9. Edge Graphs and Eccentricity Sequences

Many authors discovered edge graphs independently and gave it a different name, for example, interchange graph by Ore [177], derivative by Sabidussi [231], derived graphs by Beineke [18], edge-to-vertex dual by Seshu and Reed [233], covering graph by Kasteleyn [127] and adjoint by Menon [159].

9.1 Edge Graphs

Definition: Let $G(V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The *edge graph* $L(G)$ of G has the vertex set E and two vertices e_i and e_j are adjacent in $L(G)$ if and only if the corresponding edges e_i and e_j of G are adjacent in G. For example, in Figure 9.1, $L(G)$ is the edge graph of G. A graph G is an edge graph if it is isomorphic to the edge graph $L(H)$ of some graph H.

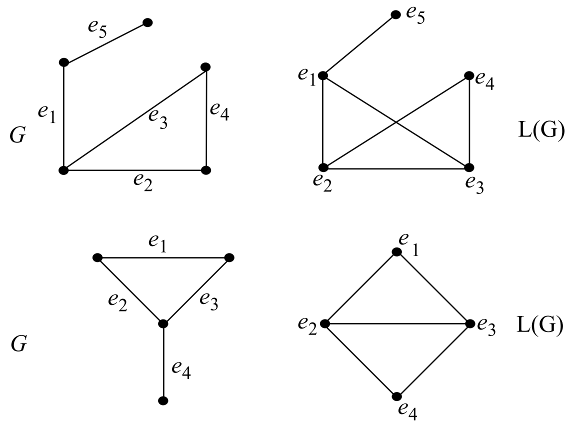


Fig. 9.1

Since isolated vertices do not contribute anything to the study of edge graphs, we assume that the graphs contain no isolated vertices. Besides this, the graphs under consideration will be without loops. We have the following observations about edge graphs.

1. A graph G is connected if and only if $L(G)$ is connected.
2. If H is a subgraph of G, then $L(H)$ is a subgraph of $L(G)$.
3. The edges incident at a vertex of G form a maximal complete subgraph of $L(G)$.
4. In G, if $e = uv$ is an edge, then the degree of e in $L(G)$ is the number of edges of G adjacent to e in G. Clearly, $d_{L(G)}(e) = d_G(u) + d_G(v) - 2$.
5. For $n > 1$, $L^n(G) = L(L^{n-1}(G))$ and $L^0(G) = G$.

The following result determines the number of edges in an edge graph.

Theorem 9.1 The number of edges m' in $L(G)$ when G has degree sequence $[d_i]_1^n$ is given by

$$m' = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

Proof Let $[d_i]_1^n$ be the degree sequence of the graph G and let $L(G)$ the edge graph of G, have m' edges.

As the degree of the vertex v_i in G is d_i, there are d_i edges incident on v_i. From these d_i edges, any two are adjacent at v_i in G. Hence the number of edges contributed by v_i to $L(G)$ is $\binom{d_i}{2}$.

Thus, $$m' = \sum_{i=1}^{n}\binom{d_i}{2} = \sum_{i=1}^{n}\frac{d_i(d_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{n}(d_i^2 - d_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \frac{1}{2}\sum_{i=1}^{n} d_i$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$ ❑

The following observation is immediate.

Theorem 9.2 The edge graph of a graph G is a path if and only if G is a path.

Proof Let G be a graph with n vertices. Assume G is a path P_n. Then $L(G)$ is the path P_{n-1} with $n-1$ vertices.

Conversely, let $L(G)$ be a path. Then no vertex of G has degree greater than two. For, if G has a vertex v of degree greater than two, the edges incident to v form a complete subgraph of $L(G)$ with at least three vertices. Therefore G is either a cycle or a path. But G cannot be a cycle, since the edge graph of a cycle is a cycle. ❑

We now have the following stronger result.

Theorem 9.3 A connected graph is isomorphic to its edge graph if and only if it is a cycle.

Proof Let G be a connected graph with n vertices, m edges and with degree sequence $[d_i]_1^n$. Let $L(G)$ be the edge graph of G. The number of vertices in $L(G)$ is m. The number of edges m' in $L(G)$ is given by

$$m' = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m.$$

Clearly, $L(G)$ is connected and $L(C_n) = C_n$.

Conversely, let $G \cong L(G)$.

Then G and $L(G)$ have the same number of vertices and edges.

So, $n = m$ and $m = \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right) - m$.

Therefore, $n = m$ and $\sum_{i=1}^{n} d_i^2 = 4m$.

Thus, variance

$$\{[d_i]\} = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2$$

$$\left[\text{Because } Var = \frac{1}{N}\sum_i f_i x_i^2 - \left(\frac{1}{N}\sum_i f_i x_i\right)^2 \text{ and we have } f_i = 1\right]$$

$$= \frac{1}{n}4m - \frac{1}{n^2}(2m)^2 = \frac{4m}{m} - \frac{4m^2}{m^2} = 4 - 4 = 0.$$

Therefore the d_i's are equal and G is regular of degree d, say.

So $nd = 2m$ implies that $d = \frac{2m}{n} = \frac{2m}{m} = 2$.

Thus G is a 2-regular connected graph, that is, C_n. ❑

The next result is about the isomorphism of edge graphs.

Theorem 9.4 If the graphs G_1 and G_2 are isomorphic, then $L(G_1)$ and $L(G_2)$ are also isomorphic.

Proof Assume (ϕ, θ) to be an isomorphism of G_1 onto G_2. Then θ is a bijection of $E(G_1)$ onto $E(G_2)$. We show that θ is an isomorphism of $L(G_1)$ to $L(G_2)$ by showing that θ preserves adjacency. Let e_i and e_j be two adjacent vertices of $L(G_1)$. So there exists a vertex v of G_1 incident to both e_i and e_j, and therefore $\phi(v)$ is a vertex incident to both $\theta(e_i)$ and $\theta(e_j)$. Thus $\theta(e_i)$ and $\theta(e_j)$ are adjacent vertices in $L(G_2)$.

Let $\theta(e_i)$ and $\theta(e_j)$ be adjacent vertices in $L(G_2)$. Then they are adjacent edges in G_2 and therefore there exists a vertex v' of G_2 incident to both $\theta(e_i)$ and $\theta(e_j)$. Then $\phi^{-1}(v')$ is a vertex of G_1 incident to both e_i and e_j, and thus e_i and e_j are adjacent vertices of $L(G_1)$.

Therefore e_i and e_j are adjacent vertices of $L(G_1)$ if and only if $\theta(e_i)$ and $\theta(e_j)$ are adjacent vertices of $L(G_2)$. Hence θ is an isomorphism of $L(G_1)$ onto $L(G_2)$. ❑

The converse of Theorem 9.4 is not true. To see this, consider the graphs $K_{1,3}$ and K_3 whose edge graphs are K_3. But $K_{1,3}$ is not isomorphic to K_3, since there is a vertex of degree three in $K_{1,3}$ while there is no such vertex in K_3. However, it was shown by Whitney [265] that the converse holds unless one is $K_{1,3}$ and the other is K_3. The proof of this result is due to Jung [123].

Theorem 9.5 Let G and G' be connected graphs with isomorphic edge graphs. Then G and G' are isomorphic unless one is K_3 and the other is $K_{1,3}$.

Proof First suppose that $n(G)$ and $n(G')$ are less than or equal to 4. A necessary condition for $L(G)$ and $L(G')$ to be isomorphic is that $m(G) = m(G')$. The only nonisomorphic connected graphs on at most four vertices are those shown in Figure 9.2.

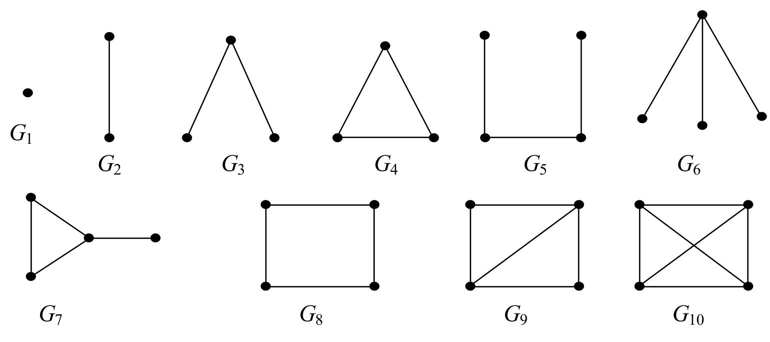


Fig. 9.2

In Figure 9.2, graphs G_4, G_5 and G_6 are the three graphs having three edges each. We know that G_4 and G_6 have isomorphic edge graphs, namely K_3. The edge graph of G_5 is a path of length 2 and hence $L(G_5)$ cannot be isomorphic to $L(G_4)$ or $L(G_6)$. Further, G_7 and G_8 are the only two graphs in the list having four edges each.

Clearly, $L(G_8) \cong G_8$ and $L(G_7)$ is isomorphic to G_9. Thus the edge graphs of G_7 and G_8 are not isomorphic. No two of the remaining graphs have the same number of edges. Hence the only non-isomorphic graphs with at most four vertices having isomorphic edge graphs are G_4 and G_6.

Now suppose that either G or G', say G, has at least five vertices and that $L(G)$ and $L(G')$ are isomorphic under an isomorphism ϕ_1. So ϕ_1 is a bijection from the edge set of G onto the edge set of G'.

We now prove that ϕ_1 transforms an induced $K_{1,3}$ subgraph of G onto a $K_{1,3}$ subgraph of G'. Let $e_1 = uv_1$, $e_2 = uv_2$ and $e_3 = uv_3$ be the edges of an induced $K_{1,3}$ subgraph of G. As G has at least five vertices and is connected, there exists an edge e adjacent to only one or all three of edges e_1, e_2 and e_3, as illustrated in Figure 9.3.

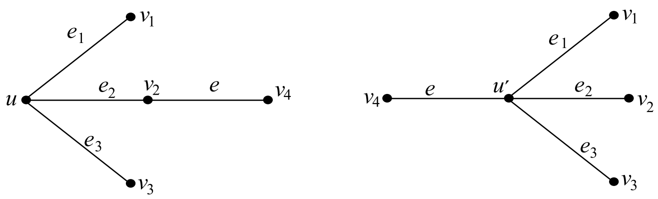


Fig. 9.3

Clearly, $\phi_1(e_1)$, $\phi_1(e_2)$ and $\phi_1(e_3)$ form either a $K_{1,3}$ subgraph or a triangle in G'. If $\phi_1(e_1)$, $\phi_1(e_2)$ and $\phi_1(e_3)$ form a triangle in G', $\phi_1(e)$ can be adjacent to precisely two of $\phi_1(e_1)$, $\phi_1(e_2)$ and $\phi_1(e_3)$ (since $L(G')$ is simple), whereas $\phi_1(e)$ must be adjacent to only one or all the three. This contradiction shows that $\{\phi_1(e_1), \phi_1(e_2), \phi_1(e_3)\}$ is not a triangle in G' and therefore forms a $K_{1,3}$ in G'.

It is clear that a similar result holds as well for ϕ_1^{-1}, since it is an isomorphism on $L(G')$ onto $L(G)$.

Let $S(u)$ denote the star subgraph of G formed by the edges of G incident at a vertex u of G. We show that ϕ_1 maps $S(u)$ onto the unique star subgraph $S(u')$ of G'.

i. First suppose the degree of u is at least 2. Let f_1 and f_2 be any two edges incident at u. The edges $\phi_1(f_1)$ and $\phi_1(f_2)$ of G' have an end vertex u' in common. If f is any other edge of G incident with u, then $\phi_1(f)$ is incident with u', and conversely, for every edge f' of G' incident with u', $\phi_1^{-1}(f')$ is incident with u. Thus $S(u)$ in G is mapped to $S(u')$ in G'.

ii. Let the degree of u be 1 and $e = uv$ be the unique edge incident with u. As G is connected and $n(G) \geq 5$, degree of v must be at least 2 in G, and therefore, by (i), $S(v)$ is mapped to a star $S(v')$ in G'. Also $\phi_1(uv) = u'v'$ for some $u' \in V(G')$. If the degree of u' is greater than 1, by (i), the star at u' in G' is transformed by ϕ_1^{-1} either to the star at u in G or to the star at v in G. But as the star at v in G is mapped to the star at v' in

G' by ϕ_1, ϕ_1^{-1} should map the star at u' to the star at u only. As ϕ_1^{-1} is $1-1$, this means that deg $u \geq 2$, a contradiction. Therefore, deg $u' = 1$ and so $S(u)$ in G is mapped to $S(u')$ in G'.

Define $\phi : V(G) \to V(G')$ by setting $\phi(u) = u'$ if $\phi_1(S(u)) = S(u')$. Since $S(u) = S(v)$ only when $u = v$ ($G \neq K_2$, $G' \neq K_2$), ϕ is $1-1$. ϕ is also onto since, for v' in G', ϕ_1^{-1} $(S(v')) = S(v)$ for some $v \in V(G)$, and by the definition of ϕ, $\phi(v) = v'$. Finally, if uv is an edge of G, then $\phi_1(uv)$ belongs to both $S(u')$ and $S(v')$, where ϕ_1 $(S(u)) = S(u')$ and ϕ_1 $(S(v)) = S(v')$. This means $u'v'$ is an edge of G'. But $u' = \phi(u)$ and $v' = \phi(v)$. Consequently, $\phi(u)$ $\phi(v)$ is an edge of G'. If u and v are nonadjacent in G, $\phi(u)$ $\phi(v)$ must be nonadjacent in G'. Otherwise, $\phi(u)$ $\phi(v)$ belongs to both $S(\phi(u))$ and $S(\phi(v))$ and hence $\phi_1^{-1}(\phi(u)\phi(v)) = uv \in E(G)$, a contradiction. Thus G and G' are isomorphic under ϕ. ❑

The following result shows that $K_{1,3}$ is not an edge graph and thus $K_{1,3}$ is of great significance in studying edge graphs as will be seen in further discussions.

Lemma 9.1 The star $K_{1,3}$ is not an edge graph.

Proof Assume that $K_{1,3}$ is an edge graph. Then $K_{1,3} = L(H)$ for some graph H. Since $K_{1,3}$ has four vertices, therefore H has four edges. Also H is connected. All the connected graphs with four edges are given in Figure 9.4.

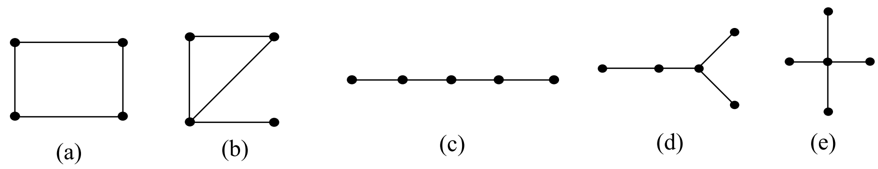


Fig. 9.4

H is neither graph (a) nor (b), because $L(C_4) = C_4$ and $L(K_{1,3} + x) = K_4 - x$. These are shown in Figure 9.5.

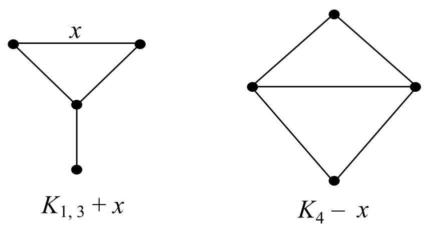


Fig. 9.5

Thus H is one of the three trees as given in (c), (d) and (e). But the edge graphs of these trees are the path P_4, the graphs K_3. K_2 and K_4, given in Figure 9.6.

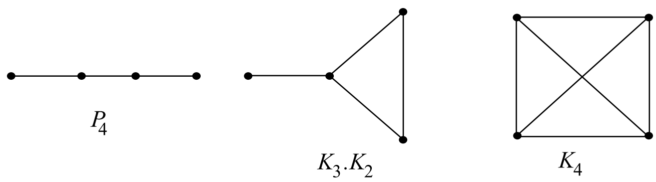


Fig. 9.6

This shows that H is none of the trees (c), (d) or (e). Hence it follows that $K_{1,3}$ is not an edge graph. ❑

Now, we proceed to give a characterisation of edge graphs which is due to Krausz [140].

Theorem 9.6 (Krausz) A graph G is the edge graph of some graph if and only if the edges of G can be partitioned into cliques such that no vertex appears in more than two cliques.

Proof

Necessity Let G be an edge graph of a graph H. Assume without loss of generality that H has no isolated vertices. Then the edges in the star $K_{1,3}$ at each vertex of H induce a clique of G and every edge of H belongs to the stars of exactly two vertices of H, therefore no vertex of G is in more than two of these cliques.

Sufficiency Let the edges of G be partitioned into the cliques $S_1, S_2, \ldots, S_k$ such that no vertex of G belongs to more than two of these cliques. We construct H such that $L(H) = G$. As isolated vertices of G become isolated edges of H, therefore assume $\delta(G) \geq 1$. Let v_1, $v_2, \ldots, v_\ell$ be the vertices of G (if any) that appear in exactly one of S_i. The vertices of H correspond to the set $S = \{S_1, S_2, \ldots, S_k, \{v_1\}, \{v_2\}, \ldots, \{v_\ell\}\}$, with one vertex for each member of S. Any two of these vertices are adjacent whenever their corresponding sets intersect. Each vertex of G appears in exactly two sets in S and no two vertices appear in the same pair of sets. Thus H is a simple graph with one edge for each vertex of G. If vertices are adjacent in G, they appear together in some S_i and the corresponding edges of H share the vertex corresponding to S_i. Hence, $G = L(H)$. ❑

Krausz characterisation of edge graphs is close to the definition. Since it characterises edge graphs by the existence of a special edge partition, it does not directly give an efficient test. This has been improved by Van Rooij and Wilf [227] by describing the structural criterion for a graph to be an edge graph. Before we take the Van Rooij and Wilf characterisation, we have the following definition.

Definition: An *induced subgraph* is a subgraph which is maximal on its vertex set. A triangle T of a graph G is said to be *odd*, if there is a vertex of G adjacent to an odd number of vertices of T, otherwise T is said to be *even*. That is, T is odd if $|V(T) \cap N(v)|$ is odd, for some $v \in V(G)$, and T is even if $|V(T) \cap N(v)|$ is even, for every $v \in V(G)$. An induced copy of $K_4 - e$ is a double triangle and clearly has two triangles with a common edge.

The following is the Van Rooij and Wilf characterisation.

Theorem 9.7 A graph G is the edge graph of some graph if and only if G does not contain an induced subgraph $K_{1,3}$ and no double triangle of G has two odd triangles.

Proof

Necessity Let $G = L(H)$. Clearly, G does not contain an induced subgraph $K_{1,3}$, since $K_{1,3}$ itself is not an edge graph. Now we observe that the vertices of a double triangle in G correspond to the edges of a $K_{1,3}+e$ in H. In particular, one of these double triangles in G is generated by a triangle in H. Obviously a triangle in G generated by a triangle in H is even, since an edge incident to a triangle in H intersects exactly two edges of the triangle in H (Fig. 9.7).

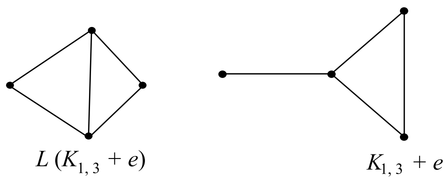


Fig. 9.7

Sufficiency Let G not contain an induced subgraph $K_{1,3}$ and let no double triangle of G have two odd triangles. Assume G is connected, for otherwise, we apply the construction to each component. In case G is $K_{1,3}-$free and has a double triangle with both triangles even, then G is one of the graphs given in Figure 9.8.

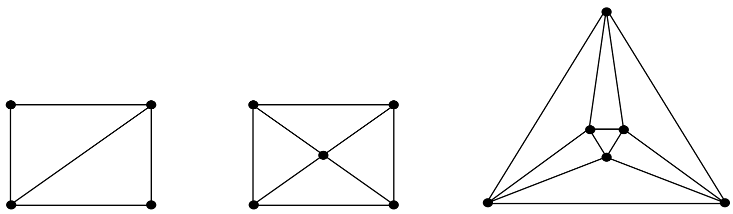


Fig. 9.8

Thus we consider the case when every double triangle of G has exactly one odd triangle. To prove the result, it suffices by Theorem 9.6, to partition $E(G)$ into cliques that cover each vertex at most twice. Now, let $S_1, S_2, \ldots, S_k$ be the maximal cliques of G that are not even triangles and let $T_1, T_2, \ldots, T_\ell$ be the edges that belong to one even triangle and no odd triangle. We claim that $B = \{S_i\} \cup \{T_j\}$ partitions $E(G)$ into cliques using each vertex at most twice.

Now, every edge appears in a maximal clique, but every triangle in a clique with more than three vertices is odd. Therefore T_j is not in any clique S_i. Also S_i and $S_{i'}$ have no common edge, because G has no double triangles with both triangles odd. Thus the cliques in B are edge-disjoint. If $e \in E(G)$, then e belongs to some S_i unless the only maximal clique containing e is an even triangle. In this case e is a T_j, since the double triangles do not have both triangles even (Fig. 9.9).

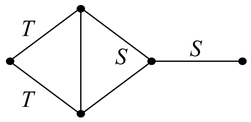


Fig. 9.9

We now show that each $v \in G$ appears at most twice in B. Assume v belongs to B_1, B_2, $B_3 \in B$. Edge-disjointness implies that v has neighbours x, y, z with each belonging to only one of B_1, B_2, B_3. Since G has no induced $K_{1,3}$, assume that xy is an edge. Now by edge-disjointness, the triangle vxy does not belong to a member of B. Therefore vxy is an even triangle. Thus z has exactly one other edge to vxy, say zx, while zy is not an edge. But now the same argument shows that zvx is an even triangle and we have a double triangle with both triangles even. This contradicts our supposition and hence each $v \in G$ appears at most twice in B. ❑

The next characterisation due to Beineke [149] displays those subgraphs which are not present in edge graphs. These subgraphs other than $K_{1,3}$ are vertex-minimal $K_{1,3}$-free graphs containing a double triangle with both triangles odd. Each such graph has a double triangle and one or two additional vertices that make the triangles odd by having one or three neighbours in the triangle.

Theorem 9.8 A graph G is an edge graph of some graph if and only if G does not contain an induced subgraph of any one of the graphs in Figure 9.10.

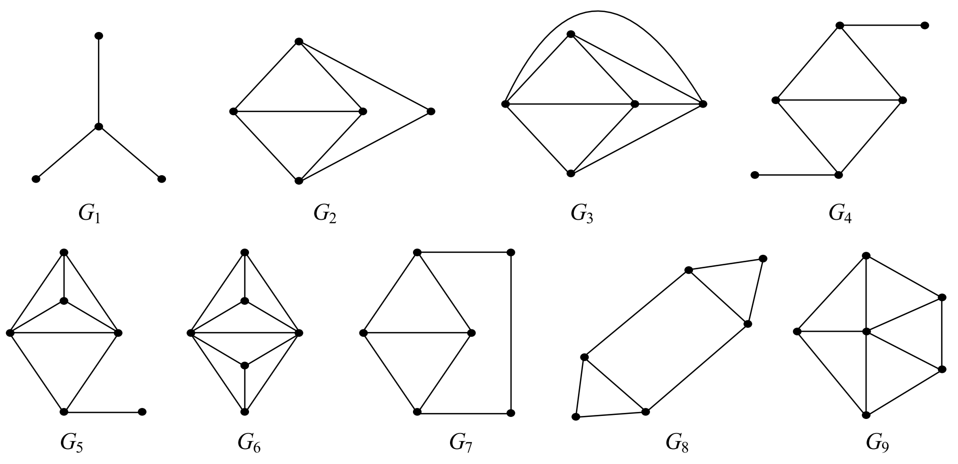


Fig. 9.10

Proof

Necessity Let G be the edge graph of some graph H so that $G = L(H)$. Then by Theorem 9.6, the edges of G can be partitioned into cliques such that every vertex appears in at most two cliques. We observe that none of these nine graphs have such a partition. Since every induced subgraph of an edge graph is itself an edge graph, G does not contain an induced subgraph of any one of the nine graphs, in Figure 9.10.

Sufficiency Let G not contain an induced subgraph of any one of these nine graphs. We prove that no double triangle of G has two odd triangles. Assume to the contrary that G has a double triangle both of which are odd. Let these triangles be abc and abd with c and d non adjacent. We discuss two cases, one in which there is a vertex v adjacent to an odd number of vertices of both odd triangles and second when there is no such vertex.

Case 1 Assume there is a vertex v which is adjacent to an odd number of vertices in each of the triangles abc and abd. Now two possibilities arise; either v is adjacent to exactly one vertex of each of these triangles, or it is adjacent to more than one vertex of one of them. If v is adjacent to exactly one vertex of each of these triangles, then either v is adjacent to a or b giving G_1, or to both c and d giving G_2. If v is adjacent to more than one vertex of one of the triangles, then v is adjacent to all four vertices of the two triangles, giving G_3 as an induced subgraph of G (Fig. 9.11).

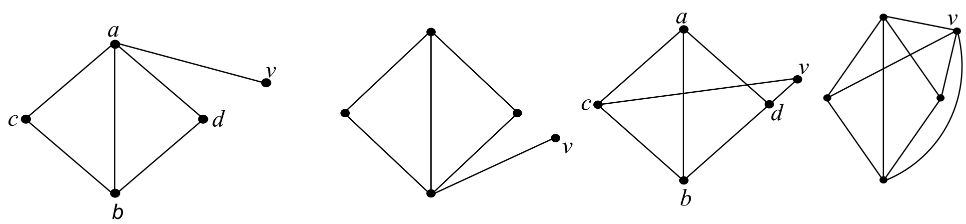


Fig. 9.11

Case 2 Now, let there be no vertex adjacent to an odd number of vertices of both triangles. Assume that the vertex u is adjacent to an odd number of vertices of triangle abc and the vertex v is adjacent to an odd number of vertices of triangle abd. We consider three subcases.

Case 2.1 u is adjacent to exactly one vertex of abc and v is adjacent to exactly one vertex of abd.

Case 2.2 One of u or v is adjacent to all three vertices of its triangle and the other to only one vertex of its triangle.

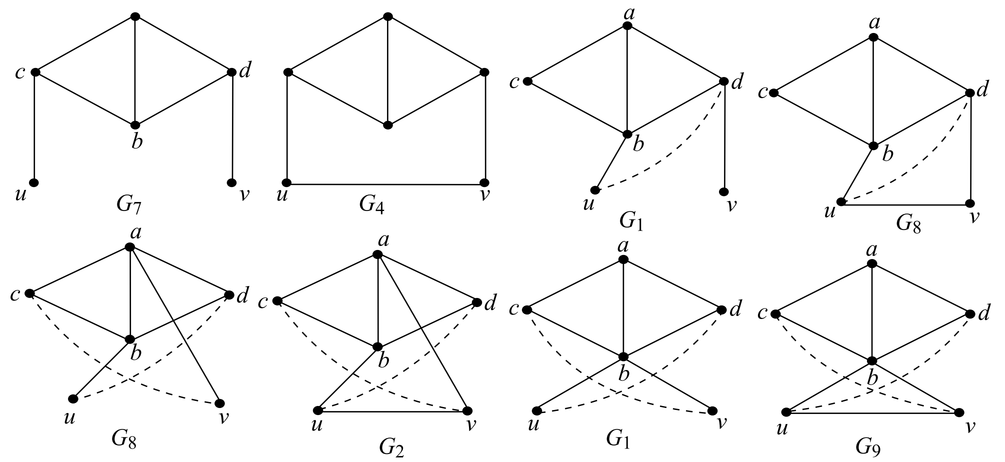
Case 2.3 u is adjacent to all three vertices of abc and v is adjacent to all three vertices of abd.

We observe that if u or v is adjacent to a or b, then it is also adjacent to c or to d, since otherwise G_1 is an induced subgraph. Also, neither u nor v is adjacent to both c and d, since otherwise G_2 or G_3 is induced.

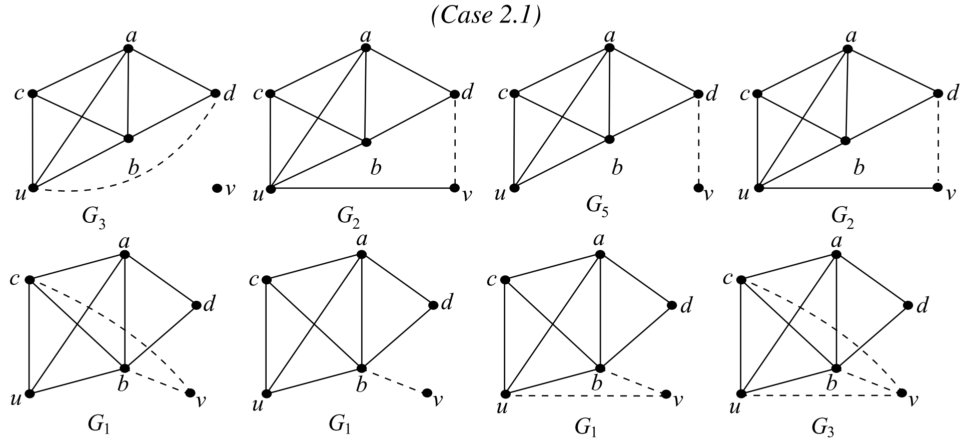
Case 2.1 Let uc, $vd \in G$. Then for $uv \in G$, G_4 is induced, and for $uv \notin G$, G_7 is induced. Now, let ub, $vd \in G$. Then it follows from the above observations that $ud \in G$ while $vc \notin G$. Therefore for $uv \notin G$, the vertices a, d, u, v induce G_1 and for $uv \in G$, the vertices a, b, c, d, u, v induce G_8. Next, let ub, $va \in G$, then clearly ud, $vc \in G$. So, when $uv \notin G$, G_8 is induced, and when $uv \in G$, G_2 is induced. Finally, let ub, $vb \in G$, then again ud, $vc \in G$. Therefore, when $uv \in G$, G_9 is induced, and when $uv \notin G$, G_1 is induced (Fig. 9.12, Case 2.1).

Case 2.2 Let ua, ub, $uc \in G$. If $ud \in G$, then G_3 is induced. Take $ud \notin G$. Then either $vd \in G$ or $vb \in G$. If $vd \in G$, then for $uv \in G$, G_2 is induced, and for $uv \notin G$, G_5 is induced. If $vb \in G$, then G_3 or G_1 is induced depending on whether or not v is adjacent to both c and u (Fig. 9.12, Case 2.2).

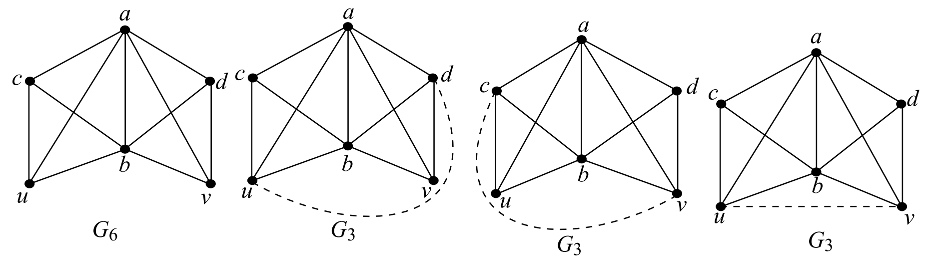
Case 2.3 If ud, vc or $uv \in G$, then G_3 is induced. The only other possibility gives G_6 (Fig. 9.12, Case 2.3). ❑



(Case 2.1)



(Case 2.2)



(Case 2.3)

Fig. 9.12

The following result due to Chartrand [53] characterises the edge graphs of a tree.

Theorem 9.9 (Chartrand [53]) A graph is the edge graph of a tree if and only if it is a connected block graph in which each cut vertex is on exactly two blocks.

Proof

Necessity Let T be any tree and let $G = L(T)$. Then G is also $B(T)$ since the edges and blocks of a tree coincide. Each cut vertex w of G corresponds to a bridge uv of T and is on exactly those two blocks of G which correspond to the stars at u and v.

Sufficiency Let G be a block graph in which each cut vertex is on exactly two blocks. Since each block of a block graph is complete, there exists a graph H such that $L(H) = G$, by Theorem 9.6. If $G = K_3$, we can take $H = K_{1,3}$. If G is any other block graph, then we show that H is a tree. Assume H is not a tree, so that it contains a cycle. If H is itself a cycle, then by Theorem 9.3, $L(H) = H$, but the only cycle which is a block graph is K_3, a case not under consideration. Thus H properly contains a cycle, implying that H has a cycle Z and an edge e adjacent to two edges of Z, but not adjacent to some edge f in Z. The vertices e and f of $L(H)$ lie on a cycle of $L(H)$ and they are not adjacent. This contradicts the fact that $L(H)$ is a block graph. Hence H is a tree. ❑

Consider the block graph G of Figure 9.13(a) in which each cut vertex lies on just two blocks. Figure 9.13(b) shows the tree T of which G is the edge graph, is constructed by first forming the block graph $B(G)$ and then adding new vertices for the non-cut vertices of G, and the edges joining each block with its non-cut vertices.

The edge graphs of complete graphs were independently characterised by Chang [47] and Hoffman [116, 117], while the edge graphs of complete bipartite graphs were characterised by Moon [163] and Hoffman [118].

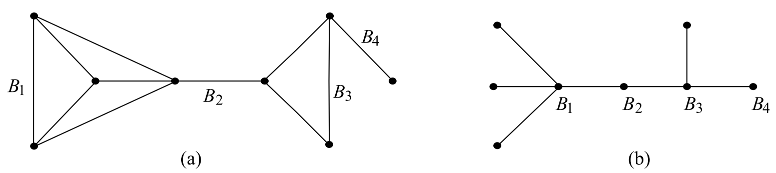


Fig. 9.13

9.2 Edge Graphs and Traversability

In this section, we study Eulerian and Hamiltonian property in edge graphs. We start with the following result.

Theorem 9.10 If G is Eulerian, then $L(G)$ is both Eulerian and Hamiltonian.

Proof Let G be Eulerian and let $\{e_1, e_2, \ldots, e_m\}$ be the edge sequence of an Euler line in G. Let the edge e_i in G be represented by the vertex v_i in $L(G)$, $1 \le i \le m$. Then $v_1 v_2 \ldots v_m v_1$ is a Hamiltonian cycle of $L(G)$. Now, if $e = u_i u_j \in E(G)$ and the vertex v in $L(G)$ represents the edge e, then $d_{L(G)}(v) = d_G(u_i) + d_G(u_j) - 2$, which is obviously even and greater than or equal to two, since both $d_G(u_i)$ and $d_G(u_j)$ are even (and ≥ 2). Thus in $L(G)$ every vertex is of even degree (≥ 2). Hence $L(G)$ is Eulerian. ❑

The converse of Theorem 9.11 is not true. To see this, consider the graph G shown in Figure 9.14. Clearly $L(G)$ is both Eulerian and Hamiltonian, but G is not Eulerian.

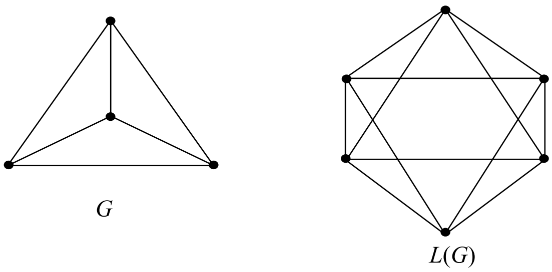


Fig. 9.14

Definition: A *dominating walk* of a graph G is a closed walk W in G (which can be just a single vertex) such that every edge of G not in W is incident with W. For example, the walk $v_1\ v_2\ v_3\ v_4$ in the graph of Figure 9.15 is a dominating walk.

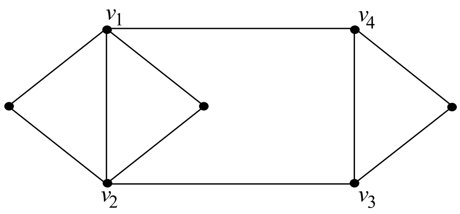


Fig. 9.15

The following characterisation of graphs that contain Hamiltonian edge graphs is due to Harary and Nash-Williams [108].

Theorem 9.11 The edge graph of a graph G with at least three edges is Hamiltonian if and only if G has a dominating walk.

Proof Let W be a dominating walk of G which is represented by the edge sequence $\{e_1, e_2, \ldots, e_k\}$. Let e_1 and e_2 be incident at v_1. Replace the subsequence $\{e_1, e_2\}$ by the sequence $\{e_1, e_{11}, e_{12}, \ldots, e_{1r_1}, e_2\}$, where $e_{11}, e_{12}, \ldots, e_{1r_1}$ are the edges other than e_1 and e_2 incident at v_1. Continuing this process for all subsequences $\{e_i, e_{i+1}\}$, $1 \le i \le k$ with $e_{k+1} = e_1$, we obtain a sequence of edges $e_1e_{11}e_{12}\ldots e_{1r_1}\ e_2e_{21}e_{22}\ldots e_{2r_2}\ e_3\ldots e_ke_{k1}e_{k2}\ldots e_{kr_k}e_1$. This clearly gives the Hamiltonian cycle $u_1u_{11}u_{12}\ldots u_{1r_1}\ u_2u_{21}u_{22}\ldots u_{2r_2}\ u_3\ldots u_ku_{k1}u_{k2}\ldots u_{kr_k}$ u_1 in $L(G)$, with u_1 being the vertex of $L(G)$ that corresponds to the edge e_1 of G, and so on.

Conversely, let $L(G)$ contain a Hamiltonian cycle $C = u_1\ u_2\ldots u_m\ u_1$ and let e_i be the edge of G corresponding to the vertex u_i of $L(G)$. Let W_0 be the edge sequence $e_1\ e_2\ldots e_m\ e_1$. We delete edges from W_0 in succession in the following way. If $e_i\ e_j\ e_k$ are the first three distinct consecutive edges of W_0 that have a common vertex, then delete e_j, and let $W_0' = W_0 - e_j = e_1e_2\ldots e_ie_k\ldots e_me_1$. Now starting with W_0', apply the same process as is applied in W, to get W_0. Continue in this way, till no such three consecutive edges exist. Clearly, the resulting subsequence of W_0 is a dominating walk or a pair of adjacent edges incident at a vertex, say v_0. In the later case, all the edges of G are incident at v_0 and hence v_0 is the dominating walk of G. ❑

The following results are simple consequences of Theorem 9.11.

Corollary 9.1 The edge graph of a Hamiltonian graph is Hamiltonian.

Proof Let G be a Hamiltonian graph with Hamiltonian cycle C. Then C is a dominating walk of G, and hence, $L(G)$ is Hamiltonian. ❑

We note that the converse of Corollary 9.1 is not true in general. To see this, consider the graph G as shown in Figure 9.16. Clearly $L(G)$ is Hamiltonian but G is not.

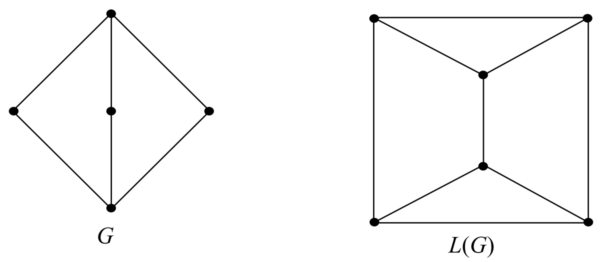


Fig. 9.16

Corollary 9.2 If G is a connected graph and each edge of G belongs to a triangle, then $L(G)$ is Hamiltonian.

Proof This follows from Theorem 9.11. ❑

The following result is due to Chartrand and Wall [55].

Theorem 9.12 If G is connected and $\delta(G) \geq 3$, then L^2 (G) is Hamiltonian.

Proof Since $\delta(G) \geq 3$, each vertex of $L(G)$ belongs to a clique of size at least three and hence each edge of $L(G)$ belongs to a triangle. Then the result follows by applying Corollary 9.2. ❑

The next result is due to Nebesky [170].

Theorem 9.13 If G is a connected graph with at least three vertices, then $L(G^2)$ is Hamiltonian.

Proof Since G is a connected graph with at least three vertices, every edge of G^2 belongs to a triangle. Hence by Corollary 9.2, $L(G^2)$ is Hamiltonian. ❑

Theorem 9.14 Let G be a connected graph in which every edge belongs to a triangle. If e_1 and e_2 are edges of G such that $G - \{e_1, e_2\}$ is connected, then there exists a spanning walk of G with e_1 and e_2 as its initial and terminal edges.

Proof Consider the longest walk W of G with e_1 and e_2 as its initial and terminal edges. Then proceed as in Theorem 9.11. ❑

The following result is due to Jaeger [121].

Theorem 9.15 The edge graph of a 4-connected graph is Hamiltonian.

Proof Let G be a 4-edge connected graph. By Theorem 9.11, it suffices to show that G contains a spanning Eulerian subgraph.

Now, G contains two edge-disjoint spanning trees T_1 and T_2. Let S be the set of vertices of odd degree in T_1. Then $|S|$ is even. Let $|S| = 2k$, $k \geq 1$. By Theorem 9.12, there exists a set of k pairwise edge-disjoint paths $\{P_1, P_2, \ldots, P_k\}$ in T_2 with the property stated in Theorem 9.12. Then $G_0 = T_1 U$ $(P_1 \cup P_2 \cup \ldots \cup P_k)$ is a connected spanning subgraph of G in which each vertex is of even degree. Hence G_0 is a spanning Eulerian subgraph of G. ❑

Let every edge in a graph G be subdivided and let $S(G)$ be the subdivision graph. If the graph obtained from G by inserting n new vertices of degree two into every edge of G be denoted by $S_n(G)$ and taking $S(G) = S_1(G)$, we define $L_n(G) = L(S_{n-1}(G))$. We see that in general $L_n(G) \ncong L^n(G)$ (Fig. 9.17).

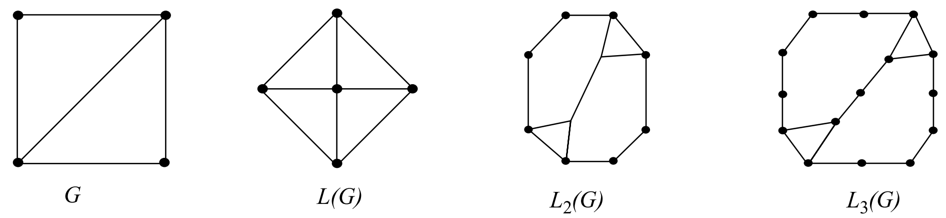


Fig. 9.17

An improvement of Theorem 9.13 is seen in the following result of Harary and Nash-Williams [108].

Theorem 9.16 A sufficient condition for $L_2(G)$ to be Hamiltonian is that G be Hamiltonian and a necessary condition is that $L(G)$ be Hamiltonian.

Now, we have the following consequence.

Corollary 9.3 A graph G is Eulerian if and only if $L_3(G)$ is Hamiltonian.

The following result is due to Chartrand [48].

Theorem 9.17 If G is a non-trivial connected graph with n vertices which is not a path, then $L^k(G)$ is Hamiltonian for all $k \geq n-3$.

9.3 Total Graphs

Let $G(V, E)$ be a graph. The total graph $T(G)$ of G has vertex set $V \cup E$ and two vertices of $T(G)$ are adjacent if and only if one of the following is true.

i. the vertices are $v_i, v_j \in V$ and $v_i v_j$ is an edge in E.

ii. one vertex is $v \in V$ and the other $e \in E$ and the edge e of G is incident with the vertex v of G.

iii. the edges are $e_i, e_j \in E$ and the edges e_i and e_j have a vertex in common in G.

Example The total graph of a graph G is shown in Figure 9.18.

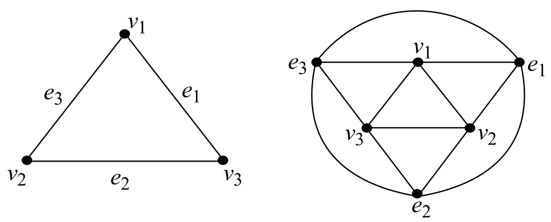


Fig. 9.18

It can be seen that G and $L(G)$ are induced subgraphs of $T(G)$ and that the remaining edges of $T(G)$ form a graph homeomorphic to G.

9.4 Eccentricity Sequences and Sets

The concept of eccentricity has been introduced in chapter 1 and has been further discussed in chapter 4. Now, we study eccentricity sequences in graphs.

Definition: A positive sequence $[e_i]_1^n$ is called an *eccentricity sequence* if it is an eccentricity sequence of some graph. The graph is said to realise the sequence. A set of positive integers is called an *eccentricity set* if it is an eccentricity set of some graph. The graph is said to realise the set. (The set of distinct eccentricities in a graph is called the eccentricity set of that graph.)

When eccentricity set is written in the increasing order $\{e_1, e_2, \ldots, e_k\}$ with $e_1 < e_2 < \ldots < e_k$, the eccentricity sequence is then expressed as $[e_1^{n_1}, e_2^{n_2}, \ldots, e_k^{n_k}]$, where $n_1, n_2, \ldots, n_k$ are respectively the number of occurrences of $e_1, e_2, \ldots, e_k$, the sequence $n_1, n_2, \ldots, n_k$ is called the *eccentricity frequency sequence* of the graph.

Here it can be noted that $e_1 = r$ (radius) and $e_k = d$ (diameter) of the graph.

Therefore, $r \leq d \leq 2r$ gives $e_1 \leq e_k \leq 2e_1$, which is a necessary condition for a positive sequence to be an eccentricity sequence.

Now, we have the following observation.

Theorem 9.18 If uv is an edge of a connected graph G, then $|e(u) - e(v)| \leq 1$.

Proof Let w be an eccentric vertex of u (i.e., w is the farthest vertex from u). Then by the triangle inequality for the metric d (distance), we have

$$d(u, w) \leq d(u, v) + d(v, w)$$

so that $e(u) \leq d(u, v) + d(v, w)$. (9.18.1)

But u and v are adjacent, therefore $d(u,v) = 1$.

Also, $e(v) \geq d(v,w)$ so that $d(v,w) \leq e(v)$.

Thus, from (9.18.1) we have

$$e(u) \leq 1 + d(v, w) \text{ so that } e(u) \leq 1 + e(v).$$

Therefore, $e(u) - e(v) \leq 1$. (9.18.2)

Similarly, by considering an eccentric vertex of v, we have

$$e(v) - e(u) \leq 1. \tag{9.18.3}$$

From (9.18.2) and (9.18.3) it follows that

$$|e(u) - e(v)| \leq 1.$$ ❑

Note The above result shows that the eccentricities of two adjacent vertices are either equal or differ by 1 as $|e(u) - e(v)| \leq 1$ gives $|e(u) - e(v)| = 0$ or $|e(u) - e(v)| = 1$.

An important consequence of Theorem 9.18 is as follows.

Corollary 9.4 If $u_0\ u_1\ u_2 \ldots u_m$ is a path in a connected graph and $e(u_0) < e(u_m)$ and k is any integer such that $e(u_0) < k < e(u_m)$, then there exists an integer j $(0 \leq j \leq m)$ such that $e(u_j) = k$.

Proof We know the difference of eccentricities of any two adjacent vertices along the path $u_0\ u_1\ u_2 \ldots u_m$ is always less or equal to 1. Therefore every integer between $e(u_0)$ and $e(u_m)$ occurs as the eccentricity of some vertex in this path. This can also be seen in the following way.

Assume, $e(u_0) < e(u_1) < \ldots < e(u_{j-1})$,

and let $j = 1 + \max\{i : e(u_i) < k\}$, that is, $j - 1 = \max\{i : e(u_i) < k\}$.

Thus, $e(u_{j-1}) < k$.

Therefore, $|e(u_j) - e(u_{j-1})| \leq 1$ gives

$$e(u_j) \leq e(u_{j-1}) + 1 < k + 1,$$

so that $e(u_j) \leq k$. (9.4.1)

But by the choice of j, we have $e(u_j) \geq k$. (9.4.2)

Hence from (9.4.1) and (9.4.2), we get $e(u_j) = k$. ❑

The following necessary condition for a positive sequence to be an eccentricity sequence is due to Lesniak [146].

Theorem 9.19 (Lesniak) If a non decreasing sequence $[e_i]_1^n$ of positive integers is an eccentric sequence then

i. $2e_1 \leq n$,

ii. $e_n \leq \min\{n-1, 2e_1\}$ and

iii. for every integer k such that $e_1 < k \leq e_n$, there exists an integer i $(2 \leq i \leq n-1)$ such that $e_i = e_{i+1} = k$.

Proof

i. Let the vertices of G be labelled as $v_1, v_2, \ldots, v_n$ such that $e(v_i) = e_i$. Then G has a spanning tree T which preserves the distance from v_1. This gives

$$e_G(v_1) = e_T(v_1) \text{ and } e_G(v_i) \leq e_T(v_i),$$

for $2 \leq i \leq n$ (since removal of edges cannot reduce distances)

Thus, if $[a_1, a_2, \ldots, a_n]$ is the eccentricity sequence of T, we have $a_1 = e_1$. So it is enough to prove that $2a_1 \leq n$. We prove this for any tree T.

Now let T be any tree with eccentricity sequence $[a_1, a_2, \ldots, a_n]$.

If $n = 2$, then $a_1 = a_2 = 1$, and the result is true. So assume $n \geq 3$.

Let u be a central vertex of T. Then $e(u) = a_1$. Also u is a cut vertex of T.

Suppose $a_1 = e(u) \geq \dfrac{n+1}{2}$. $\quad [2a_1 \geq n+1]$

Since an eccentric vertex $\overline{u}$ of u should lie in a component of $T - u$, there is at least one component C of $T - u$ with $|V(C)| \geq \dfrac{n+1}{2}$.

Now, let v be the vertex adjacent to u in C. Then for any vertex w in C, we have $d(v, w) = d(u, w) - 1$. So, $d(v, w) < e(u)$, because $d(u, w) \leq e(u)$ and so $d(u, w) - 1 < e(u)$.

For every vertex w in $V(T) - V(C)$, we have $d(v, w) - d(u, w) = 1$,

so that $d(v, w) = d(u, w) + 1$.

Total vertices in T is n, $|V(C)| \geq \dfrac{n+1}{2}$, therefore number of vertices in $V(T) - V(C)$

$$\leq n - \left(\frac{n+1}{2}\right) = \frac{n-1}{2}.$$

That is, $|V(T) - V(C)| \leq \dfrac{n-1}{2}$. Therefore, $d(u, w) \leq \dfrac{n-1}{2} - 1 = \dfrac{n-3}{2}$.

Thus, $d(v, w) \leq \dfrac{n-3}{2} + 1 = \dfrac{n-1}{2}$. So, $d(v, w) < e(u)$.

Hence for all vertices w, we have $d(v, w) < e(u)$, and thus $e(v) < e(u)$, so that $e(v) < a_1$, which is a contradiction as a_1 is the least eccentricity of a vertex of T. Thus,

$a_1 \geq \frac{n+1}{2}$ is wrong and so, $a_1 \leq \frac{n}{2}$.

ii. The maximum distance possible in an $n-$vertex graph is $n-1$. So, $e_n \leq n-1$. Also, $e_n \leq 2e_1$, $[r \leq d \leq 2r$, and here $d = e_n$, $r = e_1]$. Hence, $e_n \leq \min\{(n-1), 2e_1\}$.

iii. We have to prove that each integer between e_1 and e_n (e_n inclusive) occurs at least twice in the sequence. Let u_1 be the central vertex and u_k be the peripheral vertex of G. Then $e(u_1) = e_1$ and $e(u_k) = e_n$. Since G is connected, there exists a $u_1 - u_k$ path. Now by Corollary 9.4, if k is any integer between e_1 and e_k, there exists a vertex u_j in this path with $e(u_j) = k$. This gives the existence of a vertex whose eccentricity is k.

If $e(w) > e_1$, we show there is a vertex u other than w such that $e(u) = e(w)$. Let $\overline{w}$ be an eccentric vertex of w, that is, $d(w, \overline{w}) = e(w) = k$, say. As we have assumed that u_1 is the central vertex of G, let $P = u_1 \ldots u_m (u_m = \overline{w})$ be a $u_1 - \overline{w}$ distance path in G. Since $e(u_1) = e_1 < e(w) = d(\overline{w}, w) \leq e(\overline{w})$, applying Corollary 9.4, there is a vertex u_j in this path such that $e(u_j) = k$. But $d(\overline{w}, u_j) \leq m-1 = d(u_1, \overline{w}) \leq e(u_1) = e_1 < e(w) = d(\overline{w}, w)$. Therefore, $d(\overline{w}, u_j) < d(\overline{w}\, w)$. Thus, $u_j \neq w$ and the result is proved. ❑

The following characterisation of eccentricity sequences of trees is again due to Lesniak [146].

Theorem 9.20 (Lesniak) A non-decreasing sequence $[e_i]_1^n$ of positive integers is the eccentric sequence of a tree if and only if

i. For each integer k with $e_1 < k \leq e_n$, we have

$$e_i = e_{i+1} = k, \text{ for some } i,\ 2 \leq i \leq n-1,$$

ii. Either $e_1 = \frac{e_n}{2}$ and $e_1 \neq e_2$, or $e_1 = \frac{e_n + 1}{2}$, $e_1 = e_2$ and $e_2 \neq e_3$.

Proof

Necessity Let the nondecreasing sequence $[e_i]_1^n$ of positive integers be the eccentric sequence of a tree. Then (i) follows from condition (iii) of the previous result.

Let r be the radius and d the diameter of the tree, so that $e_1 = r$ and $e_n = d$. Since a tree is either unicentral or bicentral, we have $d = 2r$ for unicentral,

and $d = 2r - 1$ for bicentral.

In case the tree is unicentral, then the eccentricity of the center v_1 is e_1 and $e_1 \neq e_2$.

Thus, $e_n = 2e_1$ which implies that $e_1 = \frac{e_n}{2}$ and $e_1 \neq e_2$.

In case the tree is bicentral, then $e_1 = e_2$ and $d = 2r - 1$ gives $e_n = 2e_1 - 1$, so that $e_1 = \frac{e_n + 1}{2}$ with $e_2 \neq e_3$.

Sufficiency Let the nondecreasing sequence $[e_i]_1^n$ of positive integers satisfy conditions (i) and (ii). We construct a tree with eccentric sequence $[e_i]_1^n$ in the following way.

Let P be a path of length $e_n = d$. Then eccentric sequence of P is

$$S_1 = \frac{d}{2}, \left(\frac{d}{2}+1\right)^2, \left(\frac{d}{2}+2\right)^2, \ldots, \left(\frac{d}{2}+\frac{d}{2}\right)^2, \qquad \text{if } d \text{ is even,}$$

$$\text{or} \quad S_2 = \left(\frac{d+1}{2}\right)^2, \left(\frac{d+1}{2}+1\right)^2, \ldots, \left(\frac{d+1}{2}+\frac{d+1}{2}-1\right)^2, \quad \text{if } d \text{ is odd,}$$

that is, $S_1 = r, (r+1)^2, (r+2)^2, \ldots, (r+r)^2$, where $r = \dfrac{d}{2}$,

or $S_2 = r^2$, $(r+1)^2$, $(r+2)^2, \ldots, (r+r-1)^2$, where $r = \dfrac{d+1}{2}$,

where powers denote repetition of eccentricity.

Let the given sequence $[e_i]_1^n$ be written in power notation $\pi = r^{i_1}(r+1)^{i_2}\ldots d^{i_k}$, where $i_1 = 1$ or 2, according as d is even or odd. If $i_j > 2$ for any j, $1 < j \le k$, we attach $i_j - 2$ vertices to any vertex with eccentricity $r+j-2$ in the path P. This does not alter the eccentricities of the vertices of P and the resulting tree T has eccentric sequence $[e_i]_1^n$. ❑

Example Construct a tree with eccentric sequence $[4^2, 5^4, 6^3, 7^4]$.

First draw a path P say $u_0\, u_1 u_2\, u_3\, u_4\, u_5\, u_6\, u_7$ of length 7. Then the eccentricities of these vertices are 7, 6, 5, 4, 4, 5, 6, 7.

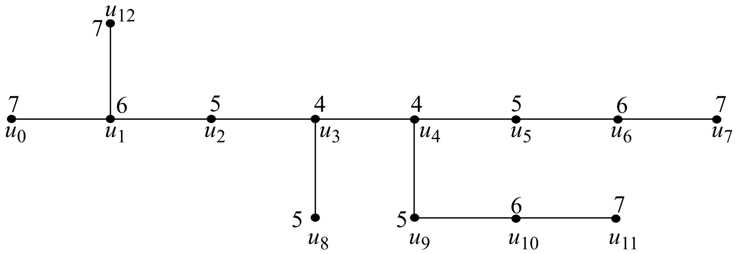


Fig. 9.19

To get two more vertices of eccentricities 5, attach a new vertex each to u_3 and u_4. Let these new vertices be u_8 and u_9. (Here $i_2 = 4 > 2$, and $i_2 - 2 = 4 - 2 = 2$). So 2 vertices one each are attached to the vertices of eccentricities $r+j-2 = r+2-2 = 4+2-2 = 4$, i.e., u_3 and u_4). Now $i_3 = 3 > 2$ and $i_3 - 2 = 3 - 2 = 1$. So one vertex is to be attached to the vertex of eccentricity $r+j-2 = 4+3-2 = 5$, i.e., the vertex u_2 or u_8 or u_9 or u_5. Let this new vertex u_{10} be attached to u_9 say. Now $i_4 = 4 > 2$ and $i_4 - 2 = 4 - 2 = 2$. So two new vertices are to be attached, one each among the vertices with eccentricities $r+j-2 = 4+4-2 = 6$, i.e., to the vertices u_1, u_6, u_{10}. Let these new vertices be u_{11} attached to u_{10} and u_{12} attached to u_1. The resulting tree is shown in Figure 9.19.

Remark Clearly, there are many trees realising this eccentric sequence.

Neighbourhood Let v be any vertex of a connected graph G. The ith *neighbourhood* of v denoted by $N_i(v)$ is the set of all those vertices in V whose distance from v is i.

i.e., $N_i(v) = \{u \in V: d(v, u) = i\}$.

We denote $N_1(v)$ by $N(v)$ and call it the *neighbourhood* of v.

Example Consider the graph in Figure 9.20.

We have $N_1(v) = N(v) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$.

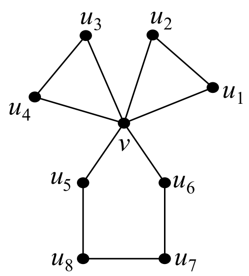


Fig. 9.20

We have the following observations.

Lemma 9.2 Any two vertices with the same neighbourhood in a graph have the same eccentricity.

Proof Let u and v be two vertices with the same neighbourhood. So $N(u) = N(v)$. Therefore the path lengths from u and v to the other vertices of the graph are equal. Clearly, u and v are not adjacent. ❑

Lemma 9.3 If u and v are adjacent in G and $N(u) - \{v\} = N(v) - \{u\}$, then u and v have the same eccentricity in G.

Proof Let G be a graph in which $uv = e$ and $N(u) - \{v\} = N(v) - \{u\}$.

Let $H = G - e$. Then u and v are not adjacent in H so that u and v have the same neighbourhood in H.

Therefore, $e(u|H) = e(v|H)$ where $(e(u|H)$ means eccentricity of vertex u in graph $H)$. If $e(u|G) = 1$, then $e(v|G) = 1$ also. If not, then $e(u|G) = e(u|H) = e(v|H) = e(v|G)$. ❑

Definition: Let v be a vertex of a graph G and let H be a graph obtained from $G - v$ by adding edge to each vertex of a new graph K_p (or $\overline{K}_p$) to every vertex of $G - v$ to which v was adjacent in G. Then H is said to be obtained from G by replacing v by K_p (or $\overline{K}_p$). This operation is illustrated in Figure 9.21.

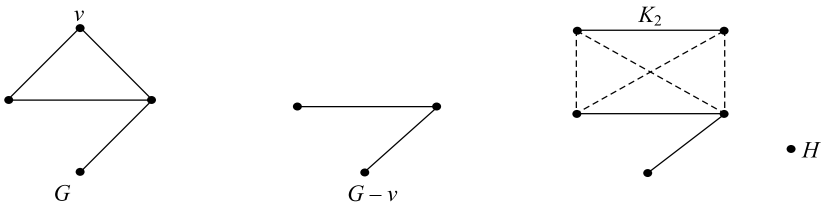


Fig. 9.21

The operation of replacing v in G by $\overline{K}_p$ is called *multiplication* of the vertex in G. The operation of replacing v in G by K_p is called the *linked multiplication* of v in G. Lesniak observed that multiplication or linked multiplication of one or more vertices of a graph is an operation preserving the eccentricity set of the graph.

Lemma 9.4 If H is the graph obtained by replacing a vertex v of a graph G with diameter greater than one, by a K_p or $\overline{K}_p$ (for any positive integer p), then G and H have the same eccentricity sets.

Proof Since v and any vertex of $\overline{K}_p$ have the same neighbourhood in H,

$$e(u|H) = e(v|G), \text{ for every } u \in \overline{K}_p.$$

For replacement by K_p, we have for any two adjacent vertices u_i and u_j in K_p,

$$N(u_i) - \{u_j\} = N(u_j) - \{u_i\}.$$

Therefore, $e(u_i|H) = e(u_j|H)$. So, $e(v|H) = e(u|G)$, for every $u \in K_p$.

Thus, $e(u|H) = e(v|G)$, for every $u \in K_p$ ($\overline{K}_p$) and obviously

$$e(w|H) = e(w|G), \text{ for all other vertices.}$$ ❑

The above result is illustrated in Figure 9.22, where we choose K_2 and $\overline{K}_2$

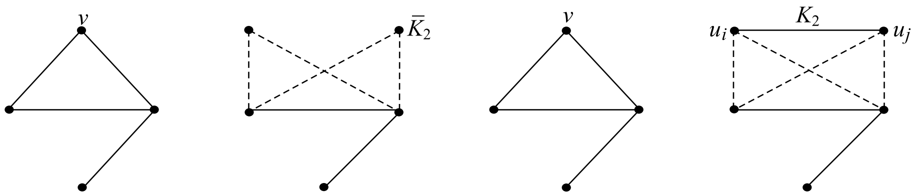


Fig. 9.22

The next gives the necessary and sufficient conditions for an eccentricity sequence of a graph, due to Lesniak [146].

Theorem 9.21 (Lesniak) A nondecreasing sequence of positive integers $[e_1^{r_1}, e_2^{r_2}, ..., e_k^{r_k}]$ is an eccentricity sequence if and only if some of its subsequence $[e_1^{s_1}, e_2^{s_2}, \ldots, e_k^{s_k}]$ with $s_i \leq r_i$ and no $s_i = 0$ for $1 \leq i \leq k$, is an eccentricity sequence.

Proof

Necessity Since every sequence is its own subsequence, necessity follows.

Sufficiency Let $[e_1^{s_1}, e_2^{s_2}, ..., e_k^{s_k}]$ be the eccentricity sequence of a graph G. Let v_i be a vertex of G with $e(v_i) = e_i, 1 \leq i \leq k$. Let $n_1 = r_1 - s_1 + 1$ so that $r_1 = n_1 + s_1 - 1$.

Now let G_1 be obtained from G by replacing v_1 by K_{n_1} or $\overline{K}_{n_1}$. Then every one of the vertices of this K_{n_1} or $\overline{K}_{n_1}$ has eccentricity $e(v_1|G)$ and the eccentricities of the vertices of G are unaltered in G_1. Thus G_1 has eccentricity sequence $[e_1^{r_1}, e_2^{s_2}, \ldots, e_k^{s_k}]$.

Let G_2 be obtained from G_1 by replacing v_2 by K_{n_2} or $\overline{K}_{n_2}$. Then by similar argument, G_2 has eccentricity sequence $[e_1^{r_1}, e_2^{r_2}, \ldots, e_k^{s_k}]$.

Proceeding in this way, by successively replacing $v_3, v_4, \ldots, v_k$ by K_{n_3} $(\overline{K}_{n_3})$, K_{n_4} $(\overline{K}_{n_4})$, $\ldots, K_{n_k}$ $(\overline{K}_{n_k})$, we get a graph G_k with eccentricity sequence $[e_1^{r_1}, e_2^{r_2}, \ldots, e_k^{r_k}]$. ❑

Remarks

1. It is assumed that $e_k > 1$.

2. The construction of G_k is not unique.

3. This result keeps unsolved the problem of characterising minimal eccentricity sequences, that is, those eccentricity sequences which have no proper eccentric subsequences.

The next result characterises eccentricity sets and is due to Behzad and Simpson [17].

Theorem 9.22 A non-empty set $S = \{e_1, e_2, \ldots, e_k\}$ of positive integers arranged in increasing order is an eccentricity set if and only if $k \leq e_1 + 1$ and $e_{i+1} = e_i + 1$ for $1 \leq i \leq k-1$.

Proof

Necessity Let S be an eccentricity set. Then by (iii) of Theorem 4.24, $e_{i+1} = e_i + 1$ for each i, $1 \leq i \leq k-1$. This gives $e_k = e_1 + k - 1$. Since $e_k \leq 2e_1$, we get $k \leq e_1 + 1$.

Sufficiency If $e_1 = 1$, then $k = 1$ or 2 and $S = \{1\}$ or $\{1, 2\}$. In this case, K_2 and $K_{1,n}$ realise the sets.

For $e_1 > 1$, let G be the graph obtained by identifying a vertex of a cycle C_{2e_1} with an end vertex of a path P_k. Let $e_1 - k + 1 = d$. Then $d \geq 0$, and the eccentricity sequence of G is easily verified to be $[e_1^{2d+1}, (e_1+1)^3, (e_1+2)^3, \ldots, (e_1+k-2)^3, (e_1+k-1)^3]$. Hence S is the eccentricity set. ❑

9.5 Distance Degree Regular and Distance Regular Graphs

Let G be a connected graph and let v be any vertex of G. Let e be the eccentricity of vertex v and $N_j(v)$ be the jth neighbourhood of v. Assume, $n_j(v) = |N_j(v)|$, for $0 \leq j \leq e$. The sequence $D(G, v) = [n_0(v), n_1(v), \ldots, n_e(v)]$ is called the *distance degree sequence* (DDS) of v in G. If all the vertices of G have the same distance degree sequence $D(G) = [n_0, n_1, \ldots, n_d]$, then G is said to be *distance degree regular*.

If G is not distance degree regular, the n vectors $D(G, v_i)$, $1 \leq i \leq n$, arranged lexicographically in an array with variable row sizes is called the *distance degree array* of G (DDA(G)).

The *distance* of vertex v is defined by

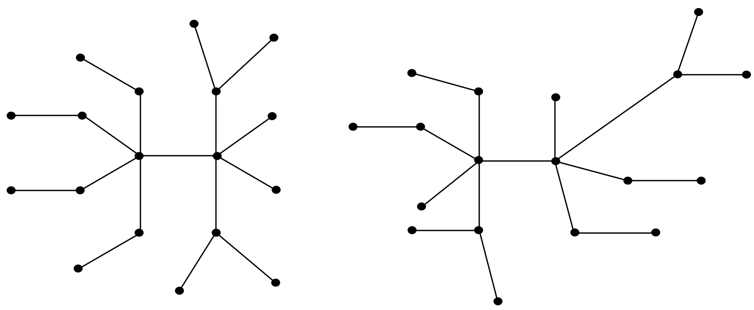
$$D(v) = \sum_{j=1}^{e(v)} n_j(v).$$

Let D_i be the distance of the vertex v_i. The sequence $DS(G) = [D_1, D_2, \ldots, D_n]$ in non-decreasing order is called the *distance sequence* of the graph.

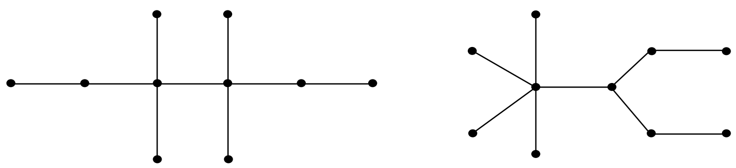
A vertex v with minimum distance $D(v)$ is called a *median* of G and the subgraph induced by the set M of median vertices of G is called the *median subgraph* of G.

Clearly, if G is distance degree regular, then all vertices in G have the same eccentricity, and G is a *self-centered graph*. Also, $n_0 = 1$ and $n_1 = |N_1(v)|$ for every $v \in G$, so that G is n_1-regular.

Randic [214] conjectured that two trees are isomorphic if and only if they have same DDA and Slater [187] disproved this giving the counter example as shown in Figure 9.23.



T_1 Non isomorphic trees with same DDA T_2



T_1 Non isomorphic trees with same DS T_2

Fig. 9.23

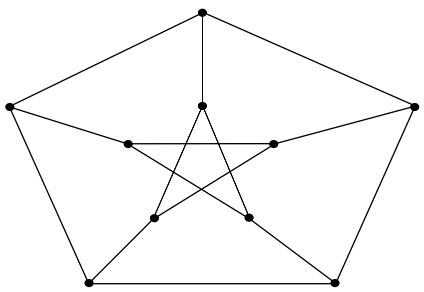
Definition: Let G be a connected graph with diameter d and let $k = b_0, b_1, \ldots, b_{d-1}$; $l = c_1, c_2, \ldots, c_d$ be $2d$ non-negative integers. Then G is said to be *distance regular* (DR) if for every pair of vertices u, v in G with $d(u, v) = j$, we have, (i) the number of vertices in $N_{j-1}(v)$ adjacent to u is c_j, $1 \le j \le d$ and (ii) the number of vertices in $N_{j+1}(v)$ adjacent to u is b_j, $0 \le j \le d-1$.

The sequence $[b_0, b_1, \ldots, b_{d-1}, c_1, c_2, \ldots, c_d]$ is called the *intersection array* of G.

Clearly, DR graphs are k-regular and self-centered. The examples of distance regular graphs are K_n, $K_{n,\,n}$ and the cubes Q_n.

A graph G is said to be *strongly regular* (SR) with parameters (n, k, λ, μ) if it is a k-regular graph of order n in which every pair of adjacent vertices are mutually adjacent to λ vertices and every pair of non-adjacent vertices are mutually adjacent to μ vertices.

The Petersen graph is strongly regular with parameters $(10, 3, 0, 1)$, that is $n = 10$, $k = 3$, $\lambda = 0$, $\mu = 1$ (Fig. 9.24).



Petersen graph

Fig. 9.24

9.6 Isometry

The concept of isometry as in Chartrand and Stewart [52] is as follows.

Let G_1 and G_2 be connected graphs with vertex sets V_1 and V_2 respectively. Then G_2 is said to be *isometric* from G_1 if for each $v \in V_1$, there is a one-one map $\phi_v : V_1 \to V_2$ such that ϕ_v preserves distances from v, that is $d_{G_2}(u,v) = d_{G_1}(\phi_v(v), \phi_v(u))$ for every $u \in V_1$.

Two graphs G_1 and G_2 are said to be *isometric* if they are isometric from each other.

Example Consider the graphs shown in Figure 9.25, we have

$$\phi_1 = \phi_4 = \phi_5\,(1 \to a,\ 2 \to b,\ 3 \to c,\ 4 \to d,\ 5 \to e),$$

$$\phi_2 = (2 \to e,\ 1 \to a,\ 3 \to d,\ 4 \to c,\ 5 \to b),\ \phi_3 = (3 \to e,\ 4 \to a,\ 2 \to d,\ 1 \to b,\ 5 \to c).$$

Here G_2 is isometric from G_1.

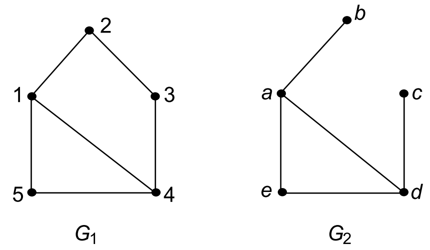


Fig. 9.25

Remarks Isometry between graphs as defined above does not imply isomorphism (Fig. 9.26). A pair of isometric graphs may even have same degree sequence and yet be non-isomorphic (Fig. 9.27).

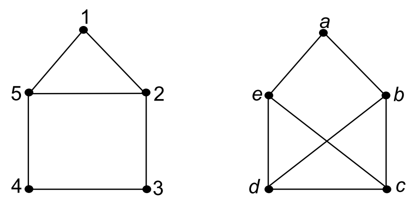


Fig. 9.26

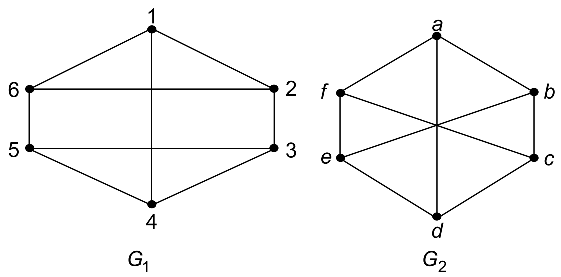


Fig. 9.27

We now have the following results.

Theorem 9.23 If G_1 and G_2 are k-regular graphs of order n, where $k \geq n-1/2$, then G_1 and G_2 are isometric.

Proof Since G_1 is a k-regular graph with $k \geq n-1/2$, $d(G_1) \leq 2$.
Let $u \in V(G_1)$ and $v \in V(G_2)$ be any two vertices and define

$$\phi_u : V(G_1) \to V(G_2) \text{ by } \phi_u(u) = v.$$

For $i = 1, 2, \ldots, k$, let $u_i \in N_1(u)$ and $v_i \in N_1(v)$ and define $\phi_u(u_i) = v_i$.

For $i = k+1, \ldots, n-1$, let $u_i \in N_2(u)$ and $v_i \in N_2(v)$ and again let $\phi_u(u_i) = v_i$.

The neighbourhoods are in the appropriate graphs. Then ϕ_u is an isometry of G_2 from G_1 at u. Since u and v are arbitrary, it is easily seen that G_2 is isometric from G_1, and G_1 is isometric from G_2. ❑

Theorem 9.24 A necessary condition for two graphs to be isometric is that they have the same degree set and the same eccentricity set.

Proof Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isometric graphs. As G_2 is isometric from G_1, let ϕ_v be the one-one mapping from $V(G_1)$ to $V(G_2)$. Therefore, $d(v|G_1) = d(\Phi_v(v)|G_2)$. Also ϕ_v has the property of preserving distance, therefore $e(v|G_1) = e(\phi_v(v)|G_2)$. So the eccentricity set of G_1 is included in the eccentricity set of G_2.

Again, as G_1 is isometric from G_2, therefore, the degree set and eccentricity set of G_2 are included respectively in the degree set and eccentricity set of G_1.

Hence the degree sets are equal in G_1 and G_2 and the eccentricity sets are equal in G_1 and G_2. ❑

9.7 Exercises

1. Show that the edge graph of $K_{1,\,n}$ is K_n.
2. Show that the edge graph of K_5 is the complement of the Petersen graph.
3. Show that if $L(G)$ is connected and regular, then either G is regular or G is a bipartite graph in which vertices of the same partite set have the same degree.
4. If G is k-edge-connected, then prove that $L(G)$ is k-connected and $(2k-2)$-edge-connected.
5. Show that the graph $L_2(G)$ is Hamiltonian if and only if G has a closed spanning walk.
6. Show that the graph $L_2(G)$ is Hamiltonian if and only if there is a closed walk in G which includes at least one vertex incident with each edge of G.
7. Prove that $T(K_n) \cong L(K_{n+1})$.
8. If G is Hamiltonian, then prove that $T(G)$ is Hamiltonian.
9. If G is Eulerian, then prove that $T(G)$ is both Eulerian and Hamiltonian.
10. Prove that $T(G)$ of every nontrivial connected graph G contains a spanning Eulerian subgraph.
11. Show that the edge graph of a graph G has a Hamiltonian path if and only if G has a walk W such that every edge of G not in W is incident with W.

12. If G is any connected graph with δ(G) $\geq$ 4, then prove that $L^2(G)$ is Hamiltonian-connected.

13. Construct graphs with eccentricity sequence

$$[2,\ 3^3,\ 4^3].$$

14. If G is a connected graph with diameter 3 and $e(u|G) = 3$, then show that $e(u|\overline{G}) = 2$.

15. If $[e_1,\ e_2,\ \ldots,\ e_n]$ with $e_n < 2e_1 - 1$ is an eccentricity sequence, then show that each central vertex lies on a cycle.

16. If $[e_i]_1^n$ is the eccentricity sequence of an $(n,\ m)$ graph, show that

$$m \leq \frac{1}{2}\left(n^2 - \sum_{i=1}^{n} e_i\right).$$

17. For a distance regular graph, prove the following

    a. If $1 \leq i \leq \dfrac{1}{2d}$, then $b_i \geq c_i$.

    b. If $1 \leq i \leq \dfrac{1}{d-1}$, then $b_1 \geq c_i$.

    c. $c_2 \geq k - 2b_1$.