H_L(R)) \geq \varepsilon$, then $\mathbb{P}_{p_2}(H_L(R')) \geq 1 - \varepsilon$ for any $x_2 n$ by $y_2 n$ rectangle R'.*

Proof. The argument is essentially the same as in [5]; we write it out for completeness. Throughout this proof n_0 will be a large constant to be chosen later, depending on all the parameters in the statement of the lemma.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two linearly independent vectors such that translations of $\mathbb{R}^2$ through $\mathbf{v}_i$ induce automorphisms of L, and let F be the corresponding fundamental region of L, i.e., the parallelogram with vertices 0, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_2$. Note that F has diameter $D = O(1)$, where the constant depends only on L, and F contains $\Theta(1)$ points of L.

Suppressing the dependence on L, suppose that $\mathbb{P}_{p_1}(H(R)) \geq \varepsilon$ for an $x_1 n$ by $y_1 n$ rectangle R with $n \geq n_0$. We may find points $\mathbf{w}_1$ and $\mathbf{w}_2$, each of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$, $a_i \in \mathbb{Z}$, within distance D of $((x_1+1)n, 0)$ and $(0, (y_2+1)n)$, respectively. Let F' be the parallelogram with vertices 0, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_1 + \mathbf{w}_2$. Then we may assume that R lies within F, and indeed that R does not come closer than a distance $n/3$ to the boundary of F. To see this, note that $\mathbb{P}_p(H(R))$ is unchanged if we translate R through a vector $\mathbf{v}_i$, $i = 1, 2$.

Let T be the graph obtained from L by quotienting by (the automorphisms whose action corresponds to) the translations of $\mathbb{R}^2$ through $\mathbf{w}_1$ and $\mathbf{w}_2$. Then T is a graph with $\Theta(n^2)$ vertices, where the implicit constants depend on L, x_1 and y_2, and T is 'locally isomorphic' to L. In particular, for rectangles R' too small to 'wrap around' T, which are the only rectangles we shall consider, each rectangle R' in L corresponds to a rectangle in T, and the induced subgraphs of L and T are isomorphic.

We write $\mathbb{P}_p^T$ for the probability measure in which each vertex of T is open with probability p, independently of all other vertices. From the remark above, there is an $x_1 n$ by $y_1 n$ rectangle R in T such that

$$\mathbb{P}_{p_1}^T(H(R)) = \mathbb{P}_{p_1}^L(H(R)) \geq \varepsilon.$$

Let E be the event that there is *some* $x_1 n$ by $y_1 n$ rectangle R' in T for which $H(R)$ holds. Then

$$\mathbb{P}_{p_1}^T(E) \geq \mathbb{P}_{p_1}^T(H(R)) \geq \varepsilon.$$

The event E is increasing and symmetric in the sense of Lemma 7; translations of T through the vectors $\mathbf{v}_i$ preserve E, and such translations map any vertex of T to a vertex in one given fundamental region. Thus the action of the group generated by these translations on T has $O(1)$ orbits each of size at least $S = cn^2$, where c depends on L, x_1 and y_2. We claim that for any constant $\eta < 1$ we have

$$\mathbb{P}^T_{p_2}(E) \geq 1 - \eta,$$

provided that n_0 is chosen large enough, which we shall assume from now on. Indeed, writing $N = |T| = \Theta(n^2)$, then as N/S is bounded, by Lemma 7 it suffices to choose n_0 large enough that for $n \geq n_0$ we have $\log N = 2\log n + O(1)$ larger than a certain constant depending on η and the parameters of the lemma.

Let R' be any $x_2 n$ by $y_2 n$ rectangle in T. Note that $x_1 > x_2$ and $y_1 < y_2$, so R' is 'shorter and fatter' than R. It follows that if n is large enough, the torus T can be covered by a bounded number M of translates R_i of R' through vectors of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$, $a_i \in \mathbb{Z}$, in such a way that any $x_1 n$ by $y_1 n$ rectangle R in T crosses some R_i horizontally, meaning that the intersection of R and R_i is an $x_2 n$ by $y_1 n$ rectangle. It follows that any horizontal open crossing of R contains a horizontal open crossing of R_i. Hence, if E holds, then so does one of the events $E_i = H(R_i)$, so $E^c \supset \cap_i E_i^c$.

The events E_i are increasing. Hence, by Lemma 2, for each i the decreasing event E_i^c is positively correlated with the decreasing event $\bigcap_{j<i} E_j^c$, and

$$\mathbb{P}^T_{p_2}(E^c) \geq \mathbb{P}^T_{p_2}\left(\bigcap_{i=1}^M E_i^c\right) \geq \prod_{i=1}^M \mathbb{P}^T_{p_2}(E_i^c) = \mathbb{P}^T_{p_2}(H(R')^c)^M.$$

For the last step we use the fact that the subgraph of T induced by each R_i is isomorphic to that induced by R'. Thus,

$$\mathbb{P}^T_{p_2}(H(R')^c) \leq \mathbb{P}^T_{p_2}(E^c)^{1/M} \leq \eta^{1/M} = \varepsilon,$$

if we choose η appropriately. Using the local isomorphism between L and T, we have

$$\mathbb{P}^L_{p_2}(H(R')) = \mathbb{P}^T_{p_2}(H(R')) \geq 1 - \varepsilon,$$

as required. □

4.2 Site percolation in the square lattice

For this subsection, let $L_\square = \mathbb{Z}^2$ be the planar square lattice viewed as a graph as in Section 3, and let $L_\boxtimes$ be the (non-planar) graph with vertex set $\mathbb{Z}^2$ in which two vertices are adjacent if they are at Euclidean distance 1 or $\sqrt{2}$. We consider the probability measure $\mathbb{P}_p$ in which each vertex $v \in \mathbb{Z}^2$ is open with probability p, independently of the other vertices. Note that we are considering two notions of site percolation involving the same probability measure. For $L = L_\square$ or $L_\boxtimes$, the open cluster $C_0(L)$ containing the origin is the set of open vertices that may be reached from the origin by a path in the graph L all of whose vertices are

open. As before, for a rectangle R with integer coordinates, we write $H_L(R)$ for the event that R has a horizontal open crossing in L, and $V_L(R)$ for the event that R has a vertical open crossing.

Note that $L_\square$ and $L_\boxtimes$ are *dual* in the following sense: let L be one of $L_\square$, and $L_\boxtimes$, and let L^* be the other. If C is a finite open cluster of L, then there is a cycle B in L^* of closed vertices each of which is adjacent (in L) to a vertex of C. Also, if B is a cycle in L^*, then any path in L from a vertex inside B to one outside B must meet B at a vertex. It follows that if R is a rectangle with integer coordinates, then R is crossed horizontally by an open path in L if and only if R is *not* crossed vertically by a closed path in L^*. Thus

$$\mathbb{P}_p(H_L(R)) + \mathbb{P}_{1-p}(V_{L^*}(R)) = 1. \tag{1}$$

The values of the critical probabilities $p_H(L, \text{site})$, $L = L_\square, L_\boxtimes$, are not known. A special case of the general result of Menshikov [19] (see also [9]) implies that for $L = L_\square$ or $L = L_\boxtimes$ there is exponential decay of the radius of $C_0(L)$ below $p_H(L, \text{site})$, and hence that $p_T(L, \text{site}) = p_H(L, \text{site})$. As noted in the introduction, it follows from the results of Kesten [16] or Aizenman and Newman [2] (see also [9]) that there is exponential decay of $|C_0(L)|$. We give a new proof of the latter, stronger result.

Theorem 9. *Let $L = L_\square$ or $L_\boxtimes$. If $p < p_H(L, \text{site})$ then there is a constant $a = a(p, L) > 0$ such that $\mathbb{P}_p(|C_0(L)| \geq n) \leq \exp(-an)$ holds for all $n \geq 0$.*

Theorem 9 implies the well known result that $p_H(L_\square) + p_H(L_\boxtimes) = 1$. Indeed, it is easy to see that $p_H(L_\square) + p_H(L_\boxtimes) \geq 1$. In the other direction, it suffices to note that we cannot have exponential decay of $|C_0(L_\square)|$ in $\mathbb{P}_p$ and of $|C_0(L_\boxtimes)|$ in $\mathbb{P}_{1-p}$. Otherwise, the $\mathbb{P}_p$-probability that a large square has either a horizontal open $L_\square$-crossing or a vertical closed $L_\boxtimes$-crossing will tend to zero, contradicting (1).

The following more general version of Lemma 5 of [5] will be useful. When there is no danger of ambiguity, we write $H(R)$ for $H_L(R)$ and $V(R)$ for $V_L(R)$.

Lemma 10. *Let $L = L_\square$ or $L_\boxtimes$, and let k, r, s and $t > r$ be positive integers. Set $R_i = [0, r] \times [(i-1)s, is]$ for $i = 1, 2, \ldots, k$, and let $R = [0, t] \times [0, ks]$. Let X_i be the event that there is an open vertical crossing of R_i joined by an open path in R to the right-hand side of R. Then for some i we have*

$$\mathbb{P}_p(X_i) \geq \mathbb{P}_p(H(R))\mathbb{P}_p(V(R_1))/k.$$

Proof. The proof is almost exactly the same as that of Lemma 5 of [5]. If $V(R_i)$ holds, we can define a left-most vertical crossing $LV(R_i)$ of R_i in such a way that the event $LV(R_i) = P_i$ does not depend on the states of vertices of R_i to the right of P_i. (One can view all vertices outside R_i as open except those L^*-adjacent to the left-hand side of R_i, which one views as closed, and consider the closed cluster in L^* containing the latter vertices. There is an open cycle in L bounding this cluster; as $V_L(R_i)$ holds, the closed cluster does not meet the

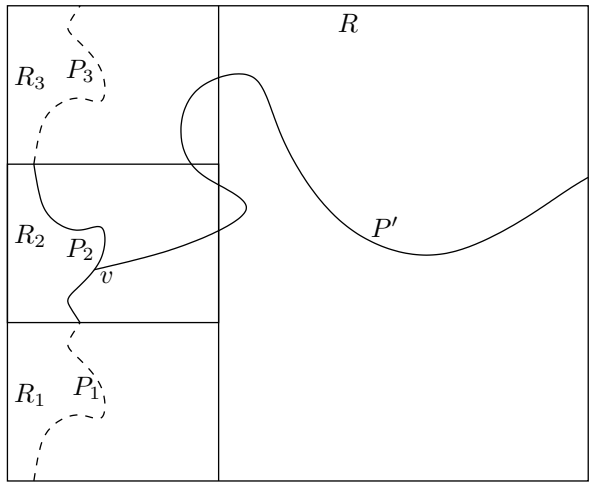

Figure 4: The rectangles R_i and rectangle R for $k = 3$: the solid paths indicate that X_2 holds.

right-hand side of R_i, so the intersection of the open bounding cycle with R_i forms the required left-most vertical open crossing.)

For a fixed i, if $V(R_i)$ holds and $LV(R_i) = P_i$, define P to be the vertical (but not necessarily open) crossing of $[0, r] \times [0, sk]$ obtained by reflecting P_i in the horizontal lines $y = is$, as shown in Figure 4. Also, let P_j, $1 \leq j \leq k$, be the sub-paths of P crossing each R_j. Note that the event that P takes a particular value is independent of the states of the vertices to the right of P.

With (unconditional) probability $\mathbb{P}_p(H(R))$ there is a horizontal open crossing P_H of R. Any such crossing must cross P; indeed, P and P_H share a vertex unless $L = L_{\boxtimes}$ and the paths cross diagonally within a grid square. It follows that P_H contains a sub-path P' with the following properties: every vertex of P' lies strictly to the right of P and is open, P' starts at a vertex adjacent to a vertex v of P, and P' ends at a vertex on the right hand side of R; see Figure 4. Let $Y_j(P)$ be the event that such a P' exists with v lying on P_j. Then we have

$$\sum_{j=1}^{k} \mathbb{P}_p(Y_j(P)) \geq \mathbb{P}_p(H(R)). \tag{2}$$

Now $Y_j(P)$ depends only on the states of the vertices to the right of P. For any possible value P_i of $LV(R_i)$, defining P and P_j as above, the event $LV(R_j) = P_j$ is independent of the states of vertices to the right of the path P. Thus,

$$\mathbb{P}_p(Y_j(P) \mid LV(R_j) = P_j) = \mathbb{P}_p(Y_j(P)),$$

and, from (2),

$$\sum_{j=1}^{k} \mathbb{P}_p(Y_j(P) \mid LV(R_j) = P_j) \geq \mathbb{P}_p(H(R)).$$

If $Y_j(P)$ holds and $LV(R_j) = P_j$, then X_j holds (see Figure 4). Thus,

$$\sum_{j=1}^{k} \mathbb{P}_p(X_j \mid LV(R_j) = P_j) \geq \mathbb{P}_p(H(R)).$$

As P_1 runs over all possible values of $LV(R_1)$, each P_j runs over all possible values of $LV(R_j)$. Hence, as $V(R_j)$ is the disjoint union of the events that $LV(R_j)$ takes each possible value,

$$\sum_{j=1}^{k} \mathbb{P}_p(X_j \mid V(R_j)) \geq \mathbb{P}_p(H(R)).$$

Since each $V(R_j)$ has the same probability, the result follows. □

As in [5], we obtain an immediate corollary concerning long thin rectangles, provided we know that certain crossings of squares exist with significant probability. We write $R_{m,n}$ for the m by n rectangle $[0,m] \times [0,n]$, and $H(R_{m,n})$ for the event that this rectangle has a horizontal open crossing in the lattice under consideration.

Corollary 11. *Let $c > 0$ and integers ρ, $k \geq 2$ be given. There is a constant $c' = c'(c,k,\rho) > 0$ such that if $L = L_\square$ or $L_\boxtimes$, and $\mathbb{P}_p(H(R_{s,s})), \mathbb{P}_p(H(R_{ks,ks})) \geq c$, then $\mathbb{P}_p(H(R_{\rho ks,ks})) \geq c'$.*

Proof. Let $h_{m,n} = \mathbb{P}_p(H(R_{m,n}))$, so $h_{s,s}$, $h_{ks,ks} \geq c$ by assumption. We claim that for $m > s$ we have $h_{2m-s,ks} \geq h_{m,ks}^2 c^3/k^2$. Applying this repeatedly, starting from $h_{ks,ks} \geq c$, gives the result.

As in [5], the claim is an immediate consequence of Lemma 10 and Harris' Lemma. To see this, choose an i, $1 \leq i \leq k$, for which Lemma 10 holds with $r = s$, $t = m$, and consider the rectangles $R = [0,m] \times [0,ks]$, $R' = [s-m,s] \times [0,ks]$ and the square $S = [0,s] \times [(i-1)s, is]$ in their intersection. Note that the square S plays the role of the rectangle R_i in Lemma 10 for the parameters ($r = s$, $t = m$) we have used.

Let us write E_1 for the event X_i defined in Lemma 10, which depends on the vertices in R, and let E_2 be the corresponding event for R', defined by reflecting in the line $x = s/2$; see Figure 5. Finally, let E_3 be the event $H(S)$. Note that if E_1, E_2 and E_3 all hold, then $H(R \cup R')$ holds, using only the fact that horizontal and vertical crossings of S must cross. By Lemma 10 and our choice of i we have

$$\mathbb{P}_p(E_1) \geq \mathbb{P}_p(H(R))\mathbb{P}_p(V(S))/k = h_{m,ks}\mathbb{P}_p(V(S))/k.$$

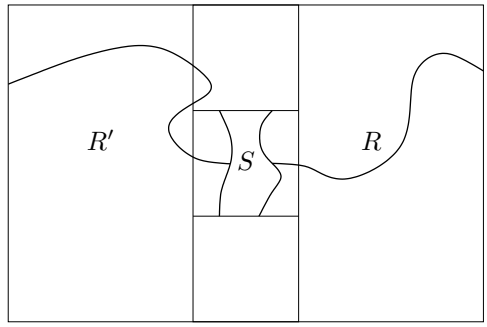

Figure 5: The overlapping rectangles R and R' with the square S in their intersection. The paths drawn show that X_i holds for R, as well as the reflected equivalent E_2 for R'. If $H(S)$ also holds, then so does $H(R \cup R')$.

By symmetry, $\mathbb{P}_p(E_2) = \mathbb{P}_p(E_1)$. As E_1, E_2 and E_3 are increasing events, by Lemma 2 we have

$$\begin{aligned} \mathbb{P}_p(H(R \cup R')) &\geq \mathbb{P}_p(E_1)\mathbb{P}_p(E_2)\mathbb{P}_p(E_3) \\ &\geq h^2_{m,ks}\mathbb{P}_p(V(S))^2\mathbb{P}_p(H(S))/k^2. \end{aligned}$$

By assumption, $\mathbb{P}_p(V(S)) = \mathbb{P}_p(H(S)) = h_{s,s} \geq c$, so

$$h_{2m-s,ks} = \mathbb{P}_p(H(R \cup R')) \geq h^2_{m,ks}c^3/k^2,$$

completing the proof of the claim and thus of the corollary. □

Using the method of Section 4 of [5], it is easy to deduce Theorem 9. The key step is to apply Lemma 8.

Proof of Theorem 9. Let $L = L_\square$ or $L_\boxtimes$. It suffices to show that for any constant $p_1 < p_2$, either percolation occurs in L at p_2 (i.e., $\theta_L(p_2) > 0$), or there is exponential decay of $|C_0(L)|$ at p_1. Fix $p_1 < p_2$ and set $p = (p_1 + p_2)/2$. Let n_0 be a large constant to be chosen later, depending only on p_1 and p_2.

Let S_i, $i = 1, 2, 4$, be squares of side length in_0. From (1) we have $\mathbb{P}_p(H_L(S_i))+\mathbb{P}_{1-p}(H_{L^*}(S_i)) = 1$, where $\{L, L^*\} = \{L_\square, L_\boxtimes\}$. It follows that either (a) there are two values of $i \in \{1, 2, 4\}$ for which $\mathbb{P}_p(H_L(S_i)) \geq 1/2$, or (b) there are two values for which $\mathbb{P}_{1-p}(H_{L^*}(S_i)) \geq 1/2$.

It follows from Corollary 11 (applied with $c = 1/2$, $\rho = 10$, and $k = 2$ or $k = 4$) that there is a $10n$ by n rectangle R, $n \geq n_0$, such that either $\mathbb{P}_p(H_L(R)) \geq c'$ or $\mathbb{P}_{1-p}(H_{L^*}(R)) \geq c'$, where c' is an absolute constant not depending on our choice of n_0.

Let R' be a $6n$ by $2n$ rectangle. As $p_1 < p < p_2$, for any constant $c_3 < 1$ it follows by Lemma 8 that if n_0 was chosen large enough, then either

$$\mathbb{P}_{p_2}(H_L(R')) \geq c_3 \tag{3}$$

or

$$\mathbb{P}_{1-p_1}(H_{L^*}(R')) \geq c_3. \tag{4}$$

If (4) holds and c_3 is chosen large enough, then as an open path in L cannot start inside and end outside a closed cycle in L^*, we can use Lemma 4 exactly as in Section 3 to obtain exponential decay of the size of $C_0(L)$ in $\mathbb{P}_{p_1}$.

If (3) holds and c_3 is chosen large enough, then $\theta_L(p_2) > 0$ follows. We shall outline the argument, which is exactly the same as in [5]. Choose $c_3 = p_0^{1/3}$, where p_0 is some constant for which Lemma 3 holds. For a $6n$ by $2n$ rectangle R, let $G(R)$ be the event that $H(R)$, $V(S_1)$ and $V(S_2)$ all hold, where the S_i are the two $2n$ by $2n$ 'end' squares of R. Note that $\mathbb{P}_{p_2}(V(S_i)) = \mathbb{P}_{p_2}(H(S_i)) \geq \mathbb{P}_{p_2}(H(R)) \geq c_3$. Thus, by Lemma 2, $\mathbb{P}_{p_2}(G(R)) \geq c_3^3 = p_0$. We define a 1-dependent bond percolation measure $\widetilde{\mathbb{P}}$ on $\mathbb{Z}^2$ by declaring the edge from (a, b) to $(a+1, b)$ to be open in $\widetilde{\mathbb{P}}$ if $G(R)$ holds in $\mathbb{P}_{p_2}$ for the $6n$ by $2n$ rectangle with bottom left corner $(2an, 2bn)$. The definition for vertical edges is analogous. By Lemma 3 we have percolation in $\widetilde{\mathbb{P}}$. The definition of the event $G(R)$ ensures that for any open path P in $\widetilde{\mathbb{P}}$ there is a corresponding open path P' in L. When P is infinite, so is P', so site percolation occurs in L in the probability measure $\mathbb{P}_{p_2}$, i.e., $\theta_L(p_2) > 0$. □

4.3 Site percolation in the triangular lattice

In this subsection we consider the equilateral triangular lattice $L_\triangle$ with edge length 1. We shall take the origin and the point $(0, 1)$ on the y-axis to be vertices of $L_\triangle$. Each vertex of $L_\triangle$ will be open independently with probability p; we write $\mathbb{P}_p$ for this site percolation probability measure. As usual, $L_\triangle$ will be viewed as a graph, in which vertices at distance 1 are adjacent.

It is well-known that $p_H(L_\triangle) = 1/2$. Indeed, the following result is another special case of the general results mentioned in the introduction.

Theorem 12. *In the triangular lattice $L_\triangle$, if $p > 1/2$ then $\theta(p) > 0$, while if $p < 1/2$, there is an $a = a(p) > 0$ such that $\mathbb{P}_p(|C_0(L_\triangle)| \geq n) \leq \exp(-an)$ holds for all $n \geq 0$.*

The arguments will be very similar to those in previous sections, so we only sketch the details.

Although the most natural equivalent of Lemma 4 in [5] (i.e., the standard starting point that the crossing probability for a square is 1/2 in $p = 1/2$ bond percolation on $\mathbb{Z}^2$) applies to a parallelogram with a 60 degree angle, we shall work with rectangles; parallelograms do not seem to fit together in the way required for the equivalent of Lemma 10. Also, while a symmetry argument shows that the crossing probability for a suitably oriented square is 1/2 at $p = 1/2$, this works only for certain orientations. These orientations will not be consistent with the symmetry required in Lemma 10.

Unlike in previous sections, the rectangles we consider will often not be aligned with the coordinate axes. Given a non-square rectangle, we define *long*

and *short* crossings of R in the obvious way, and write $L(R)$ and $S(R)$ respectively for the events that R has a long open crossing or a short open crossing.

As the neighbourhood of a vertex of $L_\triangle$ is connected, if C is a finite open cluster in $L_\triangle$ then its vertex boundary contains a closed cycle S surrounding C. Also, if a path in $L_\triangle$ starts inside and ends outside a cycle, then the path and cycle share a vertex. It follows that if R is not too small (say both sides have length at least two), then R has a long open crossing if and only if R does not have a short *closed* crossing. Hence,

$$\mathbb{P}_p(L(R)) + \mathbb{P}_{1-p}(S(R)) = 1.$$

In particular,

$$\mathbb{P}_{1/2}(L(R)) + \mathbb{P}_{1/2}(S(R)) = 1. \tag{5}$$

Most of the work needed to prove Theorem 12 is contained in the following lemma. Working in $\mathbb{Z}^2$, we took our rectangles to be aligned with the coordinate axes. Here, we do not specify the orientation of the rectangle R.

Lemma 13. *There is an absolute constant $c > 0$ such that for any n_0 there is an $n \geq n_0$ and a $6n$ by n rectangle R with*

$$\mathbb{P}_{1/2}(L(R)) \geq c. \tag{6}$$

Proof. The idea is to use an equivalent of Lemma 10 for $L_\triangle$. In fact, we have written the proof of Lemma 10 so that it goes through unchanged for $L = L_\triangle$, noting that the lines $y = is$ that we reflect in are symmetry axes of $L_\triangle$.

In order to use an argument similar to that of Corollary 11 to deduce Lemma 13, we need as a starting point that certain crossing probabilities of rectangles are not too small.

Consider a fixed integer s, and rectangles of the form $[a, b] \times [0, s]$, where a, b, $b - a > 2$ are integer multiples of $\sqrt{3}/2$. If R and R' are two rectangles of this form with $R \subset R'$, and R' is obtained by extending R horizontally by a distance of $\sqrt{3}/2$, then R' contains one extra column of lattice points. As R'

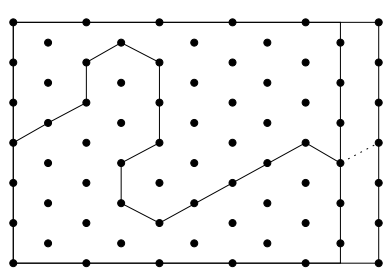

Figure 6: A rectangle R' extending a rectangle R by one column of lattice points, a path crossing R horizontally (solid lines), and an extension to a crossing of R' (dashed line).

extends R horizontally, we have $\mathbb{P}_{1/2}(H(R')) \leq \mathbb{P}_{1/2}(H(R))$. However, we also have

$$\mathbb{P}_{1/2}(H(R')) \geq \mathbb{P}_{1/2}(H(R))/2 : \tag{7}$$

$H(R)$ depends only on the states of points inside R, and if R has an open crossing then there is at least one point in $R' \setminus R$ which, if open, extends this crossing to an open crossing of R; see Figure 6.

Suppose that Lemma 13 does not hold, and in particular, that it does not hold with $c = 0.01$, say. Then there is an n_0 such that for any $n \geq n_0$ and any $6n$ by n rectangle R with any orientation, we have

$$\mathbb{P}_{1/2}(L(R)) < 0.01. \tag{8}$$

We claim that, for any integer $s \geq 6n_0$, there is a real number $t(s)$, an integer multiple of $\sqrt{3}$, such that

$$1/8 \leq \mathbb{P}_{1/2}(H([0,t] \times [0,s])) \leq 1/2$$

holds for $t = t(s)$. Indeed, as t increases, the probability above decreases from 1, and from the observation (7) above it cannot decrease by more than a factor of 4 as t increases by $\sqrt{3}$. Also, by (8), the probability above is at most 0.01 for $t = 6s$ and, using (5), at least 0.99 for $t = s/6$. Hence $s/6 \leq t(s) \leq 6s$.

This gives us a starting point for the induction used in the proof of Corollary 11: using (5) we see that for $s = 6n_0$, $S_1 = [0, t(s)] \times [0, s]$ has $\mathbb{P}_{1/2}(H(S_1))$, $\mathbb{P}_{1/2}(V(S_1)) \geq 1/8$. The same follows for $S_i = [0, t(s)] \times [(i-1)s, is]$, as each S_i is positioned in the same way with respect to the lattice as S_1. The second ingredient of the starting point is the large rectangle R, for which we may take $[0, t(40s)] \times [0, 40s]$, using $k = 40$. Note that we have $t(40s) \geq 40s/6 = (40/36)6s \geq (40/36)t(s)$. Now the proof of Corollary 11 goes through as before, noting that all the rectangles we consider have vertices that are lattice points, and that the line $x = t(s)/2$ is a symmetry axis of $L_\triangle$. □

Proof of Theorem 12. The method is similar to that for the square lattice, so we give an outline, emphasizing the differences.

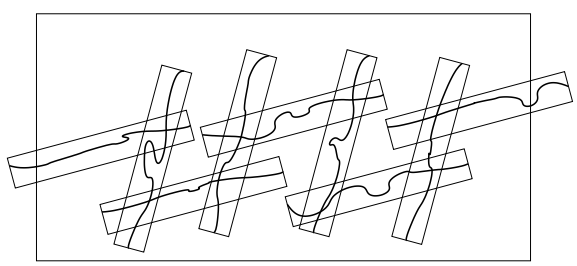

Figure 7: A path of congruent 6 by 1 rectangles crossing a large rectangle R'.

Let n_0 be a large constant, to be chosen below. Let R be a $6n$ by n rectangle with $n \geq n_0$ for which (6) holds; the existence of such an R is guaranteed

by Lemma 13. We first note that there is an absolute constant $c' > 0$ (not depending on n_0) such that if R' is a $34n$ by $10n$ rectangle with any orientation, and any position with respect to the lattice, then $\mathbb{P}_{1/2}(L(R')) \geq c'$. To see this, construct a path of rectangles R_i inside R', with each R_i congruent to R and placed similarly with respect to the grid, so that long crossings of R_i and R_{i+1} cross, and long crossings of the first and last R_i cross the opposite short sides of R', as in Figure 7. Then apply Lemma 2, noting that the number of rectangles in the path is bounded by some absolute constant. (In fact, this construction is possible starting from a rectangle R with any fixed aspect ratio larger than $(1+\sqrt{3})/2$, but with a larger aspect ratio the picture is clearer.)

The rest of the proof is as for the square lattice. Fix any $p > 1/2$ and any $c_1 < 1$. From the argument above and Lemma 8, if n_0 is chosen large enough, then for $n \geq n_0$ we have $\mathbb{P}_p(L(R'')) \geq c_1$ for every $33n$ by $11n$ rectangle R'. As in previous sections, we can apply Lemma 3 to deduce that $\theta_{L_\triangle}(p) > 0$, and Lemma 4 to deduce exponential decay of $|C_0(L_\triangle)|$ in $\mathbb{P}_{1-p}$. □

5 Non-lattice models

So far, we have considered site percolation on lattice graphs L. The lattice structure is used in two ways: firstly, the notion of percolation, or of an open crossing of a rectangle, is defined using paths in L consisting of vertices of L that are open, where the model is that the states of vertices are independent. Secondly, the symmetry of the lattice is important, principally in the application of Lemma 7. For our methods, the first use of the lattice structure is not essential, while the second is. Rather than write a very general version of Lemma 8, whose statement would be almost as long as its proof, we shall illustrate this with two examples. Before doing so, we discuss the other main ingredient of our approach, namely, a suitable equivalent of the Russo-Seymour-Welsh (RSW) Theorem.

5.1 A general weak RSW Theorem

In [4], a weak version of the RSW Theorem was proved for random Voronoi percolation, where the Voronoi cells associated to a Poisson process in the plane are coloured. Due to the more complicated setting, the proof of this result, Theorem 12 of [4], is rather long. However, as noted in [4], the result holds for a wide class of percolation models. While weaker than the natural analogue of the RSW Theorem (whose truth is not known for random Voronoi percolation), the result in [4] is enough to establish the critical probability.

Certain properties of the crossings that arise in percolation models are rather general. For example, in either bond percolation in $\mathbb{Z}^2$ or random Voronoi percolation, horizontal and vertical open crossings of the same rectangle must meet. Hence, such crossings of suitably arranged overlapping rectangles can be combined to form crossings of longer rectangles. To generalize this observation, we may consider any probability measure that assigns a *state*, open or closed,

to each point x of some set $S \subset \mathbb{R}^2$. In this setting, a horizontal open crossing of a rectangle $R \subset \mathbb{R}^2$ is a (piecewise linear) geometric path $P \subset R$ starting at a point on the left-hand side of R and ending at a point on the right-hand side, such that every point of P is open. In the bond percolation case, we may take S to be the set of points with at least one integral coordinate; this set is exactly the subset of $\mathbb{R}^2$ obtained when we draw the graph $\mathbb{Z}^2$ with straight-line segments as edges. A point of $S \setminus \mathbb{Z}^2$ is open if the corresponding edge is open. We may take the points of $\mathbb{Z}^2$ to be always open. Then crossings by open paths in the graph $\mathbb{Z}^2$ correspond to open paths P as defined above. In the random Voronoi setting we have $S = \mathbb{R}^2$, and a point of S is open if it lies in an open Voronoi cell (defined with respect to a Poisson process).

Below we shall restate Theorem 12 of [4] as Theorem 14; in [4], this result was formally stated and proved only for random Voronoi percolation, but it was noted that the proof given applies essentially without modification in a much more general setting, which we now describe.

Let us suppose that we have a probability measure $\mathbb{P}$ on assignments of a state, open or closed, to each point of some subset S of $\mathbb{R}^2$, with the following additional assumptions.

(i) The event that a point, or a measurable subset, of S is open is increasing in a suitable product space, so that Lemma 2 can be applied to events such as 'R has a horizontal open crossing'.

(ii) The whole set-up has the symmetries of $\mathbb{Z}^2$, i.e., is unchanged by translation through the vectors $(1, 0)$ and $(0, 1)$, reflection in the axes, and rotation through 90 degrees about the origin.

(iii) Disjoint regions are asymptotically independent as we 'zoom out'. To make this precise, for $R \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ let us write λR for $\{\lambda x : x \in R\}$. We assume that if R_1 and R_2 are disjoint rectangles, then for any $\varepsilon > 0$ there is a λ_0 such that for any $\lambda > \lambda_0$ and any events A_1 and A_2 defined in terms of the states of points in λR_1 and λR_2 respectively, we have $|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| \leq \varepsilon$.

(iv) Shortest paths are not too long: there is a constant C such that, for any fixed rectangle R, the probability that $H(\lambda R)$ holds but the shortest open path P crossing λR has length at least λ^C tends to zero as $\lambda \to \infty$.

Note that all these assumptions hold in the random Voronoi setting (see [4]). Also, they hold for bond percolation in $\mathbb{Z}^2$; for example, for (iv) note that any shortest open crossing of an m by n rectangle uses each vertex at most once and hence has length at most $(m + 1)(n + 1)$. We shall describe other settings in which the assumptions above hold in the subsequent subsections. We write $R_{m,n}$ for the m by n rectangle $[0, m] \times [0, n]$.

Theorem 14. *Let $c > 0$ and $\rho > 1$ be given. Under the assumptions above, if $\mathbb{P}(H(R_{n,n})) \geq c$ for all large enough n, then there is a $c' > 0$ such that for any n_0 there is an $n > n_0$ with $\mathbb{P}(H(R_{\rho n,n})) > c'$.*

Proof. Theorem 12 of [4] states that, for random Voronoi percolation, for any $\rho > 1$, $\liminf \mathbb{P}(H(R_{n,n})) > 0$ implies $\limsup \mathbb{P}(H(R_{\rho n,n})) > 0$. As noted in [4], the proof uses only the assumptions above, so the same conclusion holds in the setting here. The conclusion is exactly the claim of Theorem 14. □

5.2 Dependent bond percolation

We consider a particular model of bond percolation on $\mathbb{Z}^2$, where the states of the edges are not independent. As far as we are aware, this model has not been previously considered. Let $(\frac{1}{2}\mathbb{Z})^2$ consist of the points $x = (a, b)$ with $2a, 2b \in \mathbb{Z}$. Our underlying probability space will consist of independent identically distributed $\{-1, +1\}$-valued random variables v_x, $x \in (\frac{1}{2}\mathbb{Z})^2$, with $\mathbb{P}(v_x = +1) = p$. Let w be a function from $(\frac{1}{2}\mathbb{Z})^2$ to $\mathbb{Z}$ with the following properties: $w(a, b) \geq 0$ for all (a, b), w has finite support, $w(0, 0)$ is odd, $w(a, b)$ is even unless $a = b = 0$, and $w(a, b) = w(b, a) = w(-a, b)$ for all $(a, b) \in (\frac{1}{2}\mathbb{Z})^2$, so w has the rotational and reflectional symmetries of $\mathbb{Z}^2$. We assign states to the edges of $\mathbb{Z}^2$ as follows: an edge e of $\mathbb{Z}^2$ has a midpoint $m(e) \in (\frac{1}{2}\mathbb{Z})^2$. Let e be open if

$$\sum_{x \in (\frac{1}{2}\mathbb{Z})^2} w(x) v_{m(e)+x} > 0. \tag{9}$$

Note that the sum above is always odd, and that if $p = 1/2$ then e is open with probability $1/2$.

Let us write $\mathbb{P}_p^w$ for the probability measure defined above. As before, we write C_0 for the open cluster containing the origin, i.e., the set of vertices of $\mathbb{Z}^2$ connected to $(0, 0)$ by a path of open edges. We write $\theta^w(p)$ for $\mathbb{P}_p^w(|C_0| = \infty)$. Our next result shows that the Harris-Kesten result for (independent) bond percolation in $\mathbb{Z}^2$ extends to this particular locally-dependent setting.

Theorem 15. *Let $w : (\frac{1}{2}\mathbb{Z})^2 \to \mathbb{Z}$ satisfy the conditions above. If $p > 1/2$, then $\theta^w(p) > 0$. If $p < 1/2$, then there is an $a = a(w, p) > 0$ such that $\mathbb{P}_p^w(|C_0| \geq n) \leq \exp(-an)$ for all $n > 0$.*

We outline the proof, which is very similar to the proof of the Harris-Kesten Theorem given in [5] together with the proof of Theorem 5; note that these results are exactly the special case when $w = 0$ except at the origin.

Outline proof of Theorem 15. As usual, given a rectangle with integer coordinates we write $H(R)$ ($V(R)$) for the events that R has a horizontal (vertical) crossing by open edges. Let L^* be the dual lattice to $L = \mathbb{Z}^2$; we may realize L^* so that the dual edge e^* of each edge e of L has the same midpoint as e. As in the independent case, taking the state of e^* to be the state of e, $R = [a, b] \times [c, d]$ has a horizontal open crossing if and only if the corresponding dual rectangle $R' = [a + 1/2, b - 1/2] \times [c - 1/2, d + 1/2]$ has no closed vertical crossing; indeed the probability measure is irrelevant to this observation. In our set-up, the state of e^* is also defined by (9). Hence, e^* is closed if and only if

$$\sum_{x \in (\frac{1}{2}\mathbb{Z})^2} w(x)(-v_{m(e^*)+x}) > 0,$$

and the distribution of closed edges in $\mathbb{P}_p^w$ is exactly the distribution of open edges in $\mathbb{P}_{1-p}^w$. Taking R to be an $n+1$ by n rectangle and using the isomorphism

between $L = \mathbb{Z}^2$ and its dual that rotates R' onto R, it follows that $\mathbb{P}^w_p(H(R)) + \mathbb{P}^w_{1-p}(H(R)) = 1$. In particular, $\mathbb{P}^w_{1/2}(H(R)) = 1/2$, as in the independent case.

Writing $R_{m,n}$ for an m by n rectangle, we have

$$\mathbb{P}^w_{1/2}(H(R_{n,n})) \geq \mathbb{P}^w_{1/2}(H(R_{n+1,n})) = 1/2. \tag{10}$$

Our set-up satisfies all the conditions of Theorem 14: we define $S \subset \mathbb{R}^2$ and the states of points of S exactly as in the independent case discussed above. From (9), the event that a bond is open is increasing in the product probability space defined by the v_x, and condition (i) follows. Condition (ii) follows from our symmetry assumptions on w, and (iv) is immediate as for independent bond percolation. Finally, (iii) follows from the assumption that w has finite support – indeed, for some constant D we obtain complete independence of regions separated by a distance of at least D.

Using Theorem 14 and (10) there is a $c' > 0$ such that there are arbitrarily large n with $\mathbb{P}^w_{1/2}(R_{10n,n}) > c'$. We claim that for any $c'' < 1$, there are arbitrarily large n with $\mathbb{P}^w_{1/2}(R_{6n,2n}) > c''$. Indeed, using Lemma 7 the proof is essentially the same as that of Lemma 8, but without the complications arising from non-square lattices; we omit the details. Since the event $H(R)$ depends only on variables v_x for x within a fixed distance of R, taking n large enough we may use Lemma 3 to deduce that $\theta^w(p) > 0$: the argument is exactly that given in the last paragraph of Section 4.2, or as in [5]. Similarly, we may use Lemma 4 to deduce exponential decay for $p < 1/2$, as in Section 3. □

5.3 Random discrete Voronoi percolation

Our final example will be a discrete approximation of random Voronoi percolation. Random Voronoi percolation, described below, was introduced in the context of first-passage percolation by Vahidi-Asl and Wierman [22]. The critical probability, 1/2, was established in [4]. The proof there is rather long; the majority of the difficulties arise in attempting to compare Voronoi percolation with a suitable discrete model, to which the method of [5] can be applied. Here we shall give a much simpler proof of a discrete result.

We start with $L = \mathbb{Z}^2$. Given $0 < \pi \leq 1$, we select vertices of L independently at random, selecting each with probability π, to form a random set L_π. Given L_π, we form the Voronoi cells associated to these points: for $z \in L_\pi$ let

$$V(z) = V_{L_\pi}(z) = \{x \in \mathbb{R}^2 : d(x, z) = \inf_{y \in L_\pi} d(x, y)\},$$

where $d(.,.)$ is the Euclidean distance. Thus $V(z)$ is the set of points in the plane at least as close to z as to any other point y of L_π. We include the boundary, obtaining a set of closed convex polygons $V(z)$, $z \in L_\pi$, that tile $\mathbb{R}^2$. We say that two cells $V(z_1)$, $V(z_2)$ are *weakly adjacent* if they share at least one point, and *strongly adjacent* if they share an edge. These definitions may differ; indeed, they will do so wherever four or more cells meet at a vertex. Given L_π and $0 < p < 1$, we assign each Voronoi cell a *state*, open or closed, taking cells

to be open with probability p, independently of each other. We write $\mathbb{P}_p^\pi$ for the associated probability measure.

A strong (weak) path of open cells is a sequence of open cells in which each consecutive pair is strongly (weakly) adjacent. The strong (weak) open component of the origin is the set of cells joined by a strong (weak) open path to a cell containing the origin.

Theorem 16. *Let $0 < \pi \leq 1$ and $0 < p < 1$ be given. If $p > 1/2$ then with positive $\mathbb{P}_p^\pi$-probability the weak component of the origin is infinite. If $p < 1/2$ then there is an $a = a(\pi, p) > 0$ such that the $\mathbb{P}_p^\pi$-probability that the strong component of the origin contains more than n cells is at most* $\exp(-an)$.

As $\pi \to 0$, after suitable rescaling L_π converges to a Poisson process on $\mathbb{R}^2$, and the Voronoi tiling associated to L_π approaches that associated to a Poisson process. In such a tiling, cells meet only three at a vertex, so strong and weak connections coincide. Thus, for small π, the set-up considered in Theorem 16 is a good approximation to random Voronoi percolation, and the result strongly suggests that the critical probability for random Voronoi percolation in the plane is $1/2$, as shown in [4]. However, one cannot just deduce this result (this would amount to an unjustified exchange of the order of two limits); in fact, dealing with random Voronoi percolation is much harder.

Outline proof of Theorem 16. Let us associate a random variable v_z to each vertex z of $L = \mathbb{Z}^2$. We take $v_z = 0$ if $z \notin L_\pi$, $v_z = +1$ if $z \in L_\pi$ and $V(z)$ is open, and $v_z = -1$ if $z \in L_\pi$ and $V(z)$ is closed. Thus the v_z are independent and identically distributed, with $\mathbb{P}(v_z = i) = p_i$, where $p_{-1} = \pi(1-p)$, $p_0 = 1-\pi$ and $p_{+1} = \pi p$. Let us say that a x point of $\mathbb{R}^2$ is *open* if it lies in an open cell. Equivalently, x is open if there is a $z \in L$ with $v_z = +1$ such that no $z' \in L$ with $d(x, z') < d(x, z)$ has $v_{z'} = -1$. This event is increasing with respect to the v_z. Note that two cells $V(z)$, $V(z')$ are connected by a weak open path if and only if there is a piecewise linear path $P \subset \mathbb{R}^2$ joining z and z' with every point of P open. Given a rectangle R, let us define horizontal and vertical open crossings of R in terms of such paths P.

We claim that the conditions of Theorem 14 apply; condition (i) follows from our definition of openness for points of $\mathbb{R}^2$. Condition (ii) is immediate – our set-up inherits the symmetries of the lattice $L = \mathbb{Z}^2$ we started from. (iii) is very easy to check: for a fixed rectangle R_1, for large λ it is very likely that every circle of radius $\log \lambda$ centered within distance $\log \lambda$ of λR_1 contains at least one point of L_π; the expected number of such circles containing no points of L_π tends to 0 as $\lambda \to \infty$. It follows that with probability $1 - o(1)$ the states of all points in λR_1 are determined by the variables v_z for z within distance $2 \log \lambda = o(\lambda)$ of λR_1; asymptotic independence follows immediately. For (iv), very crudely, with probability $1 - o(1)$ the length of a shortest path crossing λR is at most λ^3, as all Voronoi cells meeting λR have diameter at most $\log \lambda$, so there are at most $O(\lambda^2)$ such cells.

Let R be any rectangle. Defining a point of $\mathbb{R}^2$ to be *closed* if it lies in a closed cell (so some points are both open and closed, if they are in the boundary

of an open cell and of a closed cell), it is easy to check that either R is crossed horizontally by an open path, or R is crossed vertically by a closed path, or both. (As usual, consider the topological boundary of the set of open points in R reachable by an open path from a point on the left-hand side of R.) It follows that for any $\pi > 0$ and any p we have $\mathbb{P}_p^\pi(H(R)) + \mathbb{P}_{1-p}^\pi(V(R)) \geq 1$. Thus, writing $R_{m,n}$ for an m by n rectangle, $\mathbb{P}_{1/2}^\pi(H(R_{n,n})) \geq 1/2$ for all n. Hence, by Theorem 14 there is a $c' > 0$ such that

$$\mathbb{P}_{1/2}^\pi(H(R_{10n,n})) > c' \tag{11}$$

for arbitrarily large n.

The rest of the argument is again similar to that in [5] and in Section 3. It suffices to show that for any fixed $p > 1/2$, $c'' < 1$ and n_0, there is an $n \geq n_0$ such that

$$\mathbb{P}_p^\pi(H(R_{6n,2n})) > c''. \tag{12}$$

Then, recalling that open paths in $\mathbb{R}^2$ correspond to weak paths of open cells, the first statement of Theorem 16 follows from Lemma 3 as usual (see the end of Subsection 4.2, or [5]), except that we must be a little careful defining the 1-dependent measure: to achieve 1-dependence, we work with a modified form $G'(R)$ of the event $G(R)$, where $G'(R)$ depends only on the variables v_z for z within distance $n/2$, say, of the $6n$ by $2n$ rectangle R. For n_0 large enough, we can find such a $G'(R)$ with probability close to that of $G(R)$; the argument is as for asymptotic independence. The same technicality arises in the Voronoi setting; see Section 8 of [4]. For the second statement, we use Lemma 4 as in Section 3, noting that if C is a weak cycle of open cells, then no strong path of closed cells starts inside and ends outside C.

To deduce (12) from (11), we argue as in the proof of Lemma 8; there are two minor complications. Firstly, it is convenient to work in the product of three element probability spaces, as above, so we need a version of Lemma 7 that applies in this setting. Such a result is given in [4]; the proof is a very simple modification of the proof of Theorem 3.2 of Friedgut and Kalai [8]. Secondly, as the event $H(R)$ depends on points outside R, it is no longer quite true that the crossing probability of a rectangle in $\mathbb{R}^2$ and of the corresponding rectangle in the torus coincide. However, the difference tends to zero as the rectangle and torus become larger in a constant ratio; the argument is the same as for asymptotic independence above. □

References

[1] M. Aizenman and D.J. Barsky, Sharpness of the phase transition in percolation models, *Comm. Math. Phys.* **108** (1987), 489–526.

[2] M. Aizenman and C.M. Newman, Tree graph inequalities and critical behavior in percolation models, *J. Statist. Phys.* **36** (1984), 107–143.

[3] P. Balister, B. Bollobás and M. Walters, Continuum percolation with steps in the square or the disc, to appear in *Random Structures and Algorithms.*

[4] B. Bollobás and O.M. Riordan, The critical probability for random Voronoi percolation in the plane is 1/2, submitted. Preprint available from `http://arXiv.org/math/0410336`.

[5] B. Bollobás and O.M. Riordan, A short proof of the Harris-Kesten Theorem, submitted. Preprint available from `http://arXiv.org/math/0410359`.

[6] J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, The influence of variables in product spaces, *Israel J. Math.* **77** (1992), 55–64.

[7] S.R. Broadbent and J.M. Hammersley, Percolation processes. I. Crystals and mazes, *Proc. Cambridge Philos. Soc.* **53** (1957), 629–641.

[8] E. Friedgut and G. Kalai, Every monotone graph property has a sharp threshold, *Proc. Amer. Math. Soc.* **124** (1996), 2993–3002.

[9] G. Grimmett, *Percolation*, Second edition. Springer-Verlag, Berlin, 1999. xiv+444 pp. ISBN 3-540-64902-6.

[10] J.M. Hammersley, Percolation processes. II. The connective constant, *Proc. Cambridge Philos. Soc.* **53** (1957), 642–645.

[11] J.M. Hammersley, Percolation processes: Lower bounds for the critical probability, *Ann. Math. Statist.* **28** (1957), 790–795.

[12] J.M. Hammersley, Bornes supérieures de la probabilité critique dans un processus de filtration, Le calcul des probabilités et ses applications. Paris, 15-20 juillet 1958, Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII (1959), pp. 17–37.

[13] T.E. Harris, A lower bound for the critical probability in a certain percolation process, *Proc. Cam. Philos. Soc.* **56** (1960), 13–20.

[14] J. Kahn, G. Kalai and N. Linial, The influence of variables on boolean functions, Proc. 29-th Annual Symposium on Foundations of Computer Science, 68-80, Computer Society Press, 1988.

[15] H. Kesten, The critical probability of bond percolation on the square lattice equals 1/2, *Comm. Math. Phys.* **74** (1980), 41–59.

[16] H. Kesten, Analyticity properties and power law estimates of functions in percolation theory, *J. Statist. Phys.* **25** (1981), 717–756.

[17] D. J. Kleitman, Families of non-disjoint subsets. *J. Combinatorial Theory* **1** (1966), 153–155.

[18] T.M. Liggett, R.H. Schonmann and A.M. Stacey, Domination by product measures, *Annals of Probability* **25** (1997), 71–95.

[19] M.V. Menshikov, Coincidence of critical points in percolation problems, *Soviet Math. Dokl.* **33** (1986), 856–859.

[20] L. Russo, A note on percolation. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **43** (1978), 39–48.

[21] P.D. Seymour and D.J.A. Welsh, Percolation probabilities on the square lattice, in Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977). *Ann. Discrete Math.* **3** (1978), pp 227–245.

[22] M.Q. Vahidi-Asl and J.C. Wierman, First-passage percolation on the Voronoi tessellation and Delaunay triangulation, in Random graphs '87 (Poznań, 1987), Wiley, Chichester (1990), pp 341–359.