A short proof of the Harris-Kesten Theorem

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Abstract

We give a short proof of the fundamental result that the critical probability for bond percolation in the planar square lattice $\mathbb{Z}^2$ is equal to $1/2$. The upper bound was proved by Harris, who showed in 1960 that percolation does not occur at $p = 1/2$. The other, more difficult, bound was proved by Kesten, who showed in 1980 that percolation does occur for any $p > 1/2$.

1 Introduction

Let us set the scene by recalling some basic notions of percolation theory, in the very special context we shall study here. Let $\mathbb{Z}^2$ be the planar square lattice, i.e., the graph whose vertices are the points of $\mathbb{Z}^2$, in which vertices at Euclidean distance 1 are joined by an edge. A *bond percolation measure* on $\mathbb{Z}^2$, or any other graph, is a probability measure on the space of assignments of a *state*, namely *open* or *closed*, to each edge $e \in E(\mathbb{Z}^2)$ (with the usual σ-field of measurable events). Most of the time we shall consider the product measure $\mathbb{P}_p = \mathbb{P}_p^{\mathbb{Z}^2}$, in which the states of the edges are independent, and every edge is open with probability p.

An *open cluster* is a maximal connected subgraph of $\mathbb{Z}^2$ all of whose edges are open. We write C_v for the open cluster containing a given vertex $v \in \mathbb{Z}^2$. Thus a vertex w lies in C_v if and only if w can be reached from v by an *open path*, i.e., a path in $\mathbb{Z}^2$ all of whose edges are open.

The fundamental question of percolation theory is 'when does percolation occur', i.e., 'for which p is there an infinite open cluster'? Of course, this question can be, and is, asked in a wide variety of contexts. Here we shall

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consider only the particular case of *bond percolation in* $\mathbb{Z}^2$ described above. Note that the question makes sense: if E_∞ is the event that there is an infinite open cluster, then by Kolmogorov's 0/1-law, $\mathbb{P}_p(E_\infty)$ is 0 or 1 for any p.

Writing $|C_v|$ for the number of vertices of C_v, let

$$\theta(p) = \mathbb{P}_p\left(|C_0| = \infty\right),$$

where $0 = (0,0)$ is the origin. Now $|C_0| = \infty$ implies E_∞, while E_∞ is the countable union of the events $|C_v| = \infty$, $v \in \mathbb{Z}^2$. From translational invariance we have $\mathbb{P}_p(|C_v| = \infty) = \theta(p)$ for all v, so $\mathbb{P}_p(E_\infty)$ is 0 if $\theta(p) = 0$ and 1 if $\theta(p) > 0$; thus, the question of when percolation occurs is precisely the question 'when is $\theta(p) > 0$?' It is easy to see that $\theta(p)$ is increasing in p, so there is a 'critical probability'

$$p_H = p_H(\mathbb{Z}^2) = \inf\{p : \theta(p) > 0\}.$$

Here, following Welsh (see [18]), the H is in honour of Hammersley. In simple cases such as the present one, it is easy to see that $0 < p_H < 1$. The problem of investigating p_H in a variety of contexts was posed by Broadbent and Hammersley [6] in 1957. Hammersley [9, 10, 11] proved general results implying in particular that $0.35 < p_H(\mathbb{Z}^2) < 0.65$. The first major progress was due to Harris [12], who proved in 1960 that $p_H(\mathbb{Z}^2) \geq 1/2$. His proof makes use of the 'self-duality' of $\mathbb{Z}^2$, and is highly non-trivial.

The *dual lattice* L^* to the lattice $L = \mathbb{Z}^2$ is the usual planar dual of $\mathbb{Z}^2$ considered as a graph drawn in the plane: there is a vertex $v \in L^*$ corresponding to each face of $\mathbb{Z}^2$, i.e., to each square $[a, a+1] \times [b, b+1]$, $a, b \in \mathbb{Z}$. For convenience, we may take $v = (a + 1/2, b + 1/2)$. There is one edge e^* of L^* corresponding to each edge e of L, joining the two vertices of L^* corresponding to the faces of L in whose boundary e lies. Of course, L^* may be defined analogously for any plane graph. When $L = \mathbb{Z}^2$, then L^* is isomorphic to L, as shown below.

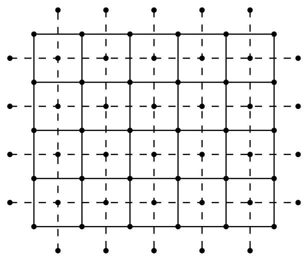

Figure 1: Portions of the lattice $L = \mathbb{Z}^2$ (solid lines) and the isomorphic dual lattice L^* (dashed lines).

For many years it was believed that $p_H = 1/2$ for bond percolation in $\mathbb{Z}^2$; see, for example [19]. However, it was only in 1980, twenty years after Harris' proved that $p_H \geq 1/2$, that Kesten [14] proved this conjecture, following significant progress by Russo [17] and Seymour and Welsh [18].

The proofs of Harris [12] and Kesten [14] are beautiful, but rather complicated and long. Here we shall give short proofs of their results for $p_H(\mathbb{Z}^2)$, the critical probability for bond percolation in $\mathbb{Z}^2$. Our methods are applicable in other contexts as well; indeed, they were developed in [3] to prove that the critical probability for random Voronoi percolation in the plane is $1/2$.

2 Preliminaries

The proofs presented here will be mostly self-contained. We shall use two results from probabilistic combinatorics, and an observation concerning percolation.

The first result is a very simple but fundamental lemma due to Harris [12]. Let X be a fixed ground set with N elements, and let X_p be a random subset of X obtained by selecting each $x \in X$ independently with probability p. For a family $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of X, let $\mathbb{P}_p^X(\mathcal{A})$ be the probability that $X_p \in \mathcal{A}$. In this context, $\mathcal{A}$ is *increasing* if $A \in \mathcal{A}$ and $A \subset B \subset X$ imply $B \in \mathcal{A}$. For example, X might be a finite set of edges of $\mathbb{Z}^2$, and X_p the subset of X consisting of the open edges. Then an event is increasing if it is preserved when the states of one or more edges are changed from closed to open.

Lemma 1. *If $\mathcal{A}$, $\mathcal{B} \subset \mathcal{P}(X)$ are increasing, then for any p we have*

$$\mathbb{P}_p^X(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p^X(\mathcal{A})\mathbb{P}_p^X(\mathcal{B}).$$

In other words, increasing events are positively correlated. Once one thinks of the statement, the proof turns out to be very simple, using induction on N. Taking complements, Lemma 1 also applies to two *decreasing* events, defined in the obvious way; similarly, an increasing event and a decreasing one are negatively correlated. The extension to infinite product spaces is immediate; we shall not need it here. In a different context, Harris' Lemma was rediscovered by Kleitman [15], and a series of important generalizations followed, culminating in the very general 'Four-functions Theorem' of Ahlswede and Daykin [1]. However, all we shall need here is the original lemma of Harris.

The next result we shall need is a sharp-threshold result of Friedgut and Kalai [7], which is itself a simple consequence of a result of Kahn, Kalai and Linial [13] (see also [5]) concerning the influences of coordinates in a product space.

With the same set-up as above, $\mathcal{A} \subset \mathcal{P}(X)$ is *symmetric* if there is a permutation group acting transitively on X whose action on $\mathcal{P}(X)$ preserves $\mathcal{A}$. In our notation, the result of Friedgut and Kalai we shall need is as follows.

Theorem 2. *There is an absolute constant c_1 such that if $|X| = N$, $\mathcal{A} \subset \mathcal{P}(X)$ is symmetric and increasing, and $\mathbb{P}_p^X(\mathcal{A}) > \varepsilon$, then $\mathbb{P}_q^X(\mathcal{A}) > 1 - \varepsilon$ whenever $q - p \geq c_1 \log(1/(2\varepsilon))/\log N$.*

Finally, we shall need an observation concerning *k-dependent percolation*. A bond percolation measure on $\mathbb{Z}^2$ is *k-dependent* if for every pair S, T of sets of edges of $\mathbb{Z}^2$ at graph distance at least k, the states (being open or closed) of the edges in S are independent of the states of the edges in T. When $k = 1$, the separation condition is exactly that no edge of S shares a vertex with an edge of T.

In static renormalization arguments, k-dependent probability measures arise very naturally. Comparisons between such measures and product measures (or arguments amounting to such comparisons) have been considered by several authors; see Liggett, Schonmann and Stacey [16] and the references therein.

Lemma 3. *There is a $p_0 < 1$ such that in any 1-dependent probability measure satisfying the additional condition that each edge is open with probability at least p_0, the probability that $|C_0| = \infty$ is positive.*

In many contexts the value of p_0 is important; the best bound known is the result of Balister, Bollobás and Walters [2] that one can take $p_0 = 0.8639$. Here, however, the value of p_0 is not important. Lemma 3 is an immediate consequence of the very general main result of [16] but, in the form above, is trivial from first principles. Indeed, if C_0 is finite, then it is surrounded by a cycle in the dual lattice L^* consisting of closed edges. Very crudely, there are at most $\ell 3^\ell$ cycles of length ℓ in L^* surrounding the origin. For each, the corresponding set S of edges of $L = \mathbb{Z}^2$ contains a subset S' of size at least $|S|/4 = \ell/4$ in which any two edges are vertex disjoint. By 1-dependence, the states of the edges in S' are independent, so the probability that all these edges are closed is at most $(1 - p_0)^{\ell/4}$. Combining these observations shows that $\mathbb{P}(|C_0| < \infty) \leq \sum_{\ell \geq 4} \ell 3^\ell (1 - p_0)^{\ell/4}$, which is less than one if p_0 is large enough.

Having collected the basic results above for future reference, we are ready to start our simple proof that $p_H(\mathbb{Z}^2) = 1/2$.

3 A short proof of Harris' Theorem

In this section we present a short proof of Harris' 1960 result that $\theta(1/2) = 0$, and hence $p_H = p_H(\mathbb{Z}^2) \geq 1/2$. The techniques are considerably simpler than Harris' original ones.

Throughout this section we shall work with 'open crossings of rectangles', a basic concept appearing in all proofs that $p_H = 1/2$. We shall identify a rectangle $R = [x_0, x_1] \times [y_0, y_1]$, where $x_0 < x_1$ and $y_0 < y_1$ are integers, with an induced subgraph of $\mathbb{Z}^2$. This subgraph includes all vertices and edges in the interior and boundary of R. If $x_1 - x_0 = k$ and $y_1 - y_0 = \ell$ then we call R a *k by ℓ rectangle.*

We write $H(R)$ for the event that there is a *horizontal open crossing* of R, i.e., a path from the left side of R to the right side consisting entirely of open edges of R. Similarly, we write $V(R)$ for the event that there is a vertical open crossing of R. Note that for disjoint rectangles R and R' (a rectangle includes its boundary) each of the events $H(R)$, $V(R)$ is independent of each of the events $H(R')$, $V(R')$.

The following starting point is very well known. For the sake of completeness we include its essentially trivial proof.

Lemma 4. *If R is an $n+1$ by n rectangle then $\mathbb{P}_{1/2}(H(R)) = 1/2$.*

Proof. Let $R = [a,b] \times [c,d]$, so $b - a = n+1$ and $d - c = n$. Recalling that we are taking the vertices of the dual lattice L^* to $L = \mathbb{Z}^2$ as the points with $1/2$-integer coordinates, let $R^* = [a+1/2, b-1/2] \times [c-1/2, d+1/2]$ be a rectangle in L^*. Defining the state of an edge e^* of L^* to be the same as the state of the corresponding edge e of L, let $V^*(R^*)$ be the event that there is a vertical crossing of R^* by *closed* edges of L^*.

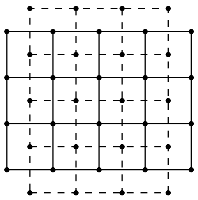

Figure 2: Portions R of the lattice $L = \mathbb{Z}^2$ (solid lines) and R^* of the dual lattice L^* (dashed lines). If one of each pair of dual (i.e., crossing) edges is selected, the selected edges contain either a left-right path across R or a top-down path across R^*, but not both.

It is well known and easy to check (see Figure 2) that $H(R)$ holds if and only if $V^*(R^*)$ does not hold, so

$$\mathbb{P}_p(H(R)) + \mathbb{P}_p(V^*(R^*)) = 1.$$

Now there is an isomorphism between L^* and L that maps R^* to R, namely a rotation through 90 degrees. A vertical crossing of R^* by closed edges maps to a horizontal crossing of R by closed edges, so $\mathbb{P}_p(V^*(R^*)) = \mathbb{P}_{1-p}(H(R))$. Taking $p = 1/2$, we obtain $2\mathbb{P}_{1/2}(H(R)) = 1$. □

It is easy to think that this lemma shows that $p_H = 1/2$. Although self-duality is of course the reason 'why' $p_H = 1/2$, a rigorous deduction is far from easy, and took twenty years to accomplish.

Note that considering an n by n square S inside an $n+1$ by n rectangle R and applying Lemma 4, we have

$$\mathbb{P}_{1/2}(V(S)) = \mathbb{P}_{1/2}(H(S)) \geq 1/2. \tag{1}$$

Lemma 5. *Let $R = [0,m] \times [0,2n]$, $m \geq n$, be an m by $2n$ rectangle. Let $X(R)$ be the event that there are paths P_1 and P_2 of open edges, where P_1 crosses $S = [0,n] \times [0,n]$ from top to bottom, and P_2 lies inside R and joins some vertex on P_1 to some vertex on the right-hand side of R. Then $\mathbb{P}_p(X(R)) \geq \mathbb{P}_p(H(R))\mathbb{P}_p(V(S))/2$.*

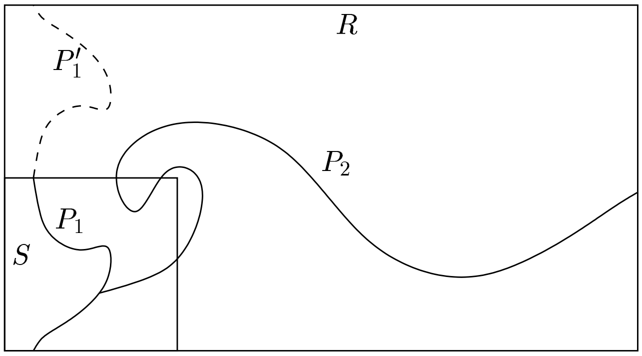

Figure 3: A rectangle R and square S inside it, drawn with paths (solid curves) whose presence as open paths would imply $X(R)$.

Proof. Suppose that $V(S)$ holds, so there is a path P of open edges crossing S from top to bottom. Note that any such P separates S into two pieces, left(P) on the left of P and right(P) on the right. When $V(S)$ holds, one can define the left-most such path P, $LV(S)$, in such a way that the event $\{LV(S) = P_1\}$ does not depend on edges to the right of P_1. There are several ways to see this; one is to define a partial order on open paths crossing S from top to bottom, where $P \prec P'$ if P is contained in $P' \cup \mathrm{left}(P')$, and to check that this partial order has a unique minimal element. Another is to define $LV(S)$ directly: start with the boundary of the set of squares (i.e., vertices of the dual lattice) in S that may be reached within S from the left of S crossing only closed edges, and remove any loops to obtain a path.

It is easy to see that for any possible value P_1 of $LV(S)$ we have $\mathbb{P}_p(X(R) \mid LV(S) = P_1) \geq \mathbb{P}_p(H(R))/2$: let P be the (not necessarily open) path formed by the union of P_1 and its reflection P_1' in the line $y = n$; see Figure 3. This path crosses $[0, n] \times [0, 2n]$ from top to bottom. With (unconditional) probability $\mathbb{P}_p(H(R))$ there is an open path P_3 of edges crossing R from left to right - this path must meet P. By symmetry, the (unconditional) probability that some such path first meets P at a point of P_1 is at least $\mathbb{P}_p(H(R))/2$. Hence, the event $Y(P_1)$ that there is a path P_2 in R to the right of P joining some point of P_1 to the right hand side of R has probability at least $\mathbb{P}_p(H(R))/2$. But $Y(P_1)$ depends only on edges to the right of P. All such edges in S are to the right of P_1 in S. As the states of these edges are independent of $\{LV(S) = P_1\}$, we have

$$\mathbb{P}_p(Y(P_1) \mid LV(S) = P_1) = \mathbb{P}_p(Y(P_1)) \geq \mathbb{P}_p(H(R))/2.$$

But $LV(S) = P_1$ and $Y(P_1)$ imply $X(R)$, so we have $\mathbb{P}_p(X(R) \mid LV(S) = P_1) \geq \mathbb{P}_p(H(R))/2$. Finally, the event $V(S)$ is a disjoint union of events of the form $\{LV(S) = P_1\}$, so $\mathbb{P}_p(X(R) \mid V(S)) \geq \mathbb{P}_p(H(R))/2$. $\square$

The following immediate corollary of Lemma 5 is standard, but is usually proved in a different way (via the Russo-Seymour-Welsh Theorem of [17] and [18]; see [8]).

Corollary 6. *Let $\rho > 1$ be a fixed integer. There is a constant $c_2(\rho) > 0$ depending only on ρ such that for any $2\rho n$ by $2n$ rectangle R we have $\mathbb{P}_{1/2}(H(R)) \geq c_2(\rho)$.*

Note that we take ρ to be an integer only because we want all our rectangles to have integer coordinates.

Proof. Writing $h_{m,n}$ for $\mathbb{P}_{1/2}(H(R))$, where R is an m by n rectangle, we claim that for $m \geq n$ we have $h_{2m-n,2n} \geq h_{m,2n}^2/2^5$. Applying this repeatedly, starting from $h_{2n,2n} \geq 1/2$ (see (1)), we obtain the result.

To see the claim, consider the rectangles $R = [0,m] \times [0,2n]$, $R' = [n-m,n] \times [0,2n]$ and the square $S = [0,n] \times [0,n]$ in their intersection, as shown in Figure 4.

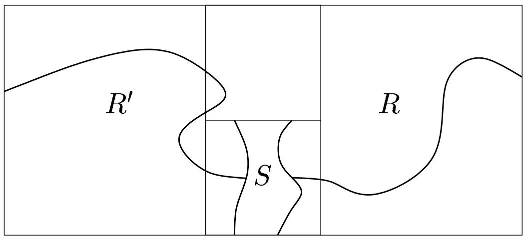

Figure 4: The overlapping rectangles R and R' with the square S in their intersection. The paths drawn show that $X(R)$ holds, as well as the reflected equivalent for R'. If $H(S)$ also holds, then so does $H(R \cup R')$.

If $E_1 = X(R)$, the corresponding (i.e., horizontally reflected) event E_2 for R', and $E_3 = H(S)$ all hold, then so does $H(R \cup R')$, using only the fact that any horizontal crossing of S meets any vertical crossing. (See Figure 4.) But the E_i are increasing events, so, by Lemma 1,

$$\mathbb{P}_{1/2}(H(R \cup R')) \geq \mathbb{P}_{1/2}(E_1)\mathbb{P}_{1/2}(E_2)\mathbb{P}_{1/2}(E_3) = \mathbb{P}_{1/2}(X(R))^2\mathbb{P}_{1/2}(H(S)).$$

Now, from (1), $\mathbb{P}_{1/2}(V(S)) = \mathbb{P}_{1/2}(H(S)) \geq 1/2$. Applying Lemma 5, it follows that

$$\begin{aligned} \mathbb{P}_{1/2}(H(R \cup R'')) &\geq \mathbb{P}_{1/2}(H(R))^2\mathbb{P}_{1/2}(V(S))^2\mathbb{P}_{1/2}(H(S))/4 \\ &\geq \mathbb{P}_{1/2}(H(R))^2/32. \end{aligned}$$

This proves the claim and thus the corollary. $\square$

Corollary 6 immediately gives Harris' result.

Theorem 7. *For bond percolation in $\mathbb{Z}^2$, $\theta(1/2) = 0$.*

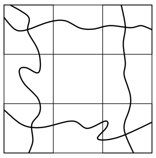

Figure 5: Four rectangles forming an annulus.

Proof. Let $c_2 = c_2(3)$ be the absolute constant given by Corollary 6. By Corollary 6 and Lemma 1, if we arrange four $6n$ by $2n$ rectangles to form a 'square annulus' as in Figure 5, then with probability at least c_2^4 each rectangle is crossed the long way by an open path. The union of any such paths contains a cycle surrounding the centre of the annulus; see Figure 5. For $k \geq 1$, let A_k be the square annulus centred on the origin with inner and outer radii 4^k and 3×4^k. Then the A_k (including their boundaries) are disjoint. Using independence of disjoint regions, at $p = 1/2$ the probability that the point $(1/2, 1/2)$ is *not* surrounded by a cycle of open edges is at most

$$\prod_{k=1}^{\infty} \mathbb{P}_{1/2}(A_k \text{ contains no open cycle}) \leq \prod_{k=1}^{\infty} (1 - c_2^4) = 0.$$

Hence, at $p = 1/2$, the probability that $(1/2, 1/2)$ is in an infinite closed cluster in the dual lattice is zero. It follows that at $p = 1 - 1/2 = 1/2$, $\theta(p)$, the probability that the origin is in an infinite open cluster, is zero. □

4 A short proof of Kesten's Theorem

Surprisingly, although it was twenty years after Harris proved that $p_H(\mathbb{Z}^2) \geq 1/2$ that Kesten finally proved that $p_H(\mathbb{Z}^2) = 1/2$, using the results mentioned in Section 2 we can deduce Kesten's theorem from Corollary 6 with very little additional work. The main idea is to apply Theorem 2.

Lemma 8. *Let $p > 1/2$, $c < 1$ and an integer $\rho > 1$ be fixed. There is a constant $n_0 = n_0(p, c, \rho)$ such that, if $n \geq n_0$ and R_n is a $4\rho n$ by $4n$ rectangle in $\mathbb{Z}^2$, then $\mathbb{P}_p(H(R_n)) \geq c$.*

Proof. To simplify the notation we shall prove the lemma in the case $\rho = 3$; the general case follows by Lemma 1.

Fix $p > 1/2$. From Corollary 6, applied with $\rho = 7$, there is an absolute constant $c_2 > 0$ such that $\mathbb{P}_{1/2}(H(R)) \geq c_2$ for any $14n$ by $2n$ rectangle R. If $H(R)$ were a symmetric event then we could just apply Theorem 2, but $H(R)$ is not symmetric, so we shall introduce symmetric auxiliary events.

For $n \geq 3$, let $\mathbb{T}_n$ be the n by n discrete torus, i.e., the graph $C_n \times C_n$, which may be obtained from $\mathbb{Z}^2$ by identifying all pairs of vertices for which the corresponding coordinates are congruent modulo n. Thus $\mathbb{T}_n$ has n^2 vertices and

$2n^2$ edges. For $1 \leq k, \ell \leq n-2$, a *k by ℓ rectangle* R in $\mathbb{T}_n$ is an induced subgraph of $\mathbb{T}_n$ corresponding to a k by ℓ rectangle $R' = [a, a+k] \times [b, b+\ell]$ in $\mathbb{Z}^2$. Note that our rectangles in the torus are always too small to 'wrap around', so the induced subgraph of $\mathbb{T}_n$ is isomorphic to the corresponding induced subgraph of $\mathbb{Z}^2$.

We shall take the edges of $\mathbb{T}_n$ to be open independently with probability p, writing $\mathbb{P}_p^{\mathbb{T}_n}$ for the corresponding probability measure. Most of the time, we shall suppress the dependence on n.

Let E_n be the event that $\mathbb{T}_{16n}$ contains *some* $14n$ by $2n$ rectangle with a horizontal crossing by open edges, or some $2n$ by $14n$ rectangle with a vertical crossing. Note that E_n is symmetric as a subset of $\mathcal{P}(X)$, where X is the edge set of $\mathbb{T}_{16n}$, which has cardinality $N = 512n^2$.

Considering one fixed $14n$ by $2n$ rectangle R in the torus, which we may identify with a corresponding rectangle in $\mathbb{Z}^2$, and writing $\mathbb{T}$ for $\mathbb{T}_{16n}$, we have

$$\mathbb{P}_{1/2}^{\mathbb{T}}(E_n) \geq \mathbb{P}_{1/2}^{\mathbb{T}}(H(R)) = \mathbb{P}_{1/2}(H(R)) \geq c_2.$$

Hence, by Theorem 2, for *any* constant $c_3 < 1$ we have

$$\mathbb{P}_p^{\mathbb{T}}(E_n) \geq c_3 \tag{2}$$

if n is large enough: we merely need to chose n so that $N = 512n^2$ has logarithm larger than a certain constant depending on c_2 and c_3.

Clearly, there are $12n$ by $4n$ rectangles $R_1, \ldots, R_{64}$ covering $\mathbb{T} = \mathbb{T}_{16n}$ such that any $14n$ by $2n$ rectangle R in $\mathbb{T}$ crosses one of the R_i, in the sense that the intersection of R and R_i is a $12n$ by $2n$ subrectangle of R_i. For example, we may take the $12n$ by $4n$ rectangles R_i whose bottom left coordinates are all possible multiples of $2n$. Similarly, there are $4n$ by $12n$ rectangles $R_{65}, \ldots, R_{128}$ so that any $2n$ by $14n$ rectangle in $\mathbb{T}$ crosses one of these from top to bottom.

It follows that if E_n holds, then so does one of the events $E_{n,i}$, $i = 1, \ldots, 128$, where each $E_{n,i}$ is the event that R_i is crossed the long way by an open path. Thus E_n^c, the complement of E_n, contains the intersection of the $E_{n,i}^c$.

By Lemma 1, applied to the product measure $\mathbb{P}_p^{\mathbb{T}}$, for each i the decreasing events $E_{n,i}^c$ and $\cap_{j<i} E_{n,j}^c$ are positively correlated. Hence

$$\mathbb{P}_p^{\mathbb{T}}(E_n^c) \geq \mathbb{P}_p^{\mathbb{T}}\left(\bigcap_{i=1}^{128} E_{n,i}^c\right) \geq \prod_{i=1}^{128} \mathbb{P}_p^{\mathbb{T}}(E_{n,i}^c) = \mathbb{P}_p^{\mathbb{T}}(E_{n,1}^c)^{128}.$$

Thus, from (2),

$$\mathbb{P}_p^{\mathbb{T}}(E_{n,1}^c) \leq \mathbb{P}_p^{\mathbb{T}}(E_n^c)^{1/128} \leq (1-c_3)^{1/128}.$$

(Sometimes this is called the 'square-root trick'.) For any constant $c > 0$, taking $c_3 = 1 - (1-c)^{128}$ we see that for n sufficiently large, $\mathbb{P}_p^{\mathbb{T}}(E_{n,1}^c) \leq 1 - c$, i.e., that $\mathbb{P}_p^{\mathbb{T}}(E_{n,1}) \geq c$.

But $\mathbb{P}_p^{\mathbb{T}}(E_{n,1})$ is just the probability of crossing a $12n$ by $4n$ rectangle R in the grid the long way. In other words, returning to $\mathbb{Z}^2$, we have

$$\mathbb{P}_p(H(R)) = \mathbb{P}_p^{\mathbb{T}_{16n}}(H(R)) \geq c \tag{3}$$

whenever R is a $12n$ by $4n$ rectangle and n is large enough, completing the proof of the lemma. $\square$

It is now easy to deduce Kesten's Theorem.

Theorem 9. *For bond percolation in $\mathbb{Z}^2$, if $p > 1/2$ then $\theta(p) > 0$.*

Proof. Let $p > 1/2$ be fixed, and let $p_0 < 1$ be a constant for which Lemma 3 holds. Set $c = p_0^{1/3}$ and fix an $n \geq n_0(p, c, 3)$, where n_0 is as in Lemma 8. We shall write m for $4n$.

Given a $3m$ by m rectangle R, let S' and S'' be the two end squares when R is cut into three squares. Note that $H(R)$ certainly implies $H(S')$, so

$$\mathbb{P}_p(V(S'')) = \mathbb{P}_p(V(S')) = \mathbb{P}_p(H(S')) \geq \mathbb{P}_p(H(R)) \geq c,$$

where the final inequality is from Lemma 8. Let $G(R)$ be the event $H(R) \cap V(S') \cap V(S'')$; see Figure 6.

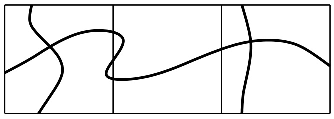

Figure 6: A $3m$ by m rectangle R such that $G(R)$ holds.

By Lemma 1,

$$\mathbb{P}_p(G(R)) \geq \mathbb{P}_p(H(R))\mathbb{P}_p(V(S'))\mathbb{P}_p(V(S'')) \geq c^3 = p_0.$$

For an m by $3m$ rectangle R' we define $G(R')$ similarly, so $G(R')$ is the event that $G(R)$ holds after rotating R' by 90 degrees to obtain R. Thus $\mathbb{P}_p(G(R')) = \mathbb{P}_p(G(R)) \geq p_0$.

We define a 1-dependent bond percolation measure $\widetilde{\mathbb{P}}$ as follows: the edge from (x, y) to $(x+1, y)$ is open in $\widetilde{\mathbb{P}}$ if and only if $G(R)$ holds in $\mathbb{P}_p$ for the $3m$ by m rectangle $[2mx, 2mx + 3m] \times [2my, 2my + m]$. Similarly, the edge from (x, y) to $(x, y+1)$ is open in $\widetilde{\mathbb{P}}$ if and only if $G(R')$ holds in $\mathbb{P}_p$ for the m by $3m$ rectangle $[2mx, 2mx + m] \times [2my, 2my + 3m]$.

This probability measure is indeed 1-dependent, as $G(R)$ depends only on the states of edges in R, and vertex disjoint edges of $\mathbb{Z}^2$ correspond to disjoint rectangles.

By Lemma 3 we have percolation in $\widetilde{\mathbb{P}}$, i.e., $\widetilde{\mathbb{P}}(|C_0| = \infty) > 0$. However, we have defined $G(R)$ in such a way that a $\widetilde{\mathbb{P}}$-open path guarantees a corresponding

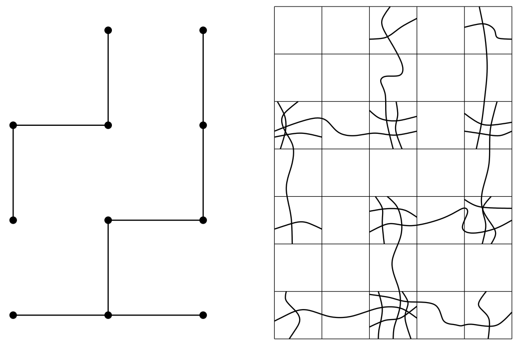

Figure 7: A set of open edges in $\widetilde{\mathbb{P}}$ (left), and corresponding rectangles R drawn with $G(R)$ holding in $\mathbb{P}_p$.

(much longer) open path in the original bond percolation, using the fact that horizontal and vertical crossings of a square must meet; see Figure 7. Hence, with positive probability some point of $[0,m]\times[0,m]$ is in an infinite $\mathbb{P}_p$-open cluster. It follows that $\theta(p)>0$, completing the proof of Theorem 9. $\square$

Together, Theorems 7 and 9 show that $p_H(\mathbb{Z}^2)=1/2$.

5 Extensions

The argument above gives a very short proof of Kesten's Theorem, using Theorem 2 as a key ingredient. In fact, as we shall show in future work [4], this argument, like those of Harris and Kesten, easily gives further results. For example, our method gives exponential decay of $|C_0|$ when $p<1/2$, showing that p_T, the critical probability at which $\mathbb{E}|C_0|$ becomes infinite, is also equal to $1/2$, another result of Kesten [14]. Furthermore, although we have written everything for bond percolation in $\mathbb{Z}^2$, the same method gives similar results in other contexts; for example, we obtain new, simple proofs that $p_T=p_H$ for site percolation in the square lattice, and that both critical probabilities are equal to $1/2$ for site percolation in the triangular lattice. The basic method used in the previous section is much more general, applying to many other two-dimensional contexts. Indeed, as remarked earlier, it was developed in [3] to prove that the critical probability for random Voronoi percolation in the plane is $1/2$.

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