8. Matchings and Factors

Consider the formation of an executive council by the parliament committee. Each committee needs to designate one of its members as an official representative to sit on the council, and council policy states that no senator can be the official representative for more than one committee. For example, let there be five committees 1, 2, 3, 4 and 5. Let A, C, D, E be members of committee 1; B, C, D be members of committee 2; A, B, E, F be members of committee 3; and A, F be members of committee 5. Here the executive council can be formed by members A, C, B, F and E, representing committees 1, 2, 3, 4 and 5 respectively. The problem of formation of the executive council and many related problems can be solved using the concept of matchings in graphs.

8.1 Matchings

Definition: A *matching* in a graph is a set of independent edges. That is, a subset M of the edge set E of a graph $G(V, E)$ is a matching if no two edges of M have a common vertex. A matching M is said to be *maximal* if there is no matching M' strictly containing M, that is, M is *maximal* if it cannot be enlarged. A matching M is said to be *maximum* if it has the largest possible cardinality. That is, M is maximum if there is no matching M' such that $|M'| > |M|$.

Consider the graph G shown in Figure 8.1. The examples of matchings in G are $M_1 = \{v_1v_5, v_2v_6, v_3v_4, v_7v_8\}$, $M_2 = \{v_1v_5, v_2v_6, v_3v_4, v_7v_9\}$, $M_3 = \{v_1v_5, v_2v_6, v_3v_4, v_7v_{10}\}$, $M_4 = \{v_1v_5, v_2v_6, v_3v_4\}$ and $M_5 = \{v_4v_7, v_1v_6, v_2v_3\}$. Clearly, M_1 is a maximum matching, whereas M_5 is maximal but not maximum.

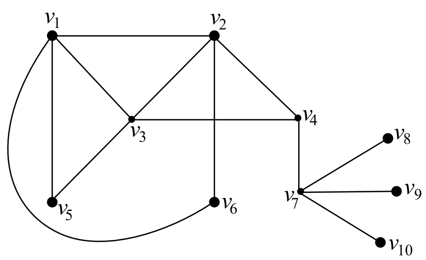


Fig. 8.1

The cardinality of a maximum matching is denoted by $\alpha_1(G)$ and is called the *matching number* of G (or the edge-independence number of G).

Definition: Let M be a matching in a graph G. A vertex v in G is said to be *M-saturated* (or saturated by M) if there is an edge $e \in M$ incident with v. A vertex which is not incident with any edge of M is said to be *M-unsaturated*. In other words, given a matching M in a graph G, the vertices belonging to the edges of M are M-saturated and the vertices not belonging to the edges of M are M-unsaturated. Consider the graph shown in Figure 8.2. Clearly, $M = \{v_1v_2,\ v_3v_7,\ v_4v_5\}$ is a matching and the vertices $v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_7$ are M-saturated but v_6 is M-unsaturated.

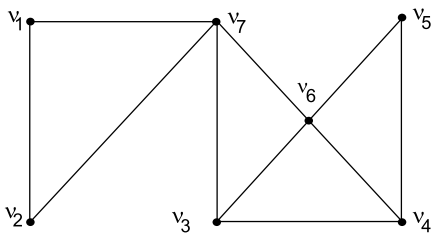


Fig. 8.2

Definition: A matching M in a graph G is said to be a *perfect matching* if M saturates every vertex of G. In Figure 8.3(a), G_1 has a perfect matching $M = \{v_1v_2,\ v_3v_6,\ v_4v_5\}$, but G_2 has no perfect matching.

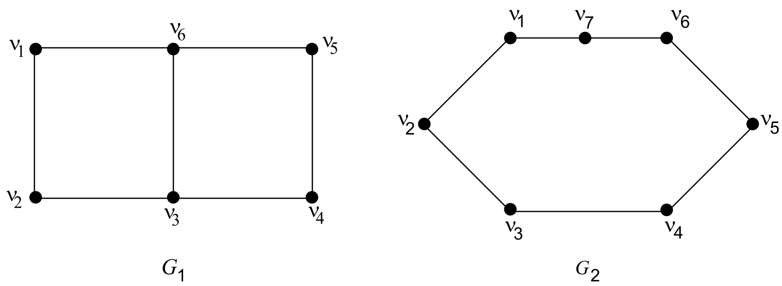


Fig. 8.3(a)

Definition: If G is a bipartite graph with bipartition $V = V_1 \cup V_2$ and $|V_1| = n_1,\ |V_2| = n_2$, then a matching M of G saturating all the vertices in V_1 is called a *complete matching* (or V_1 matched into V_2).

Definition: Let H be a subgraph of a graph G. An H-alternating path (cycle) in G is a path (cycle) whose edges are alternately in $E(G) - E(H)$ and $E(H)$.

If M is matching in a graph G, then an *M-alternating path* (cycle) in G is a path (cycle) whose edges are alternately in $E(G) - M$ and M. That is, in an M-alternating path, the edges alternate between M-edges and non-M-edges. An M-alternating path whose end vertices are M-unsaturated is said to be an *M-augmenting path.*

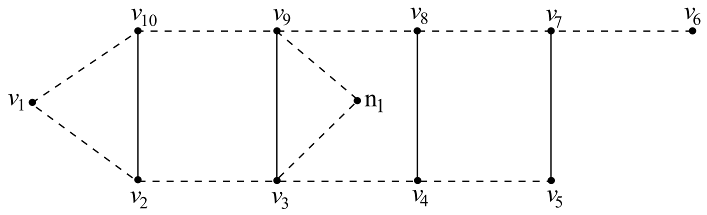


Fig. 8.3(b)

Consider the graph G shown in Figure 8.3(b). An example of a matching in G is $M = \{v_2v_{10}, v_3v_9, v_4v_8, v_5v_7\}$. Clearly $v_{10}, v_2, v_3, v_9, v_8, v_4$ is an M-alternating path and $v_1, v_{10}, v_2, v_9, v_3, v_{11}$ is an M-augmenting path.

The following result due to Berge [21] characterises a maximum matching in a graph. This forms the foundation of an efficient algorithm for obtaining a maximum matching.

Theorem 8.1 (Berge) A matching M of a graph G is maximum if and only if G contains no M-augmenting paths.

Proof Let M be a maximum matching in a graph G. Assume $P = v_1v_2 \ldots v_k$ is an M-augmenting path in G. Due to the alternating nature of M-augmenting path, we observe that k is even and the edges $v_2v_3, v_4v_5, \ldots, v_{k-2}v_{k-1}$ belong to M. Also the edges $v_1v_2, v_3v_4, \ldots, v_{k-1}v_k$ do not belong to M (Figure 8.4).

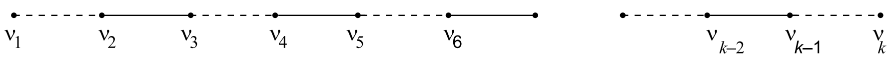


Fig. 8.4

Now, let M_1 be the set of edges given by

$$M_1 = [M - \{v_2v_3, \ldots, v_{k-2}v_{k-1}\}] \cup \{v_1v_2, \ldots, v_{k-1}v_k\}.$$

Then M_1 is a matching and clearly M_1 contains one more edge than M. This contradicts the assumption that M is maximum. Thus G contains no M-augmenting paths.

Conversely, assume that G has no M-augmenting paths. Let M' be a matching such that $|M'| > |M|$. Let H be a subgraph of G with $V(H) = V(G)$ and $E(H)$ be the set of edges of G that appear in exactly one of M and M'. Since every vertex of G lies on at most one edge from M and at most one edge from M', therefore degree (in H) of each vertex of H is at most 2. This implies that each connected component of H is either a single vertex, or a path, or a cycle (Fig. 8.5). If a component is a cycle, then it is an even cycle, because the edges alternate between M-edges and M'-edges. Since $|M'| > |M|$, there is at least one component in H that is a path which begins and ends with edges from M'. Clearly, this path is an M-

augmenting path, contradicting the assumption. Thus no such matching M' can exist and hence M is maximum. ❑

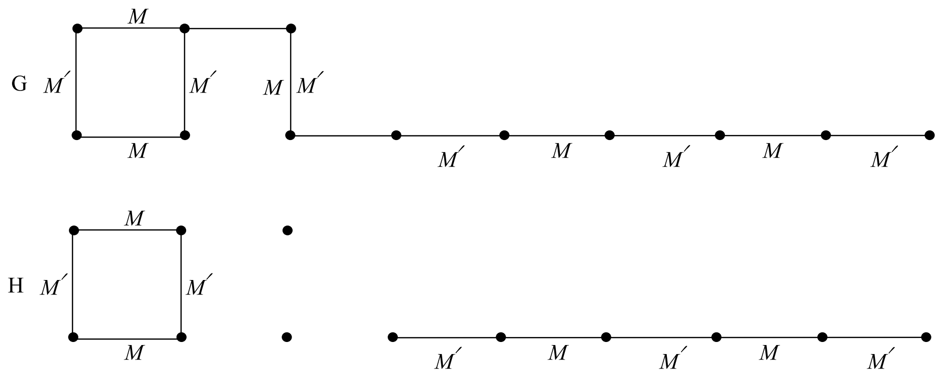


Fig. 8.5

Definition: The *neighbourhood* of a set of vertices S, denoted by $N(S)$, is the union of the neighbourhood of the vertices of S.

The following classic result due to P. Hall [101] characterises a complete matching in a bipartite graph.

Theorem 8.2 (Hall) If $G(V_1,\, V_2,\, E)$ is a bipartite graph with $|V_1| \leq |V_2|$, then G has a matching saturating every vertex of V_1, if and only if $|N(S)| \geq |S|$, for every subset $S \subseteq V_1$.

Proof Let G have a complete matching M that saturates all the vertices of V_1 and let S be any subset of V_1. Then every vertex in S is matched by M into a different vertex in $N(S)$, so that $|S| \leq |N(S)|$ (Fig. 8.6 (a)).

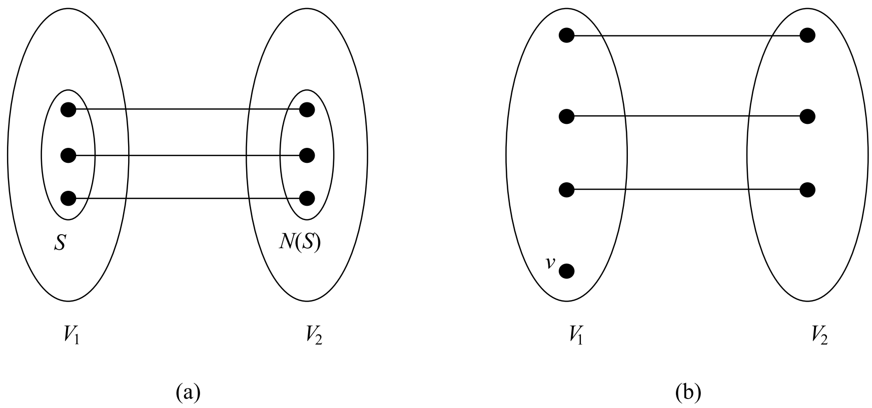


Fig. 8.6

Conversely, let $|N(S)| \geq |S|$ for all subsets S of V_1 and let M be a maximum matching. Assume G has no complete matching. Then there exists a vertex, say $v \in V_1$, which is M-unsaturated (Fig. 8.6(b)). Let Z be the set of vertices of G that can be joined to v by M-alternating paths.

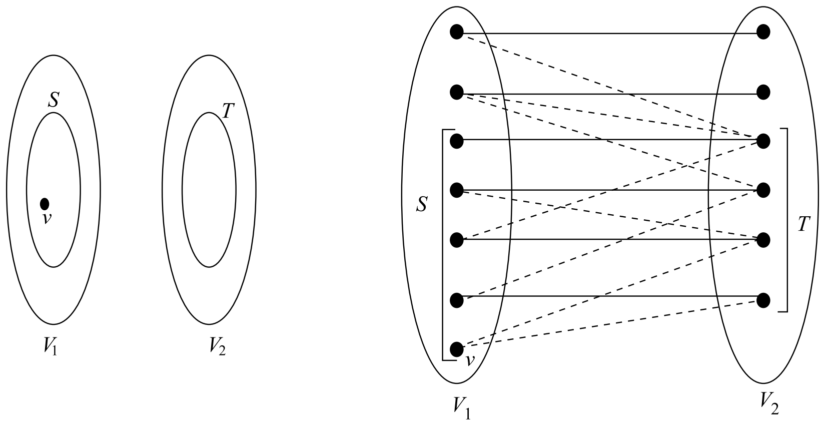


Fig. 8.7

Since M is a maximum matching, by Theorem 8.1, G has no M-augmenting path. This implies that v is the only vertex of Z that is M-unsaturated. Let $S = Z \cap V_1$ and $T = Z \cap V_2$ (Fig. 8.7). Then every vertex of T is matched under M to some vertex of $S - \{v\}$ and conversely and $N(S) = T$. Since $|T| = |S| - 1$, therefore $|N(S)| = |T| = |S| - 1 < |S|$, contradicting the given assumption. Hence G has a complete matching. ❑

The aim of the above theorem was basically to obtain the conditions for the existence of a system of distinct representatives (SDR) for a collection of subsets of a given set. Before giving such conditions, we have the following definition.

Definition: If $A = \{A_i : i \in N\}$ is a family of sets, then a system of distinct representatives (SDR) of A is a set of elements $\{a_i : i \in N\}$ such that $a_i \in A_i$ for every $i \in N$ and $a_i \neq a_j$ whenever $i \neq j$. For example, if $A_1 = \{2, 8\}$, $A_2 = \{8\}$, $A_3 = \{5, 7\}$, $A_4 = \{2, 4, 8\}$ and $A_5 = \{2, 4\}$, then the family $\{A_1, A_2, A_3, A_4\}$ has an *SDR* $\{2, 8, 7, 4\}$, but the family $\{A_1, A_2, A_4, A_5\}$ has no SDR.

Theorem 8.3 (Hall's SDR Theorem) A family of finite nonempty sets $A = \{A_i : 1 \leq i \leq r\}$ has an SDR if and only if for every k, $1 \leq k \leq r$, the union of any k of these sets contains at least k elements.

Proof Clearly $\bigcup_{i=1}^{r} A_i$ is finite, since each of the sets $A_1, A_2, \ldots, A_r$ is finite. Let $\bigcup_{i=1}^{r} A_i = \{a_1, a_2, \ldots, a_n\}$.

Construct a bipartite graph G with partite sets $X = \{A_1, \ldots, A_r\}$ and $Y = \{a_1, \ldots, a_n\}$, as shown in Figure 8.8. We take an edge between A_i and a_j if and only if $a_j \in A_i$.

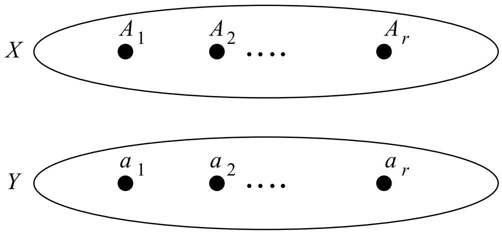


Fig. 8.8

Clearly, A has an *SDR* if and only if G has a matching that saturates all the vertices of X. Now, Theorem 8.2 implies that G has such a matching if and only if $|S| \leq |N(S)|$ for all subsets S of X, that is, if and only if $|S| \leq |\underset{a_i \in S}{\cup} A_i|$ proving the result. ❑

Remarks

1. Hall's theorem is often referred to as Hall's marriage theorem.
2. Since Hall's theorem, there has been remarkable progress in the theory of SDR, and besides other references the reader can refer to the book of Mirsky [161].

The following result due to M. Hall [100] counts the number of complete matchings.

Theorem 8.4 If $G(V_1, V_2, E)$ is a bipartite graph with $|V_1| = n_1 \leq n_2 = |V_2|$ and satisfying Hall's condition $|S| \leq |N(S)|$ for all $S \subseteq V_1$, then G has at least

$$r(\delta, n_1) = \prod_{1 \leq i \leq \min(\delta, n_1)} (\delta + 1 - i)$$

complete matchings, where $\delta = \min\{d(u) : u \in V_1\}$.

Proof Induct on n_1. The result is trivial for $n_1 = 1$. Assume the result to be true for all values of $n_1 \leq m - 1$ and let G be a bipartite graph satisfying the given conditions and $|V_1| = m$.

Case 1 For each nonempty proper subset S of V_1, $|S| < |N(S)|$. Now, for a given $u \in V_1$, choose $v \in N(u)$. This v can be chosen in at least δ ways, since $\delta = \min\{d(u) : u \in V_1\}$. Obviously, $G - \{u, v\}$ is a bipartite graph with $|V_1| = m - 1$, and $\min\{d(x) : x \in V_1\} \geq \delta - 1$. Therefore by induction hypothesis, $G - \{u, v\}$ has at least $r(\delta - 1, m - 1) = \prod_{1 \leq i \leq \min(\delta - 1, m - 1)} (\delta - i)$ complete matchings. These together with uv, give at least

$$\delta[r(\delta - 1, m - 1)] = \overset{\delta}{\Lambda} \prod_{1 \leq i \leq \min(\delta - 1, m - 1)} (\delta - i) = \prod_{1 \leq i \leq \min(\delta, m)} (\delta + 1 - i) = r(\delta, m)$$

complete matchings in G.

Case 2 There is a non empty proper subset S of V_1 with $|S| = |N(S)|$. Let $|S| = s$, and $G_1 =< S \cup N(S) >$ and $G_2 = G- < S \cup N(S) >$. Then the values of n_1 and δ in G_1 are s and δ, and in G_2 are $m-s$ and $\delta_2 = \max\{\delta - s,\ 1\}$. Therefore G_1 and G_2 are disjoint graphs with at least $r(\delta,\ s)$ and $r(\delta_2,\ m-s)$ complete matchings, by induction hypothesis. Thus G has at least $r(\delta,\ s)\ r(\delta_2,\ m-s) = r(\delta,\ m)$ complete matchings. ❑

The following result is due to Ore [175].

Theorem 8.5 If a bipartite graph $G(V_1,\ V_2,\ E)$ satisfies $|S| \leq d + |N(S)|$ for all $S \subseteq V_1$, where d is a given positive integer, then G contains a matching M with at most d vertices of V_1 being M-unsaturated.

Proof Let $H(V_1,\ U_2,\ E')$ be the bipartite graph with $U_2 = V_2 \cup W_2$, $|W_2| = d$ and $E' = E \cup [V_1,\ W_2]$. Then the given condition of G implies Hall's condition for H and so H has a complete matching M. Obviously, at most d of the edges of M can be in $E' - E$, proving the result. ❑

The following observation is an immediate consequence of Theorem 8.5.

Corollary 8.1 For a bipartite graph $G(V_1,\ V_2,\ E)$, with $|V_1| = n_1$, $\alpha_1(G) = n_1 - \max\{|S| - |N(S)| : S \subseteq V_1\} = \min\{|V_1 - S| + |N(S)| : S \subseteq V_1\}$.

The following result is a consequence of Hall's theorem.

Theorem 8.6 A $k(\geq 1)$ regular bipartite graph has a perfect matching.

Proof Let G be a k regular bipartite graph with partite sets V_1 and V_2. Then $E(G)$ is equal to the set of edges incident to the vertices of V_1 and also $E(G)$ is equal to the set of edges incident to the vertices of V_2. Therefore, $k|V_1| = k|V_2| = E(G)$ and thus $|V_1| = |\ V_2|$. If S$\subseteq V_1$, then $N(S) \subseteq V_2$ and thus $N(N(S))$ contains S. Let E_1 and E_2 be the sets of edges of G incident respectively to S and $N(S)$. Then $E_1 \subseteq E_2$, $|E_1| = k|S|$ and $|E_2| \geq k|N(S)|$. Since $|E_2| \geq |E_1|$,$|N(S)| \geq |S|$. Thus by Theorem 8.2, G has a matching that saturates all the vertices of V_1, that is G has a complete matching M. Now deleting the edges of M from G gives a $(k-1)$-regular bipartite graph G', with $V(G) = V(G')$. By the same argument, G' has a complete matching M'. Deletion of the edges of M' from G' results in a $(k-2)$-regular bipartite graph. Repeating this process $k-1$ times, we arrive at a 1-regular bipartite graph G^* such that $|V(G^*)| = |V(G)|$. Clearly G^* has a perfect matching. ❑

Definition: A set C of vertices in a graph G is said to *cover* the edges of G if every edge of G is incident with at least one vertex of C. Such a set C is called a *covering* of G. Consider the graphs G_1 and G_2 shown in Figure 8.9. Clearly, $\{v_1,\ v_2,\ v_4,\ v_5\}$ and $\{v_1,\ v_5,\ v_6\}$ are coverings of G_1. Also, we observe that there is no covering of G_1 with fewer than three vertices. In G_2, each of $\{v_1,\ v_2,\ v_6\}$ and $\{v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\}$ is a covering.

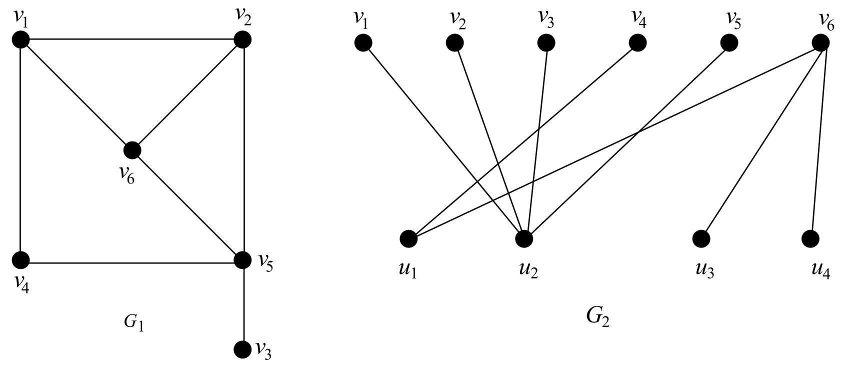


Fig. 8.9

The number of vertices in a minimum covering of G is called the *covering number* of G and is denoted by $\beta(G)$.

Definition: An *edge covering* of a graph G is a subset L of E such that every vertex of G is incident to some edge of L. Clearly, an edge covering of G exists if and only if $\delta > 0$. The cardinality of a minimum edge covering of G is denoted by $\beta_1(G)$. For example, in the wheel W_5 of Figure 8.10, the set $\{v_1v_5,\ v_2v_3,\ v_4v_6\}$ is a minimum edge covering.

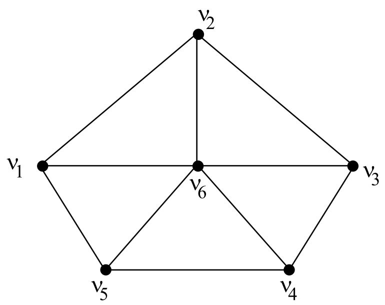


Fig. 8.10

The following results are immediate.

Lemma 8.1 A subset S of V is independent if and only if $V-S$ is a covering of G.

Proof Clearly, S is independent if and only if no two vertices in S are adjacent in G. Thus every edge of G is incident to a vertex of $V-S$. This is possible if and only if $V-S$ is a covering of G. ❑

Lemma 8.2 For any graph G, $\alpha+\beta=n$.

Proof If S is a maximum independent set of G, then by Lemma 8.1, $V-S$ is a covering of G. Therefore, $|V-S|=n-\alpha\geq\beta$. Similarly, if C is a minimum covering of G, then $V-C$

is independent and therefore, $|V-S| = n-\beta \leq \alpha$. Combining the two inequalities, we obtain $\alpha+\beta = n$. ❑

Remark Consider the graph of Figure. 8.11. Clearly the set $E_1 = \{e_3, e_4\}$ is independent and $E - E_1 = \{e_1, e_2, e_5\}$ is not an edge covering of G. Also, $E_2 = \{e_1, e_3, e_4\}$ is an edge covering of G and $E - E_2$ is not independent in G. These observations imply that the edge analogue of Lemma 8.1 is not true in general.

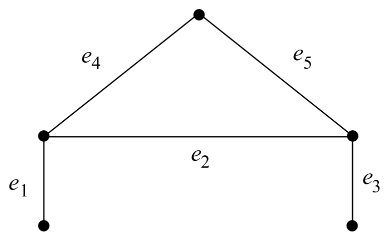


Fig. 8.11

The following result gives the relation between α_1 and β_1 of a graph G.

Theorem 8.7(a) In a graph G with n vertices and $\delta > 0$, $\alpha_1 + \beta_1 = n$.

Proof Let M be a maximum matching in G, so that $|M| = \alpha_1$. Let X be the set of M-unsaturated vertices in G. Since M is maximum, X is an independent set of vertices with $|X| = n - 2\alpha_1$. As $\delta > 0$, choose one edge for each vertex in X incident with it and let F be the set of these edges chosen. Then $M \cup F$ is an edge covering of G. Thus, $|M \cup F| = |M| + |F| = \alpha_1 + n - 2\,\alpha_1 \geq \beta_1$, and so

$$n \geq \alpha_1 + \beta_1. \tag{8.7.1}$$

Now assume that L is a minimum edge covering of G, so that $|L| = \beta_1$. Let $H = G(L)$, the edge subgraph of G defined by L and let M_H be a maximum matching in H. Let the set of M_H-unsaturated vertices in H be denoted by X. Since L is an edge covering of G, therefore, H is a spanning subgraph of G. Thus, $|L| - |M_H| = |L - M_H| \geq |X| = n - 2|M_H|$, and therefore $|L| + |M_H| \geq n$. Since M_H is a matching in G, so $|M_H| \leq \alpha_1$, and hence

$$n \leq \beta_1 + \alpha_1. \tag{8.7.2}$$

Combining (8.7.1) and (8.7.2), we obtain $\alpha_1 + \beta_1 = n$. ❑

Let M be any matching of a graph G and C be any vertex covering of G. Then in order to cover each edge of M, at least one vertex of C is to be chosen. Thus, $|M| \leq |C|$. In case, M' is a maximum matching and C' is a minimum covering of G, then $|M'| \leq |C'|$.

We have the following obvious result.

Lemma 8.3 If C is any covering and M any matching of a graph G with $|C| = |M|$, then C is a minimum covering and M is a maximum matching.

Proof Let M' be a maximum matching and C' be a minimum covering of G. Then $|M| \leq |M'|$ and $|C| \geq |C'|$. Therefore, $|M| \leq |M'| \leq |C'| \leq |C|$. As $|M| = |C|$ we have $|M| = |M'| = |C'| = |C|$, and the proof is complete. ❑

The next result is due to Konig [135].

Theorem 8.7(b) (Konig) In a loopless bipartite graph G, the maximum number of edges in a matching of G is equal to the minimum number of vertices in an edge cover of G, that is, $\alpha_1 = \beta$.

Proof Let G be a bipartite graph with partite sets V_1 and V_2 and let M be a matching of G. Let X be the set of all M-unsaturated vertices of V_1, so that $|M| = |V_1| - |X|$ (Fig. 8.12).

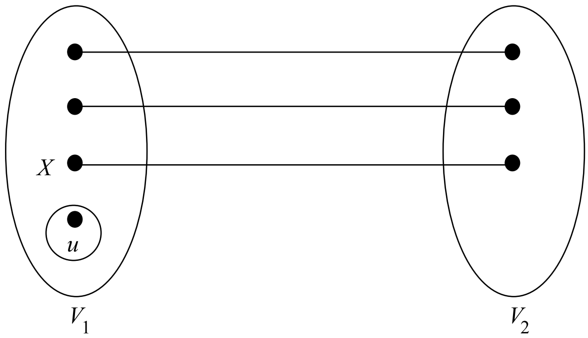


Fig. 8.12

Let A be the set of those vertices of G which are connected to some vertex of X by an M-alternating path. Further, let $S = A \cap V_1$ and $T = A \cap V_2$. Obviously, $N(S) = T$ and $S - X$ is matched to T so that $|X| = |S| - |T|$. Clearly, $C = (V_1 - S) \cup T$ is a covering of G, because if there is an edge e not incident to any vertex in C, then one of the end vertices of e is in S and the other in $V_2 - T$, which contradicts the fact that $N(S) = T$. Now, $|C| = |V_1| - |S| + |T| = |V_1| - |X| = |M|$. Then by Lemma 8.3, M is a maximum matching and C is a minimum covering of G. ❑

The following is a new proof of Konig's theorem due to Rizzi [224].

Second proof of Theorem 8.7 (Rizzi [224]) Let G be a minimal counter-example. Then G is connected but is not a cycle, nor is a path. Therefore G has a vertex of degree at least three. Let u be such a vertex and v one of its neighbours. If $\alpha_1(G - v) < \alpha_1(G)$, then by minimality, $G - v$ has a cover W' with $|W'| < \alpha_1(G)$. Thus $W' \cup \{v\}$ is a cover of G with cardinality at most $\alpha_1(G)$. Therefore assume there exists a maximum matching M of G having no edge incident at v. Let f be an edge of $G - M$ incident at u but not at v. Let W'

be a cover of $G-f$ with $|W'| = \alpha_1(G)$. Since no edge of M is incident at v, it follows that W' does not contain v. Therefore W' contains u and is a cover of G. ❑

8.2 Factors

We start this section with the following definitions.

Definition: A *factor* F of a graph G is a spanning subgraph of G. A factor F is said to be an *F–factor* if the components of F are some graphs in a given collection $F = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of G. If F contains a single graph H, then F is called an *H-factor*. Consider the graph G shown in Figure 8.13. Clearly G_1 is a spanning subgraph whose components belong to the set $F = \{H_1, H_2, H_3\}$ and therefore is an F-factor. Each of the components of the spanning subgraph G_2 is a graph K_3 and so G_2 is a K_3-factor.

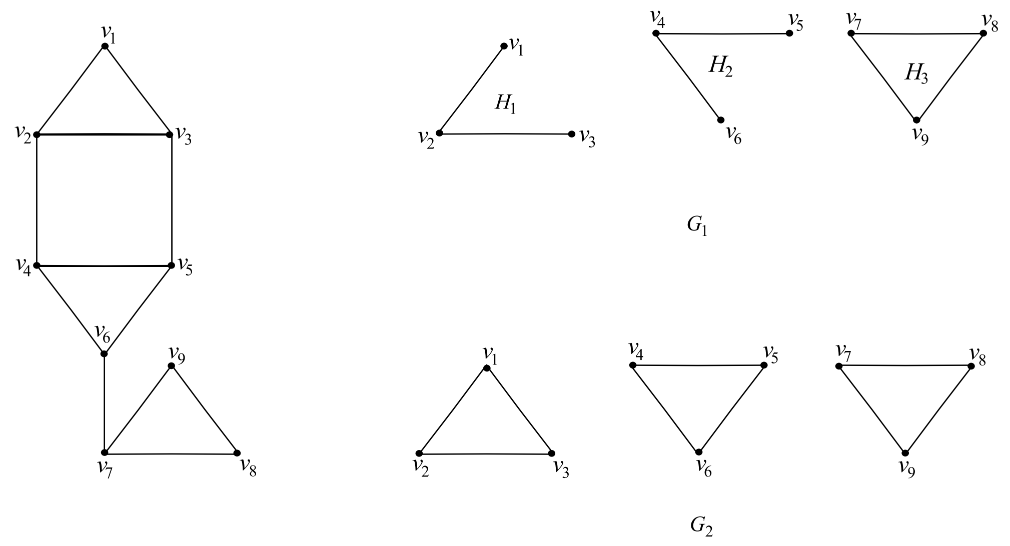


Fig. 8.13

We observe that a perfect matching is a K_2-factor, since each of the components in a perfect matching is a graph K_2.

A *vertex function* of a graph $G(V, E)$ is a function $f: V \to N_o$ (N_o being the set of non-negative integers).

Definition: A spanning subgraph F of a graph $G(V, E)$ is called an *f-factor* of G if for a vertex function f on V, $d(v|F) = f(v)$ for all $v \in V$. For example, consider the graph G shown in Figure 8.14. Define $f: V \to N_0$ by $f(v_i) = 1$ or 2, for all $v \in V$ in G. Then the f-factor of G is given by graph H.

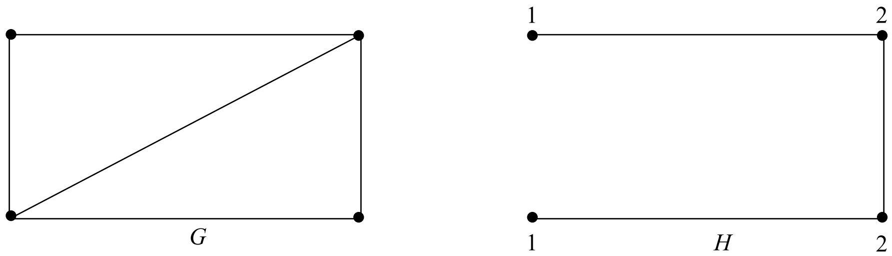


Figure 8.14

Further, F is said to be a *partial f–factor* (or f-matching) if $d(v|F) \leq f(v)$, for all $v \in V$ and F is said to be a covering f-factor of G if $d(v|F) \geq f(v)$, for all $v \in V$. Some of the details of this can be seen in Graver and Jurkat [88]. We note that an f-factor exists only if $\sum\limits_{v \in V} f(v)$ is even.

Definition: A *k-factor* of a graph G is a factor of G that is k-regular. Clearly, a 1-factor is a perfect matching and exists only for graphs with an even number of vertices. A 2-factor of G is a factor of G that is a disjoint union of cycles of G and connected 2-factor is a Hamiltonian cycle.

Definition: If the edge set of a graph G is partitioned by the edge sets of a set of factors $F_1, F_2, \ldots, F_q$ of G, then these factors are said to constitute a *factorisation* or decomposition of G and we write $G = F_1 \cup F_2 \cup \ldots \cup F_q$. If each factor F_i in the factorisation is isomorphic to the same spanning subgraph F of G, then the factorisation is called an *isomorphic factorisation* or an F-decomposition of G.

The factorisation is called an *f-factorisation* or *k-factorisation* according as each factor f_i of a factorisation is an f-factor or k-factor. A graph G is f-factorable or k-factorable if G has respectively f-factorisation or k-factorisation.

Example In Figure 8.15, the three distinct 1-factors of G_1 are $< e_1, e_6 >$, $< e_2, e_4 >$ and $< e_3, e_5 >$. The two distinct 2-factors of G_2 are $< e_1, e_2, e_3, e_4, e_5 >$ and $< e_6, e_7, e_8, e_9, e_{10} >$.

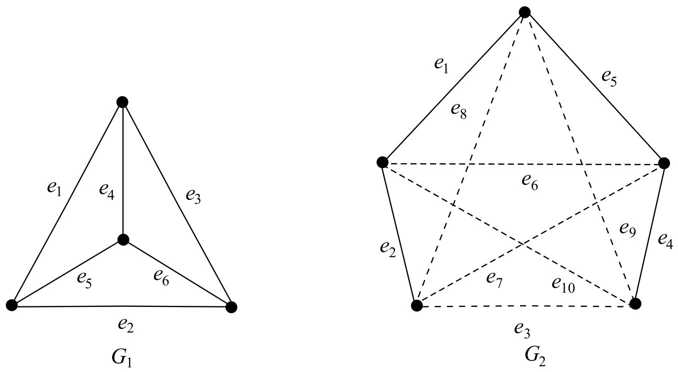


Fig. 8.15

Definition: A component of a graph is *odd or even* according to whether it has an odd or even number of vertices. The number of odd components of a graph G is denoted by $k_0(G)$.

The following result of Tutte [249] characterises the graphs with a 1-factor. Several proofs of this result exist in the literature and the proof given here is due to Lovasz [150].

Theorem 8.8 (Tutte) A graph G has a 1-factor if and only if

$$k_0(G-S) \leq |S|, \text{ for all } S \subseteq V. \tag{8.8.1}$$

Proof Let G be a graph having 1-factor M. Let S be an arbitrary subset of V and let O_1, O_2, $\ldots, O_k$ be the odd components of $G-S$. For each i, the vertices in O_i can be adjacent only to other vertices in O_i and to vertices in S. Since G has a 1-factor (perfect matching), at least one vertex from each O_i is to be matched with a different vertex in S. Thus the number of vertices in S is at least k, the number of odd components. Hence $|S| \geq k$, so that $k_0(G-S) \leq |S|$ (Fig. 8.16).

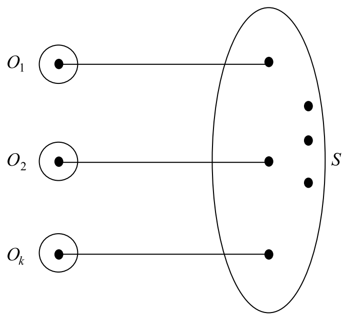


Fig. 8.16

Conversely, assume that the condition (8.8.1) holds. Let G have no 1-factor. Join pairs of non-adjacent vertices of G and continue this process till we get a maximal super graph G^* of G having no 1-factor. Since joining two odd components by an edge results in an even component, therefore

$$k_0(G^*-S) \leq k_0(G-S). \tag{8.8.2}$$

Thus for G^* also, we have $k_0(G^*-S) \leq |S|$.

When $S = \phi$ in (8.8.1), we observe that $k_0(G) = 0$ so that $k_0(G^*) = 0$. Therefore, $|V(G^*)| = |V(G)| = n$ is even. Also, for every pair of non-adjacent vertices u and v of G^*, $G^* + uv$ has a 1-factor and any such 1-factor definitely contains the edge uv.

Now, let K be the set of those vertices of G^* which are having degree $n-1$. Clearly, $K \neq V$, since otherwise $G^* = K_n$ has a perfect matching.

Claim that each component of $G^* - K$ is complete. Assume, on the contrary, that in $G^* - K$, there exists a component G_1 which is not complete. Then in G_1, there are vertices

x, y and z such that xy, $yz \in E(G^*)$ and $xy \notin E(G^*)$. Also, $y \in V(G_1)$ so that $d(y|G^*) < n-1$ and thus there exists a vertex w of G^* with $yw \notin E(G^*)$. Evidently, $w \notin K$ (Fig. 8.17).

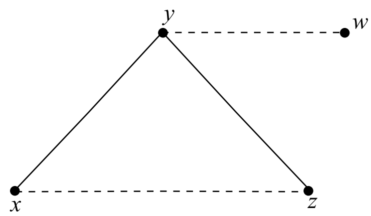
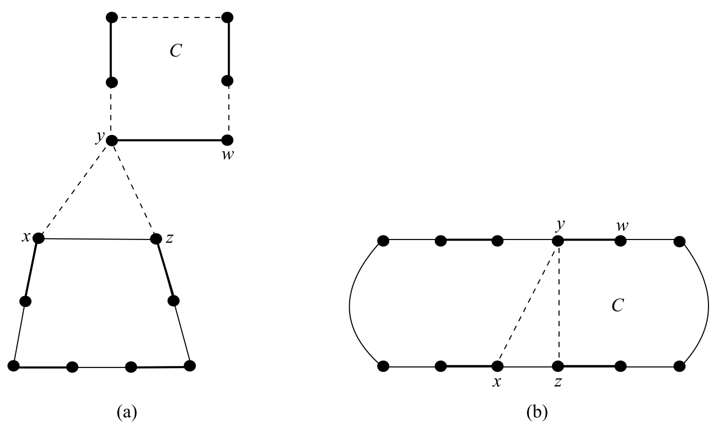


Fig. 8.17

Now, by the choice of G^*, both $G^* + xz$ and $G^* + yw$ have 1-factors, say M_1 and M_2 respectively. Clearly, $xz \in M_1$ and $yw \in M_2$. Let H be the subgraph of $G^* + \{xz, yw\}$ induced by the edges that are in M_1 and M_2 but not in both (that is, in the symmetric difference of M_1 and M_2). Since M_1 and M_2 are 1-factors, each vertex of G^* is saturated by both M_1 and M_2, and H is a disjoint union of even cycles in which the edges alternate between M_1 and M_2. We have two cases to consider.

Case 1 Let xz and yw belong to different components of H, as shown in Figure 8.18 (a). If yw belongs to the cycle C, then the edge of M_1 in C together with the edges of M_2 not belonging to C form a 1-factor of G^*, contradicting the choice of G^*.



Ordinary lines correspond to the edges of M_1 and bold lines to the edges of M_2

Fig. 8.18

Case 2 Let xz and yw belong to the same component C of H. Since each component of H is a cycle, C is a cycle (Fig. 8.18 (b)). By symmetry of x and z, it can be assumed that the vertices x, y, w and z appear in that order on C. Then the edges of M_1 belonging to the

$yw\ldots z$ section of C, together with the edge yz, and the edges of M_2 not in the $yw\ldots z$ section of C form a 1-factor of G^*, again contradicting the choice of G^*.

Thus each component of $G^* - K$ is complete.

By conditions (8.8.2), $k_0(G^* - K) \le |K|$. Therefore one vertex of each of the odd components of $G^* - K$ is matched to a vertex of K. This is because each vertex of K is adjacent to every other vertex of G^*. Also the remaining vertices in each of the odd and even components of $G^* - K$ can be matched among themselves (Fig. 8.19). The total number of vertices thus matched is even. As $|V(G^*)|$ is even, the remaining vertices of K can be matched among themselves, thus giving a 1-factor of G^*, which is a contradiction. ❑

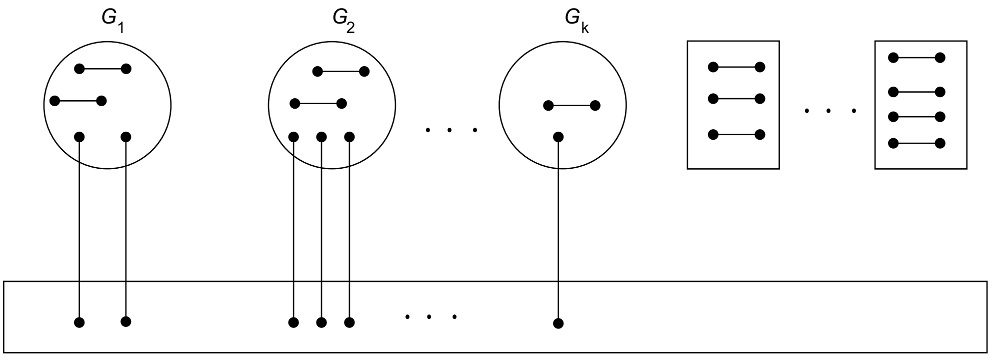


Fig. 8.19

Remark Among the various proofs of Theorem 8.8, one proof due to Anderson [4] uses Hall's theorem.

Tutte's characterisation of graphs admitting 1-factors has been used to obtain several results including those for regular graphs and many of these are reported in the review article by Akiyama and Kano [1]. We present some of these results. The first is due to Peterson [181].

Theorem 8.9 (Peterson) Every cubic graph without cut edges has a 1-factor.

Proof Let G be a cubic graph without cut edges and let $S \subseteq V$. Let $O_1, O_2, \ldots, O_k$ be the odd components of $G - S$ and let m_i be the number of edges of G having one end in $V(O_i)$ and the other end in S. As G is cubic,

therefore, $$\sum_{v \in V(O_i)} d(v) = 3\,|V(O_i)| \tag{8.9.1}$$

and $$\sum_{v \in S} d(v) = 3\,|S|. \tag{8.9.2}$$

Now, $E(O_i) = [V(O_i),\, V(O_i) \cup S] - [V(O_i),\, S]$, where [A, B] denotes the set of edges having one end in A and the other end in B, $A \subseteq V$, $B \subseteq V$. Thus, $m_i = |[V(O_i),\, S]| = \sum_{v \in V(O_i)} d(v) - 2\,|E(O_i)|$. Since $d(v) = 3$ for each v and $V(O_i)$ is odd component, m_i is odd for each i. Also, since G has no cut edges, $m_i \geq 3$. Therefore, $k_0(G-S) = k \leq \frac{1}{3}\sum_{v\in S} d(v) < \frac{1}{3}3\,|S| = |S|$. Hence by Theorem 8.8, G has a 1-factor. ❑

Remark A cubic graph with cut edges may not have a 1-factor. To see this, consider the graph G_1 of Figure 8.20 (a). If we take $S = \{v\}$, then $k_0(G-S) = 3 > 1 = |S|$ and hence G_1 has no 1-factor.

Also, a cubic graph with a 1-factor may have cut edges. Consider the graph G_2 of Figure 8.20 (b). Clearly, $< e_1, e_2, e_3, e_4, e_5 >$ is a 1-factor and e_3 is a cut edge of G_2.

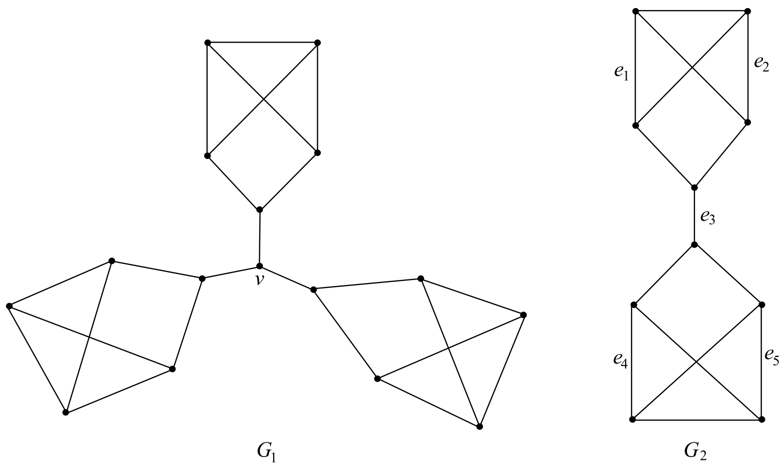


Fig. 8.20

The next result is due to Cunningham [62].

Corollary 8.2 The edge set of a simple 2-edge-connected cubic graph G can be partitioned into paths of length three.

Proof By Theorem 8.9, G is a union of a 1-factor and 2-factor. Orient the edges of each cycle of the 2-factor in any manner so that each cycle becomes a directed cycle. Then, if e is any edge of the 1-factor and f_1, f_2 are the two arcs of G having their tails at the end vertices of e, then $\{e, f_1, f_2\}$ form typical 3-path of the edge partition of G (Fig. 8.21). ❑

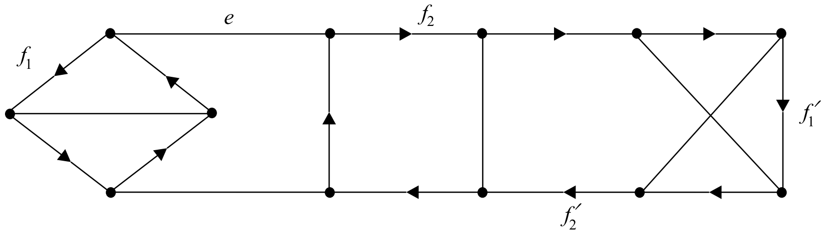


Fig. 8.21

Berge [23] obtained the following estimate for $\alpha_1(G)$ of a graph G, the proof of which is given by Bollobas [29].

Theorem 8.10 For any simple graph G, $\alpha_1(G) = (n-d)/2$, where $d = \max\{k_0(G-S)-|S| : S \subseteq V\}$.

Proof Clearly at most $n-d$ vertices can be saturated in any matching of G. In order to show that this bound is attained, we use the following construction. Let $H = GVK_d$. Then H satisfies Tutte's condition. For if S' is any subset of $V(H)$, then $k_0(H-S') = 0$ or 1 if $S' \not\supseteq V(K_d)$. Also, if $S' \supseteq V(K_d)$, then $k_0(H-S') = k_0(G-S)$, where $S = S' \bigcap V(G)$, so that $k_0(H-S')-|S'| = k_0(G-S) - |S| - d = 0$, by definition of d. Therefore, by Tutte's theorem, H has a 1-factor M. The restriction of M of G is a matching in G which does not cover at most d vertices of G. ❑

Remark Berge actually used the theory of alternating paths to prove Theorem 8.10 and then deduced the Tutte's theorem.

The restatement of Theorem 8.10 is as follows.

Corollary 8.3 If G is a graph with n vertices and $\alpha_1(G) = a$, then there is a set S of s vertices such that $k_0(G-S) = n+s-2a$.

The following result is due to Babler [10].

Theorem 8.11 Every k-regular, $(k-1)$-edge-connected graph with even order has a 1-factor.

Proof Let G be a k-regular, $(k-1)$-edge-connected graph with even order and let S be any subset of V. Let $O_1, O_2, \ldots, O_r$ be the odd components of $G-S$ with order n_i and size m_i, and let k_i edges join O_i to S, for $1 \le i \le r$. Then $k_i \ge k-1$, (8.11.1)

$$\sum_{v \in V(O_i)} d(v) = kn_i = 2m_i + k_i \tag{8.11.2}$$

and $$\sum_{v \in S} d(v) = k|S| \ge \sum_{1}^{r} k_i + 2|E(S)|. \tag{8.11.3}$$

Now, (8.11.2) gives $k_i = kn_i - 2m_i$, so that if k is even, so is k_i, and thus $k_i > k-1$ implies $k_i \ge k$. Similarly, if k is odd, so is k_i, and again $k_i > k-1$ implies $k_i \ge k$. Summing this over r odd components, we obtain by using (8.11.3)

$$rk \le \sum_{i=1}^{r} k_i \le \sum_{i=1}^{r} k_i + 2|E(S)| \le k|S| \, .$$

Therefore, $k_0(G-S) = r \le |S|$ and by Tutte's theorem, G has a 1-factor. ❑

8.3 Antifactor Sets

Definition: A subset $S \subseteq V$ of a graph $G(V, E)$ such that $k_0(G-S) > |S|$ is called an *antifactor* set. An antifactor set S such that no proper subset of S is an antifactor set is called a *minimal antifactor set*. For example, in the graph of Figure 8.20(a), $\{v\}$ is an antifactor set.

Definition: A vertex $v \in V$ is a *claw centre* if and only if at least three vertices of G adjacent to v form an independent set. Clearly, vertex v is a claw centre in the graph of Figure 8.20(a).

Now assume that G is a graph of even order n and let S be an antifactor set of G. Let $k_0(G-S) = k$ and $O_1, O_2, \ldots, O_k$ be the odd components of $G-S$. Since n is even, $|S|$ and k have the same parity. Choose a vertex $u_i \in V(O_i)$, $1 \le i \le k$. Then $|S \cup \{u_1, u_2, \ldots, u_k\}|$ is even, that is, $k+|S| \equiv 0 \pmod 2$. This implies that $k \equiv |S| \pmod 2$ and therefore $k_0(G-S) \equiv |S| \pmod 2$. Thus we have the following result.

Lemma 8.4 If S is an antifactor set of a graph G of even order, then $k_0(G-S) \ge |S|+2$.

Tutte's theorem implies that a graph G has either a 1-factor or an antifactor set. It is necessary to mention here that the computational complexity is high in the verification of Tutte's condition for every subset S of V. In order to reduce the computations, the antifactor sets are of great significance. The following result on antifactors is due to Sumner [236].

Lemma 8.5 If S is a minimal antifactor set of a connected graph G of even order and having no 1-factor, and if $k_0(G-S) = r$ and $|S| = s$, then

i. $r \ge s+2$,

ii. each vertex of S is adjacent to vertices in at least $r-s+1$ distinct odd components of $G-S$,

iii. every vertex of S is a claw centre.

Proof

i. Already established (Lemma 8.4).

ii. If $v \in S$ is adjacent to h distinct odd components say $O_1, O_2, \ldots, O_h$ with $h \le r-s$, then $G - \{S - \{v\}\}$ has the odd components $O_{h+1}, O_{h+2}, \ldots, O_r$. Then, if h is odd, $k_0(G-S') = r-h \ge s > s-1 = |S'|$, and if h is even, then $\{O_1 \cup O_2 \cup \ldots \cup O_h\} \cup \{v\}$ is also an odd component of $G-S'$ so that $k_0(G-S') = r-h+1 \ge s+1 > s-1 = |S'|$, where $S' = S - \{v\}$. That is, S' is an antifactor set, contradicting the minimality of S. Thus, $h \ge r-s+1$.

iii. Since any $v \in S$ is adjacent to at least $r-s+1 \ge 3$ (by (i)) distinct odd components, v is a claw centre, by definition. ❑

Remark The above observations imply that Tutte's theorem is equivalent to the following.

A graph G has a 1-factor if and only if G does not have a set S of claw centres such that $k_0(G-S) > |S|$. Obviously, this form of Tutte's theorem decreases the number of subsets S of V to be verified in Tutte's condition.

An improvement of Theorem 8.11 is due to Plesnik [209].

Theorem 8.12 If G is a k-regular, $(k-1)$-edge-connected graph of even order and G_1 is obtained from G by removing any $k-1$ edges, then G_1 has a 1-factor.

Proof Assume G_1 has no 1-factor. Then G_1 has an antifactor set S with $k_0(G_1-S) > |S|$. Therefore by Lemma 8.4,

$$k_0(G_1-S) \geq |S|+2. \tag{8.12.1}$$

Let $O_1, O_2, \ldots, O_r$ be the odd components and $O_{r+1}, O_{r+2}, \ldots, O_{r+s}$ be the even components of G_1-S. For a component O_i, let a_i and m_i be the number of edges respectively of $E(G)-E(G_1)$ and $E(G_1)$ joining it to S and b_i be the number of edges of $E(G)-E(G_1)$ joining it to the other O_j's. Then the total number of edges going out of O_i is $a_i+b_i+m_i$. Since G is $(k-1)$-edge-connected, $a_i+b_i+m_i \geq k-1$. By the argument used in Theorem 8.11, this implies that $a_i+b_i+m_i \geq k$, for an odd component O_i. Summing over the r odd components, we obtain

$$\sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i + \sum_{i=1}^{r} m_i \geq rk. \tag{8.12.2}$$

The number of edges between the O_i's, and between O_i's and S, removed from G is $\sum\limits_{i=1}^{t} a_i + \left(\frac{1}{2}\right) \sum\limits_{i=1}^{t} b_i$, where $t = r+s$. As this is at most $k-1$, we have

$$2\sum_{i=1}^{t} a_i + \sum_{i=1}^{t} b_i \leq 2(k-1). \tag{8.12.3}$$

Also, $\sum\limits_{i=1}^{t} a_i + \sum\limits_{i=1}^{t} m_i \leq \sum\limits_{v\in S} d(v) = k|S|.$ (8.12.4)

Adding (8.12.3) and (8.12.4), we get

$$3\sum_{i=1}^{t} a_i + \sum_{i=1}^{t} b_i + \sum_{i=1}^{t} m_i \leq k(|S|+2)-2. \tag{8.12.5}$$

As the sum on the left of (8.12.2) is less than the sum on the left of (8.12.5), we have

$rk \leq k(|S|+2)-2 < k(|S|+2)$, giving r $< |S|+2$, which contradicts (8.12.1). ❑

We now have the following observation.

Corollary 8.4 A $(k-1)$-regular simple graph on $2k$ vertices has a 1-factor.

Proof Assume G is a $(k-1)$-regular simple graph with $2k$ vertices having no 1-factor. Then G has an antifactor set S and by Lemma 8.4, $k_0(G-S) \geq |S|+2$. Therefore, $|S|+(|S|+2) \leq 2k$ and so $|S| \leq k-1$. Let $|S| = k-r$. Then $r \neq 1$. For if $r = 1$, then $|S| = k-1$ and therefore $k_0(G-S) = k+1$. Thus each odd component of $G-S$ is a singleton and therefore each such vertex is adjacent to all the $k-1$ vertices of S, as G is $(k-1)$ regular. This implies that every vertex of S is of degree at least $k+1$, a contradiction. Thus, $|S| = k-r$, $2 \leq r \leq k-1$. If G_1 is any component of $G-S$ and $v \in V(G_1)$, then v can be adjacent to at most $|S|$ vertices of S. Since G is $(k-1)$-regular, v is adjacent to at least $(k-1)-(k-r) = r-1$ vertices of G_1. So, $|V(G_1)| \geq r$. Counting the vertices of S, we obtain $(|S|+2)r+|S| = 2k$, or $(k-r+2)r+(k-r) \leq 2k$. This gives $(r-1)(r-k) \geq 0$, violating the condition on r. ❑

The next result is due to Sumner [236].

Theorem 8.13 If G is a connected graph of even order n and claw-free (that is, contains no $K_{1,3}$ as an induced subgraph), then G has a 1-factor.

Proof If G has no 1-factor, then G contains a minimal antifactor set S of G. Also, there is an edge between S and each odd component of $G-S$. If $v \in S$ and vx, vy and vz are edges of G with x, y and z belonging to distinct odd components of $G-S$, then a $K_{1,3}$ is induced in G. This is not possible, by hypothesis.

Since $k_0(G-S) > |S|$, there exists a vertex v of S and edges vu and vw of G, with u and w in distinct odd components of $G-S$. Assume O_u and O_w to be the odd components containing u and w respectively. Then $< O_u \cup O_w \cup \{v\} >$ is an odd component of $G-S_1$, where $S_1 = S-\{v\}$. Also, $k_0(G-S_1) = k_0(G-S)-1 > |S|-1 = |S_1|$ and therefore S_1 is an antifactor set of G with $|S_1| = |S|-1$, a contradiction to the choice of S. So G has a 1-factor. (Clearly, by Lemma 8.4, the case $|S| = 1$ and $k_0(G-S) = 2$ does not exist). ❑

The following results are immediate, as each type of graph in these cases does not contain $K_{1,3}$ as an induced subgraph.

Corollary 8.5 Every connected even order edge graph has a 1-factor.

Corollary 8.6 If G is a connected graph of even order, then G^2 has a 1-factor.

Corollary 8.7 Every connected total graph of even order has a 1-factor.

Corollary 8.8 Every connected cubic graph in which every vertex lies on a triangle has a 1-factor.

Corollary 8.9 If a connected even order graph G has less than $\kappa(G)$ claw centres, then G has a 1-factor.

The following result is the generalisation of Theorem 8.13.

Theorem 8.14 If G is n-connected even order graph with no induced $K_{1,n+1}$, then G has a 1-factor.

Proof Assume G has no 1-factor. Then G has a minimal antifactor set S. Let $|S| = s$ and let the odd components of $G-S$ be $O_1, O_2, \ldots, O_k$. If $S_i' = \{v \in S : v$ is adjacent to some vertex to $O_i\}$, then S_i' is a vertex cut of G and therefore $|S_i'| \geq n$. Thus there are at least n edges from O_i incident with n distinct vertices of S. The number of such edges as i ranges from 1 to k is at least nk and this is greater or equal to $n(s+2)$, by Lemma 8.4. Thus at least one vertex v of S is incident with at least $n+1$ of these edges. But then, v is the centre of an induced $K_{1,n+1}$, contradicting the assumption. ❑

It can be observed that a graph can have more than one 1-factors and different 1-factors can either have no edge in common or have some edges in common. Depending on the nature of edges, we have the following definition.

Definition: In a given graph, two 1-factors are said to be *disjoint* if they have no edge in common and two 1-factors are said to be *distinct* if they have at least one different edge. For example, in the graph of Figure 8.22, the 1-factors $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ are disjoint, while as the 1-factors $\{e_1, e_2, e_3\}$ and $\{e_1, e_7, e_5\}$ are distinct.

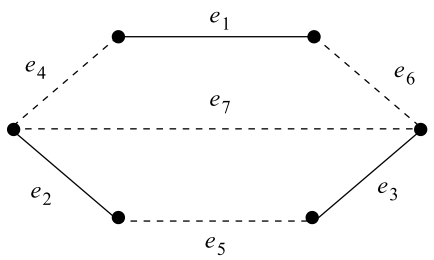


Fig. 8.22

Consider maximal sets S which satisfy Tutte's condition in graphs admitting a 1-factor. If such a graph is connected, then definitely there exist subsets S such that $k_0(G-S) = |S|$. For such a graph will have a vertex v which is not a cut vertex, and $G-v$ being connected and of odd order, $k_0(G-v) = 1 = |v|$. The maximal subsets S for which $k_0(G-S) = |S|$ holds is called a maximal factor set.

A result on counting distinct 1-factors for bipartite graphs is given by Theorem 8.4. The next result as reported in Bollobas [29] gives the bounds for distinct 1-factors in a graph..

Theorem 8.15 Let G be a graph admitting a 1-factor and let S_0 be a maximal factor set of G with $|S_0| = m$ and $O_1, O_2, \ldots, O_m$ be the odd components of $G-S_0$. Further, let $d = \min$ $\{|N(O_i) \cap S_o| : 1 \leq i \leq m\}$ and d' be the minimum number of edges joining O_i to S_0, $1 \leq i \leq m$. Then the number $F(G)$ of 1-factors of G satisfies

i. $F(G) \geq r(d,m) = \prod_{i=1}^{\min(d,m)} (d+1-i)$ and

ii. $F(G) \geq \tilde{r}\ (d',m)$, where $\tilde{r}\ (d',m)$ is the minimum number of 1-factors in a bipartite multigraph with m vertices in each class, in which d' is the minimum degree in the first vertex class.

Proof Let S_0 be a maximal factor set of G so that $k_0(G-S_0) = |S_0|$.

Using Hall's theorem for a complete matching in a bipartite graph, we can show that $|S_0|$ odd components of $G-S_0$ can be paired with the vertices of S_0. The sufficient condition is that any k such odd components should be joined in G to at least k vertices in S_0. Clearly, this is satisfied here. For if some set of k odd components are joined in G only to a subset T of S_0 with h vertices, $h < k$, then $k_0(G-T) \geq k > h = |T|$, contradicting the hypothesis. Therefore one vertex each of $|S_0|$ odd components can be matched to one vertex of S_0. Let O be an odd component of which a vertex a has been matched with a corresponding vertex s of S_0. Clearly, $O' = O-a$ is a graph on an even number of vertices. Now we verify Tutte's condition on O'. If there is a $T \subseteq V' = V(O')$ such that $k_0(O'-T) > |T|$, then $k_0(O'-T) > |T|+2$. Let $S = S_0 \cup T \cup \{a\}$. Then $k_0(G-S) = k_0(O'-T) + k_0(G-S_0) - 1 = k_0(O'-T) + |S_0| - 1 > |T| + |S_0| + 1 = |S|$, contradicting the choice of S_0. Thus Tutte's condition is satisfied for O' and O' has a 1-factor.

If $G-S_0$ has an even component C and $a \in V(C)$, then $G-(S_0 \cup \{a\})$ has at least one more odd component than $G-S_0$, contradicting the choice of S_0 or the Tutte condition. Thus $G-S_0$ has no even components.

Therefore we have seen that for each $o_i \in O_i$ such that there is an $s_i \in S_0$ with $o_i s_i \in E$, each $O_i - o_i$ has a 1-factor forming part of a 1-factor of G. The number of such choices of m independent edges $o_i s_i$ equals the number of 1-factors in a bipartite graph $H(V_1 \cup V_2,\ E)$, where V_1 has m vertices, one corresponding to each odd component O_i of $G-S_0$ and V_2 has the m vertices of S_0. If we take the edge set E to be such that whenever $s_j \in N(O_i) \cap S_0$, we take exactly one $o_i s_i \in E$, then we get an ordinary bipartite graph for H in which the minimum degree in V_1 is $d = \min\{|N(O_i) \cap S_0|\}$ and then (i) follows from Theorem 8.4. However, if we take E to be such that for each edge joining O_i to s_j we take an edge $o_i s_j$, then we get a bipartite multigraph for H in which the least degree of a vertex in V_i is d' as defined and then (ii) follows. ❑

The following observations can be seen immediately.

Corollary 8.10 For a k-edge-connected graph with a unique 1-factor, $F(G) \geq k$.

Corollary 8.11 If G is a connected graph with a unique 1-factor, then G has a cut edge which belongs to a 1-factor.

The details of the following results can be found in Bollobas [29].

Theorem 8.16 If $\delta(n)$ is the minimum degree of a graph of order $2n$ with exactly one 1-factor, then $\delta(n) \leq [\log 2(n+1)]$.

Theorem 8.17 If $e(n)$ is the maximum size of a graph of order $2n$ with exactly one 1-factor, then $e(n) = n^2$.

8.4 The f-factor Theorem

The f-factor Theorem gives a set of necessary and sufficient conditions for the existence of an f-factor in a general graph. This result was proved by Tutte [250] for loopless graphs, using the theory of alternating paths, and later again by Tutte [251], using the 1-factor Theorem.

First of all, we have the following definitions.

Definition: A *vertex function* for a general graph $G(V, E)$ is a function $f : V \to N_0$ such that $0 \le f(v) \le d(v)$, for every $v \in V$.

For any function $f : V \to N_0$ and for any $U \subseteq V$, we set $f(U) = \sum\limits_{v \in U} f(v)$.

Definition: Three pair-wise disjoint (possibly empty) subsets S, T and U of the vertex set V of a general graph G, such that $V = S \cup T \cup U$, is called a *decomposition* $D = (S, T, U)$ of G. The components, say $C_1, C_2, \ldots, C_k$, of the vertex-induced subgraph $< U >$ of G are called the components of U. For each C_i, let $h_i = f(C_i) + q_G[T, C_i]$, where f is a given vertex function of G, and $q_G[T, C_i]$ denotes the number of edges of G, whose one end is in T and the other end is in C_i. Then C_i is said to be an odd or even component of U, depending upon h_i being odd or even, and the number of odd components of U is denoted by $k_0'(D, f)$.

The *f-deficiency* of D for the given G is defined as

$$\delta(D, f) = k_0'(D, f) - f(S) + f(T) - d(T) + q[S, T].$$

The decomposition D is defined to be an *f-barrier* of G if $\delta(D, f) > 0$.

The term decomposition is used by Graver and Jurkat [88], and Tutte calls it a *G-triple*. These decompositions play the same role as the antifactor sets in the theory of 1-factors.

Before going into the details, we first state the f-factor Theorem.

Theorem 8.18 If f is a vertex function for a graph G, then either G has an f-factor or an f-barrier but not both.

It can be seen that both the statement and the proof of the f-factor Theorem as in Tutte [253] definitely have similarities to those of the 1-factor Theorem. Since there are few complications in the constructions involved which make the proof given by Tutte [253] some what lengthy, we follow here the earlier proof by Tutte [251].

Tutte [250] defines an ordinary graph G' corresponding to a given graph G such that G has an f-factor if and only if G' has a 1-factor and shows that the 1-factor condition corresponds to the f-factor condition, and derives the f-factor Theorem as a consequence of the 1-factor Theorem. Berge [23] uses the same graph construction to derive the maximum

f-matching Theorem. Since we have general graphs under consideration, we construct G' with suitable changes as reported by Parthasarthy [180], and then find that the proof still holds.

Let G be a general graph with the vertex function f. This defines for each vertex v_i, its degree $d_i = d(v_i|G)$, $f_i = f(v_i)$ and $k_i = d_i - f_i$. For each vertex v_i of G, take two subsets X_i and Y_i of vertices of G', where $|X_i| = d_i$ and $|Y_i| = k_i$. Let the vertices of Y_i be labelled y^i_j, $1 \leq j \leq k_i$, arbitrarily.Now, label the vertices of X_i in the following manner.

Label the edges of G arbitrarily as $1, 2, \ldots, m$. Since $|X_i| =$ number of link edges l_i incident with v_i plus twice the number of loops ℓ^0_i incident with v_i, a vertex $x^i_j \in X_i$ is taken corresponding to each link edge j incident with v_i and two vertices x^i_h, $x^i_{h'}$ are taken corresponding to each loop h incident at v_i. As for edges, if link j joins v_i and v_k, then $x^i_j x^k_j$ is made an edge. Corresponding to loop h incident with x^i_h the edge $x^i_h\, x^i_{h'}$ is taken. In addition, all the edges of the complete bipartite subgraph with vertex partition $\{X_i,\, Y_i\}$ are taken for each i. These constitute the graph G'. The induced subgraph $< X_i \cup Y_i >$ of G is called the *star graph* at i and is denoted by $St(i)$. For multi graph G, it is a complete bipartite graph, but for general graphs it will contain edges in $< X_i >$ which we call *loop edges*. The other edges of $St(i)$ are called *star edges* and the remaining edges of G' are called as *link edges*.

This construction is illustrated in Figure 8.23. In the general graph G, $d(v_1) = 5$, $d(v_2) = 3$ and $d(v_3) = 2$. Let $f = (3,\, 2,\, 1)$ be the vertex function, so that $f(v_1) = 3$, $f(v_2) = 2$ and $f(v_3) = 1$. Therefore, $k_1 = 2$, $k_2 = 1$ and $k_3 = 1$. Here $|X_1| = 5$, $|X_2| = 3$ and $|X_3| = 2$, and $|Y_1| = 2$, $|Y_2| = 1$ and $|Y_3| = 1$.

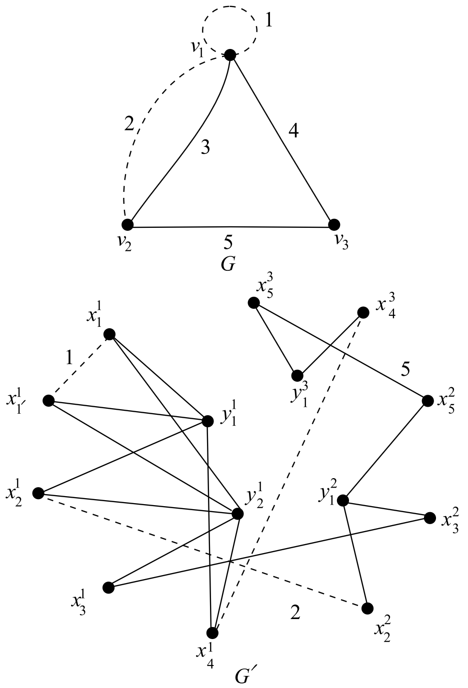


Fig. 8.23

Label the edges of G as 1, 2, 3, 4 and 5 as shown. Then $Y_1 = \{y_1^1, y_2^1\}$, $Y_2 = \{y_1^2\}$ and $Y_3 = \{y_1^3\}$, and we have $X_1 = \{x_1^1, x_{1'}^1, x_2^1, x_3^1, x_4^1\}$, $X_2 = \{x_2^2, x_3^2, x_5^2\}$ and $X_3 = \{x_4^3, x_5^3\}$. The vertex set of G' is $X_i \cup Y_i$, $1 \le i \le 3$. The edges of G' are $x_1^1x_{1'}^1$, $x_2^1x_2^2$, $x_3^1x_3^2$, $x_4^1x_4^3$, $x_5^2x_5^3$ (shown by dotted lines) and the star edges (shown by bold lines). The dotted lines in G form an f-factor with $f = (3, 2, 1)$.

The following result gives a relation between f-factors of G and the 1-factors of G'.

Theorem 8.18 A general graph G has an f-factor if and only if G' has a 1-factor.

Proof First, assume G has an f-factor F with link edges L and loop edges L^0. Let F have g_i link edges and f_i^o loop edges at the vertex v_i. Then $f_i = g_i + 2f_i^0$ $|X_i| = d_i = l_i + 2l_i^0$ and $|Y_i| = k_i = d_i - f_i = (l_i + 2l_i^0) - (g_i + 2f_i^0) = (l_i - g_i) + 2(2l_i^0 - f_i^0)$, where l_i is the number of link edges incident with v_i and l_i^0 is the number of loops incident with v_i. Let L' and $(L^0)'$ be the set of corresponding set of link and loop edges in G'. The matching consisting of these edges, keeps unsaturated all the vertices of Y_i and a set of $l_i - g_i + 2(l_i^o - f_i^o) = k_i$ vertices of X_i, for $1 \le i \le n$. Now match the vertices of Y_i with these unsaturated vertices of X_i using the star edges $St(i)$ and let S' be this set of edges. Clearly $L'U(L^0)' \cup S'$ is a 1-factor in G'.

Conversely, assume G' has a 1-factor H. As the vertices in Y_i are matched only to vertices in X_i, there are exactly $d_i - (d_i - f_i) = f_i = g_i + 2f_i^0$ vertices of X_i which are not matched by H to vertices in Y_i. Clearly, $2f_i^0$ vertices from these unmatched vertices of X_i correspond to loop edges of $< X_i >$ and thus are matched in pairs amongst themselves. The remaining g_i vertices are matched by the link edges of H to X_j vertices of other star graphs $St(i)$ of G'. If F is the set of edges of G corresponding to the loop edges and link edges, then d $(v_i| < F >) = g_i + 2f_i^0$. Thus f is an f-factor of G. ❑

The following definition will be required for the graph G' constructed as above.

Definition: A subset W of the vertex set V' of the graph G' is said to be *simple* if the following conditions get satisfied.

i. $(X_i \cap W) \neq \varphi$ implies $X_i \subseteq W$,

ii. $(Y_i \cap W) \neq \varphi$ implies $X_i \subseteq W$ and

iii. At most one of X_i and Y_i is a subset of W, implies that if $Y_i = \varphi$, then $Y_i \subseteq W$, that is $X_i \cap W = \varphi$.

Now assume W be a simple set of V', S be the set of vertices v_i of G, for each $X_i \subseteq W$ and T be the set of vertices v_j of G for each $Y_j \subseteq W$. We observe that $S \cap T = \varphi$, and $S \cup T \cup (V - S - T)$ is a decomposition D of V.

Denoting by $d(G, S) = k_0(G - S) - |S|$, for any subset S of the vertex set V of a graph, Theorem 8.8 can be restated in the following form.

Theorem 8.19 A simple graph G has a 1-factor if and only if it has no subset $S \subseteq V$ with $d(G, S) > 0$.

Theorem 8.20 If W is a simple subset of V' and D is the corresponding decomposition of V, then $d(G', W) = \delta(D, f)$.

Proof Since W is a simple subset of V' and D is the corresponding decomposition of V,

$$|W| = \sum_{v_j \in S} |X_j| + \sum_{v_j \in T} |Y_j| = \sum_{v_j \in S} d_j + \sum_{v_j \in T} (d_j - f_j) = d(S) + d(T) - f(T). \tag{8.20.1}$$

Thus, $d(G', W) = k_0(G' - W) - d(S) - d(T) + f(T)$.

We proceed to find $k_0(G' - W)$ and in doing so we consider the components K of $G' - W$. We have the following possibilities.

i. The component K contains a single vertex of the form x_r^i. This is possible when $Y_i \subseteq W$ and the edge r is a link edge whose other end is in $Y_j \subseteq W$. Thus, in this case, K is an odd component, and the number of such odd components is equal to the number of such link edges and equals $q[S, T]$.

ii. The component K contains a single vertex of the form y_α^i, and this is possible when $X_i \subseteq W$. This again is an odd component and the number of such odd components is equal to

$$\sum_{i \in S} |Y_i| = \sum_{i \in S} (d_i - f_i) = d(S) - f(S).$$

iii. K contains a single edge of the form $x_\alpha^i\, x_\alpha$ corresponding to a loop of G at α. This is possible when $Y_i \subseteq W$. Clearly, this is an even component of $G' - W$.

iv. K contains a single edge of the form $x_r^i x_r^j$, where r is the link edge of G joining vertices v_i and v_j. Clearly, this happens when $Y_i \subseteq W$ and $Y_j \subseteq W$. This also is an even component.

v. K contains more than one edge and we call such components large components. Clearly, they have one star graph $St(i)$ as a subgraph. If K contains more than one such star graph as a proper subgraph, it will also contain all edges of G' between such star graphs. The other edges of K are of the form $x_r^i x_r^j$, where r is an edge of G joining v_i and v_j, and $St(i) \subseteq K$ and $Y_j \subseteq T$. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ be the vertices of G corresponding to the star graphs $St(i_j)$, $1 \le j \le k$, which are subgraphs of K. Let C be the induced subgraph $< v_{i_1}, v_{i_2}, \ldots, v_{i_k} >$ of G. Then C is a component of $G - S - T$ and therefore the number of vertices of the large component K is given by

$$n(K) = |V(K)| = \sum_{v_i \in C} |V(St(i))| + q[C, T]$$

$$= \sum_{v_i \in C} (d_i + d_i - f_i) + q[C, T]$$

$$= \sum_{v_i \in C} f_i + q[C,\, T] \text{ (mod 2)}$$

$$= f(C) + q[C,\, T] \text{ (mod 2)}.$$

Therefore K is an odd component of $G' - W$ if and only if C is an odd component of $G - S - T$. So the number of large odd components of $G' - W$ is $k'_0\,(D,\, f)$.

Combining the above results, we observe that the number of odd components of $G' - W$ is given by

$$k_0(G' - W) = q[S,\, T] + d(S) - f(S) + k'_0(D,\, f). \tag{8.20.2}$$

Now, $\delta(D,\, f) = k'_0(D,\, f) - f(S) + f(T) - d(T) + q[S,\, T]$

$= k_0(G' - W) - d(S) + f(T) - d(T),$ (by using (8.20.2))

$= d(G',\, W).$ (by using (8.20.1)) ❑

Now, we have the following main result.

Theorem 8.21 A general graph G has an f-factor if and only if it has no decomposition D with $\delta(D,\, f) > 0$.

Proof

Necessity Assume there are decompositions D for which $\delta(D,\, f) > 0$ and choose one such decomposition for which $|S|$ is least. Claim that for each $v_i \in S,\ f(v_i) < d(v_i)$. If not, there is a $v \in S$ such that $f(v) = d(v)$. Consider the decomposition $D' = (S',\, T',\, U')$, where $S' = S - \{v\},\ T' = T \cup \{v\}$ and $U' = U$. Then the components of $< V - S' - T' >$ are the same as those of $< V - S - T >$. Also, for such a component C, $f(C)$ is unchanged, but $q[C,\, T'] = q[C,\, T] + q[C,\, v]$. Therefore at most $q[v,\, U]$ components can change from odd to even. That is, $k'_0(G - S' - T') \geq k'_0(G - S - T) - q[v,\, U]$.

Thus, $\delta(D',\, f) = k'_0(G - S' - T') + f(T') - f(S') - d(T') + q[S',\, T']$

$= k'_0(G - S' - T') + f(T) + f(v) - f(S) + f(v) - d(T) - d(v) +$

$q[S,\, T] - q[v,\, T] + q[v,\, S]$

$= k'_0(G - S' - T') + f(T) - f(S) - d(T) + q[S,\, T] + 2f(v) - d(v)$

$-q[v,\, T] + q[v,\, S]$

$\geq k'_0(G - S - T) - q[v,\, U] + f(T) - f(S) - d(T) + q[S,\, T] + d(v)$

$$-q[v, T] + q[v, S] \qquad (\text{as } d(v) = f(v))$$

$$= \delta(D, f) + d(v) - q[v, U] - q[v, T] + q[v, S]$$

$$= \delta(D, f) + 2q[v, S] > \delta(D, f) > 0.$$

This contradicts the choice of D and thus the claim is established.

Now, suppose $W = \{X_i : v_i \in S\} \cup \{Y_j : v_j \in T\}$.

Then W is simple, as S and T are disjoint. Therefore by Theorem 8.20, $d\,(G', W) = \delta(D, f) > 0$ and G' has no 1-factor. Thus by Theorem 8.18, G has no f-factor.

Sufficiency Assume that G has no f-factor, so that G' has no 1-factor. Therefore, by Theorem 8.19, there is a subset W of V' with $d(G', W) > 0$. We choose such a W with least cardinality and prove that W is simple.

Claim 1 $Y_j \cap W \neq \varphi$ and $Y_j \subseteq W$.

If this is not true, then there is a Y_j such that $Y_j \cap W \neq \varphi$, and $Y_j \cap (V' - W) \neq \varphi$. Define $Q = W - (Y_j \cap W)$ (Fig. 8.24).

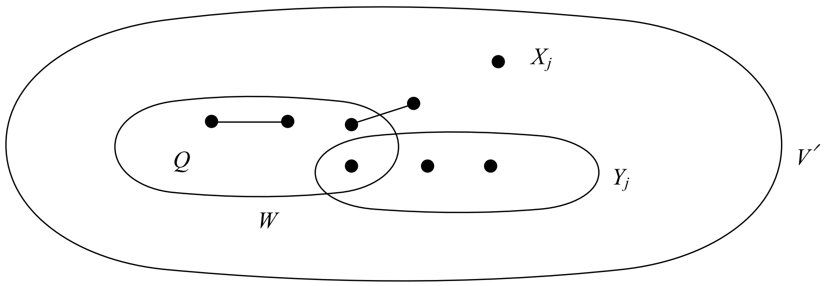


Fig. 8.24

If $X_j \not\subseteq W$, then $G' - W$ and $G' - Q$ differ in one component only, the component star edges.

If $X_j \subseteq W$, then each component of $G' - W$ is also a component of $G' - Q$.

In either case, $k_0(G' - Q) \geq k_0(G' - W) - 1$ and $|Q| \leq |W| - 1$. Therefore, $d(G', Q) > 0$, contradicting the choice of W.

Claim 2 $X_i \cap W \neq \varphi$ implies $Y_i \subseteq W$.

If this is not true, then there is an i such that $X_i \cap W \neq \varphi$ and $Y_i \subseteq W$. Let $a \in X_i \cap W$ and define $Q = W - \{a\}$. Then $G' - W$ can have at most one component with a vertex belonging to Y_i which is adjacent to a in G. Thus,

$$k_0(G' - W) \leq k_0(G' - Q) + 1 \text{ and } |Q| = |W| - 1.$$

Therefore, $d(G', Q) \geq d(G', W)$, contradicting the choice of W.

Now claim (1) and claim (2) together imply that both X_i and Y_i are not subsets of W.

Claim 3 $X_i \cap W \neq \varphi$ implies $X_i \subseteq W$.

If not, there is an i such that $X_i \cap W \neq \varphi$ and $X_i \cap (V' - W) \neq \varphi$. Let a $\in X_i \cap W$ and define $Q = W - \{a\}$. By claims (2) and (1), $Y_i \subseteq G' - W$, so that all vertices of Y_i belong to a single component of $G' - W$. Also there is at most one component of $G' - Q$ which has a vertex not belonging to Y_i which is adjacent to an a in G' (Fig. 8.25). Therefore $G' - W$ and $G' - Q$ differ in at most two components.

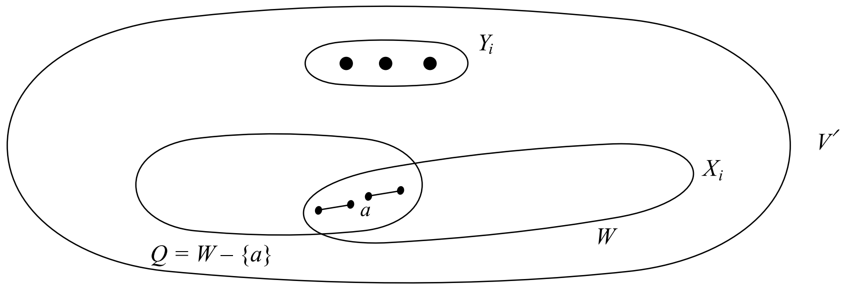


Fig. 8.25

Thus, $k_0(G' - Q) \geq k_0(G' - W) - 2$

and $d(G', Q) \geq d(G', W) - 1$. (8.21.1)

But $k_0(G' - W) + |W| \equiv |V'|$ and $k_0(G' - Q) + |Q| \equiv |V'|$.

So, $k_0(G' - W) - |W| \equiv |V'| \equiv k_0(G' - Q) - |Q|$ and $d(G', W) \equiv d(G', Q)$.

Therefore, (8.21.1) becomes $d(G', Q) \geq d(G', W) > 0$, contradicting the choice of W.

Hence we have proved that W is a simple set and by Theorem 8.20, for the corresponding decomposition D of G, $\delta(D, f) > 0$. ❑

Remark Theorem 8.21 shows that the 1-factor Theorem implies the f-factor Theorem. The converse is also true as can be seen in Theorem 8.22. Before that we have the following definition.

Definition: Suppose $D = (S, T, U)$ is a decomposition of V and $x \in S$, $y \in T$. The decomposition $D_1 = (S - \{x\}, T, U \cup \{x\})$ is said to be obtained by an x-transfer from S and the decomposition $D_2 = (S, T - \{y\}, U \cup \{y\}$ is said to be obtained by y-transfer from T.

The following result is an immediate consequence of the above definition and can be easily established.

Lemma 8.6 An x-transfer from S does not reduce the deficiency of D if $f(x) = d(x)$ or $d(x) - 1$ and a y-transfer does not reduce the deficiency of D if $f(y) = 0$ or 1.

Theorem 8.22 The f-factor Theorem implies the 1-factor Theorem.

Proof Clearly, by the f-factor Theorem with $f(v) = 1$, for all $v \in V$, G has a 1-factor if and only if it has a 1-barrier $D = (S, T, U)$. But by Lemma 8.6, $D_1 = (S, \varphi, V - S)$ has no less deficiency than D (by y-transfer from T). Therefore G has no 1-factor if and only if there is a 1-barrier of the form D_1. But $\delta(D_1, f) = k_0(G-S) - f(S) - d(T) + f(T) + q[S, T] = k_0(G-S) - |S|$ and D_1 is a 1-barrier if and only if $\delta(D_1, f) > 0$, that is $k_0(G-S) > |S|$. Hence the 1-factor Theorem. ❑

Corollary 8.12 The Erdos-Gallai Theorem 2.4 on degree sequences follows from the f-factor Theorem.

Proof Theorem 2.4 gives a necessary and sufficient condition for a non-increasing sequence $d = [d_i]_1^n$ of non-negative integers to be the degree sequence of a simple graph. In order to get this condition, we observe that d is graphic if and only if the complete graph K_n has a d-factor, where $d(v_i) = d_i$, given $\sum\limits_{i=1}^{n} d_i = 2m$.

By the f-factor Theorem, d fails to be graphic if and only if K_n has a d-barrier $D = (S, T, U)$. Since for K_n, U can have at most one component, $k_0(D, d) = 0$ or 1, further $\delta(D, d) > 0$ is even, as $d(K_n) = \sum d_i = 2m$ is even. Therefore the condition $\delta(D, d) > 0$ becomes

$$q_{k_n} = [S, T] - d(S) - (n-1)|T| + d(T) > 0. \tag{8.12.1}$$

Now transfer of elements t from T to U, or elements s from S to U does not decrease the deficiency $\delta(D, d)$. So elements $t \in T$ with $d(t) < |T| + |U|$ and elements $s \in S$ with $d(s) > |T|$ in S can be transferred to U without decreasing the deficiency. Also it can be seen that elements $u \in U$ with $d(u) \geq |T| + |U|$ can be transferred to S without decreasing the deficiency. Therefore the f-barrier can be made such that $d(t) > |T| + |U|$ for all $t \in T$, $|T| < d(u) < |T| + |U|$ for all $u \in U$ and $d(s) \leq |T|$ for all $s \in S$, and d is non-graphical if and only if K_n has such a d-barrier D. But for such a D, $q[S, T] = r(n - r - |U|)$, where $|T| = r$, $d(T) = \sum\limits_{1}^{r} d_i$, and thus condition (8.12.1) becomes

$$r(n - r - |U|) - d(S) - r(n-1) + d(T) > 0.$$

That is, $\sum\limits_{1}^{r} d_i > r(n-1) + d(S) - r(n - r - |U|)$

$$= r(r-1) + r(n-r) + d(S) - r(n - r - |U|)$$

$$= r(r-1) + r|U| + d(S)$$

$$> r(r-1) + \sum_{i=r+1}^{n} \min\{r,\, d_i\},$$

which is the Erdos-Gallai condition. ❑

8.5 Degree Factors

Let $d = [d_i]_1^n$ and $r = [d_i - k_i]_1^n$ be graphical sequences. The problem we consider here is the existence of a graph G realising d with a spanning subgraph (factor) realising $k = [k_i]_1^n$. This problem was asked by Rao and Rao [215] as k-factor conjecture, when $k \le k_i \le k+1$ for each i, and solved by Kundu [143]. An improvement of this was due to Kleitman and Wang [129] and further improvement by Kundu [141]. For $k_i = 1$, $1 \le i \le n$, the problem was earlier posed by Grunbaum [93] and proved by Lovasz [149], and this proof also applies when $k_i = k$, $1 \le i \le n$.

Theorem 8.23 (k-factor Theorem) If $[d_i]_1^n$ is a graphical sequence such that $[d_i - k]_1^n$ is also graphical, then there is a realisation of $[d_i]_1^n$ with a k-factor.

Proof Let G_1 and G_2 be two graphs on $V = \{v_1,\, v_2,\, \ldots,\, v_n\}$ such that $d(v_i|G_1) = d_i$ and $d(v_i|G_2) = d_i - k$. Assume that G_1 and G_2 are chosen such that G_2 has maximum edges common with G_1. We show that all edges of G_2 are in G_1. Let $E_1 = E(G_1)$ and $E_2 = E(G_2)$.

Claim

If $v_iv_j \in E_2 - E_1$ and $v_iv_k,\, v_jv_r \in E_1 - E_2$ ($i,\ j,\ k,\ r$ being distinct), then $v_kv_r \in E_1 - E_2$.

If $v_kv_r \in E_2$, then an EDT in G_2 switching the pair $v_iv_j,\, v_kv_r$ to the positions $v_iv_k,\, v_jv_r$ gives a graph degree-equivalent to G_2 with more edges common with G_1, contradicting the choice of G_1 and G_2. If $v_kv_r \notin E_1$, then an EDT in G_1 switching the pair $v_iv_k,\, v_jv_r$ to the positions $v_iv_j,\, v_kv_r$ gives a graph degree-equivalent to G_1, such that G_2 has more common edges with this graph, again a contradiction. Thus, the claim is established.

Let G_2 has some edges which are not in G_1 and let v_1 be a vertex with a maximum number t of edges of $E_2 - E_1$ incident with it. By the degree stipulation on G_1 and G_2, there are $t+k$ vertices $v_2,\, v_3,\, \ldots,\, v_{t+k+1}$ which are adjacent to v_1 in G_1, but not in G_2. Assume $v_1v_k \in E_2 - E_1$ and $v_kv \in E_1 - E_2$, and consider the following two cases.

Case 1 Let $v \ne v_i$ for i, $2 \le i \le t+k+1$. Then by the above claim, $vv_i \in E_1 - E_2$ for $2 \le i \le t+k+1$, so that there are $t+k+1$ edges in $E_1 - E_2$ adjacent with v. By the degree-stipulation on G_1 and G_2, there are $t+1$ edges in $E_2 - E_1$ incident with v, contradicting the choice of v.

Case 2 Let $v = v_i$ for some i, $2 \le i \le t+k+1$, and without loss of generality, take $v = v_2$. Then again, by the above claim, $vv_i \in E_1 - E_2$ for $3 \le i \le t+k+1$. Also, $vv_1,\, vv_k \in E_1 - E_2$. Therefore there are $t+k+1$ edges in $E_1 - E_2$ incident with v, again a contradiction.

Hence G_2 has no edges which are not in G_1. But then $< E(G_1) - E(G_2) >$ is a k-factor of G_1. ❑

Lemma 8.7 If $[k_i]_1^n$ is a sequence of non-negative integers such that $k \leq k_i \leq k+1$ for $1 \leq i \leq n$ and for some k, $0 \leq k \leq n-2$, and $\sum\limits_1^n k_i$ is even, then $[k_i]_1^n$ is a graphical sequence.

Proof Induct on $S = \sum\limits_1^n k_i$. Clearly, the result is trivial for $S = 0$. Assume it is true for all values of $\sum\limits_1^n k_i < S-1$ and let $[k_i]_1^n$ be a sequence with $\sum\limits_1^n k_i = S$ (even). Then by Theorem 2.1, $[k_i]_1^n$ is graphical if and only if $[k_2-1,\ k_3-1, \ldots,\ k_{k_1+1}-1,\ k_{k_1+2}, \ldots,\ k_n]$ is graphical, where $k_1 \geq k_2 \geq \ldots \geq k_n$. But the sum of the terms in this sequence is $S - 2k_1$, which is even, and by induction hypothesis is graphical. Hence by Theorem 2.1, $[k_i]_1^n$ is graphical. ❑

The proof given here of the following k-factor Theorem of Kundu, is due to Chen [57].

Theorem 8.24 If $[d_i]_1^n$ and $[d_i - k_i]_1^n$ are two graphical sequences satisfying $k \leq k_i \leq k+1$, for $1 \leq i \leq n$ and some $k > 0$, then there is a graph realising $[d_i]_1^n$ and having a k-factor $k = [k_i]_1^n$.

Proof Since $[d_i]_1^n$ and $[d_i - k_i]_1^n$ are both graphical, $[k_i]_1^n$ is also graphical by Lemma 8.7.

Clearly, there exists a graph with degree sequence $[d_i]_1^n$ containing a k-factor $k = [k_i]_1^n$ if and only if there exists a graph with degree sequence $[n-1-k_i]_1^n$ containing a k'-factor $k' = [n-1-d_i]_1^n$. This can be easily established by taking the complementary graphs. Thus, the conditions can be shifted from k_i's to d_i's, so that it is enough to prove that if $[d_i]_1^n$ and $[d_i - k_i]_1^n$ are graphical with $k \leq d_i \leq k+1$ for $1 \leq i \leq n$ and the k_i's unrestricted, then there is a graph with degree sequence $[d_i]_1^n$ and a k-factor $k = [k_i]_1^n$.

Let G_1 and G_2 be graphs with degree sequences $[k_i]_1^n$ and $[d_i - k_i]_1^n$ respectively, such that the multigraph G, obtained by superimposing G_1 and G_2 has a minimum number of multiple edges. Obviously, in the superimposed graph, there are at most two edges between any pair of vertices, one of G_1 and one of G_2. If there are no such multiple edges, then G_1 is a factor of G.

Assume there are multiple edges in G and let uv be one such multiple edge. Since $d(u|G) = d(u|G_1) + d(u|G_2) \leq n-1$ and uv is a multiple edge, there is a vertex w not adjacent to u in G. Let V be the common vertex set of G_1, G_2 and G. If for every $x \in V - u$, $q[v,\ x] \geq q[w,\ x]$, then $d(v|G) - 2 \geq d(w|G)$, contradicting the choice that each d_i is k or $k+1$. Therefore there is a vertex $x \in V - u$ such that $q[v,\ x] < q[w,\ x]$ (Fig. 8.26).

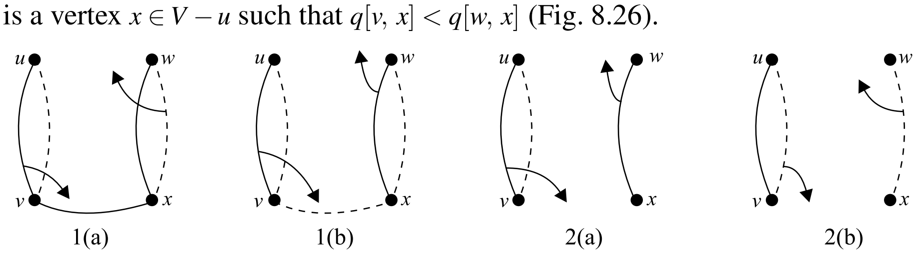


Fig. 8.26

The following two cases arise:

1. $q[v, x] = 1$ and $q[w, x] = 2$,
2. $q[v, x] = 0$ and $q[w, x] = 0$.

In case 1, we have either $1(a)$ $vx \in G_1$, or $1(b)$ $vx \in G_2$, and in case 2, either 2(a) $wx \in G_1$, or $2(b)$ $wx \in G_2$. For all these cases, make an EDT switching the pair of edges vu, wx to the positions vx, uw. In 1(a) and 2(b), this is done in G_2 to obtain a graph G_2' degree-equivalent to G_2 and with less number of multiple edges in $G_1 \cup G_2'$. In 1(b) and 2(a), this is done in G_1 to get a graph G_1' degree-equivalent to G_1 and with less multiple edges in $G_1' \cup G_2$. This contradicts the choice of G_1 and G_2. ❑

Remark If d_i's and k_i's being allowed to be arbitrary with $[d_i]_1^n$, $[d_i - k_i]_1^n$ are graphical, and so $[k_i]_1^n$ is graphical, then $[d_i]_1^n$ need not be realised with a k-factor. To see this, consider the following example of Lovasz [149].

Let $d = [8, 8, 3, 2, 2, 2, 2, 2, 3]$ and $k = [4, 4, 0, 0, 0, 1, 2, 2, 3]$ so that $d - k = [4, 4, 3, 2, 2, 1, 0, 0, 0]$. Then d and $d - k$ are graphical, realised by graphs G_1 and G_2 of Figure 8.27, and evidently k is same as $d - k$ with a different labeling. Now, it is easy to see that any realisation of $k = d - k$ needs an edge joining the two vertices of degree 4 in it. But if this uv is an edge in G_2, it will not be an edge in $G_1 - G_2$, so that there is no realisation of d with a k-factor.

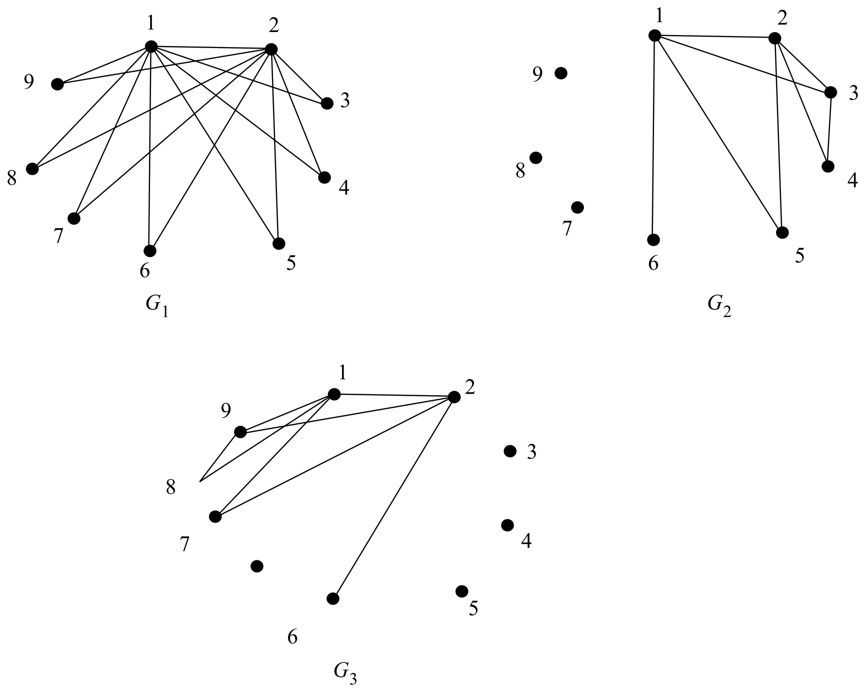


Fig. 8.27

8.6 (g, f) and [a, b]-factors

In this section, we give a brief description of $(g,\ f)$ and $[a,\ b]$ factors in a graph.

Definition: If f and g are two vertex functions of a graph G such that $g(v) \leq f(v)$ for all $v \in V$ and F is a factor of G such that $g(v) \leq d(v|F) \leq f(v)$ for all $v \in V$, then F is called a *(g, f)-factor* of G.

If H is a $(g,\ f)$-factor of G, its *f-deficiency* is defined as $\Delta_H(G,\ f) = f(V) - d(H)$, where $d(H) = \sum\limits_{v \in V} d(v|H)$. Clearly, $\Delta_H(G, f) \equiv f(V)(\mathrm{mod}2)$.

If $D = (S,\ T,\ U)$ is a decomposition of G, then $k''(D,\ f,\ g)$ denotes the number of odd components of U with respect to D and f for which $f(c) = g(c)$, for every vertex c of the component.

We now state the $(g,\ f)$-factor Theorem of Lovasz [148].

Theorem 8.25 If f and g are vertex functions of a graph $G\ (V,\ E)$ such that $0 \leq g(v) \leq f(v) \leq d(v|G)$ for all $v \in V$, then either G has a $(g,\ f)$-factor or it has a decomposition $D = (S,\ T,\ U)$ satisfying $k''\ (D,\ f,\ g) > f(S) + d(T) - g(T) - q[S,\ T]$, but not both, where $k''(D,\ f,\ g)$ is the number of odd components of $< U >_G$ with $f(v) = g(v)$ for every vertex v of the component.

Next, we state the $(g,\ f)$-factor Theorem due to Las Vergnas [255].

Theorem 8.26 Let $g,\ f$ be vertex functions for a graph $G(V,\ E)$ such that $0 \leq g(v) \leq 1 \leq f(v) \leq d(v|G)$ for all $v \in V$. For $S \subseteq V$, let $k'(G-S)$ be the number of odd components C of $G-S$ which are such that either $C = \{v\}$ and $g(v) = 1$ or $< V(C) >$ is odd and at least 3, with $g(v) = f(v) = 1$ for all $v \in V(C)$. Then G has a $(g,\ f)$-factor if and only if $f(S) \geq k'(G-S)$ for all $S \subseteq V$.

The following variation of the (g, f)-factor Theorem is due to Kano and Saito [124].

Theorem 8.27 If g and f are vertex functions of a graph $G(V,\ E)$ and θ a real number such that $0 \leq \theta \leq 1$, and if $g(v) < f(v)$, $g(v) \leq d(v|G) \leq f(v)$ for all $v \in V$, then G has a $(g,\ f)$-factor.

Definition: If a and b are non-negative integers with $a \leq b$ and F is a factor of G such that $a \leq d(v|F) \leq b$ for all $v \in V$, then F is called an *[a, b]-factor* of G. We note that an $[a,\ b]$-factor is a $(g,\ f)$-factor for constant vertex functions $g(v) = a$ for all $v \in V$ and $f(v) = b$ for all $v \in V$.

The following $[a,\ b]$-factor Theorem can easily be derived from the $(g,\ f)$-factor Theorem.

Theorem 8.28 If $G(V,\ E)$ is a graph, and a and b are distinct integers with $a < b$, then either G has an $[a,\ b]$-factor H or a decomposition $D = (S,\ T,\ U)$ such that $q[S,\ T] > d(T) + b|S| - a|T|$, but not both.

The following result is due to Las Vergnas [255].

Theorem 8.29 If G is a graph and $k(\geq 2)$ an integer, then G has a $[1, k]$-factor if and only if $k|N(S)| \geq |S|$, for all independent sets S of G.

The next result can be found in Kano and Saito [125].

Theorem 8.30 If k, r, s and t are integers such that $0 \leq k \leq r$, $0 \leq s$, $1 \leq t$ and $ks \leq rt$, then every $[r,\ r+s]$-graph has a $[k,\ k+1]$-factor.

The following result on the existence of $[a,\ b]$-factors is given by Kouider and Long [139].

Theorem 8.31 Let $b \geq a+1$ and let G be a graph with minimum degree 8.

$$\text{If} \qquad a(G) \leq \begin{cases} \dfrac{4b(\delta - a+1)}{(a+1)^2}, & \text{for } a \text{ odd} \\ \dfrac{4b(\delta - a+1)}{a(a+2)}, & \text{for } a \text{ odd} \end{cases}$$

then G has an $[a,\ b]$-factor.

8.7 Exercises

1. Prove that a tree can have at most one perfect matching.
2. Give an example of a cubic graph having no 1-factor.
3. Show that $K_{n,\,n}$ and K_{2n} are 1-factorable, and further show that the number of 1-factors of $K_{n,\,n}$ and K_{2n} are respectively $n!$ and $\dfrac{(2n)!}{2^n n!}$.
4. Prove that the Peterson graph is not 1-factorable.
5. Let $G = (V_1,\ V_2,\ E)$ be a bipartite graph with $|V_1| = n_1$ and $|V_2| = n_2$, and the vertices $x_i \in V_1$ and $y_j \in V_2$ indexed such that $d(x_1) \leq d(x_2) \leq \ldots \leq d(x_n)$ and $d(y_1) \leq d(y_2) \leq \ldots \leq d(y_n)$. Then show that a sufficient condition for V_1 to be matched into V_2 is that $n_1 \leq n_2$, $d(x_1) > 0$ and $\sum\limits_{i=1}^{k} d(x_i) > \sum\limits_{i=1}^{k-1} d(y_j)$, $2 \leq k \leq n_1$.
6. Prove that a bipartite graph $G = (V_1,\ V_2,\ E)$ has a matching saturating all the vertices of maximum degree.
7. Show that the following statements are equivalent for a bipartite graph $G = (V_1,\ V_2,\ E)$.
   a. G is connected and each edge of G is contained in a 1-factor.
   b. $|V_1| = |V_2|$ and for each non-empty subset S of V_1, $|S| < |N(S)|$.

c. For each $u \in V_1$ and $v \in V_2$, $G - u - v$ has a 1-factor.

8. Show that a tree T has a perfect matching if and only if $k_0(T - v) = 1$ for every vertex v of T.

9. If G is a cubic graph without cut edges or with all cut edges on a single suspended path, then prove that G has a 1-factor.

10. Prove directly that if G is a connected graph with no induced subgraph isomorphic to $K_{1,\,3}$, then G has an edge uv such that $G - \{u,\, v\}$ is connected.

11. If M and N are matchings in a graph G and $|M| > |N|$, then prove that there exist matchings M' and N' in G such that $|M'| = |M| - 1$, $|N'| = |N| + 1$ and M' and N' have the same union and intersection (as edge sets) as M and N.

12. Prove that deleting any perfect matching from the Peterson graph leaves the subgraph $C_5 + C_5$.

13. Determine the minimum size of a maximal matching in the cycle C_n.

14. If $G = (V_1,\, V_2,\, E)$ is a bipartite graph that has a matching of V_1 into V_2, then prove that G has at most $\binom{|V_1|}{2}$ edges belonging to no matching of V_1 into V_2.

15. Describe complete tripartite graphs of the form $K_{n,\,n,\,n}$ that have perfect matching.

16. For what values of $n \geq 4$ does the wheel W_n have a perfect matching?

17. Prove that a graph has an f-factor if and only if it has an f'-factor, where $f'(v) = d(v) - f(v)$ for every $v \in V$.

18. Show that a k-regular, $(k-1)$-edge connected graph of even order has a 1-factor.

19. If G is a connected graph of even order, then show that G^2 has a 1-factor.

20. If G is a connected cubic graph without a 1-factor and S is a minimal antifactor set, then prove that S is an independent set of claw centers and $G - S$ has exactly $|S| + 2$ odd components and no even components.