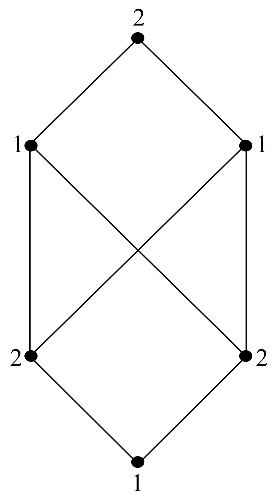
7. Colourings

Colouring is one of the important branches of graph theory and has attracted the attention of almost all graph theorists, mainly because of the four colour theorem, the details of which can be seen in Chapter 12.

7.1 Vertex colouring

A vertex colouring (or simply colouring) of a graph G is a labelling $f : V(G) \to \{1, 2, \ldots\}$; the labels called *colours*, such that no two adjacent vertices get the same colour and each vertex gets one colour. A k-colouring of a graph G consists of k different colours and G is then called *k-colourable*. A 2-colourable and a 3-colourable graph are shown in Figure 7.1. It follows from this definition that the k-colouring of a graph $G(V, E)$ partitions the vertex set V into k independent sets $V_1, V_2, \ldots, V_k$ such that $V = V_1 \cup V_2 \cup \ldots \cup V_k$. The independent sets $V_1, V_2, \ldots, V_k$ are called the *colour classes* and the function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(v) = i$ for $v \in V_i$, $1 \le i \le k$, is called the *colour function*.



2-colourable graph

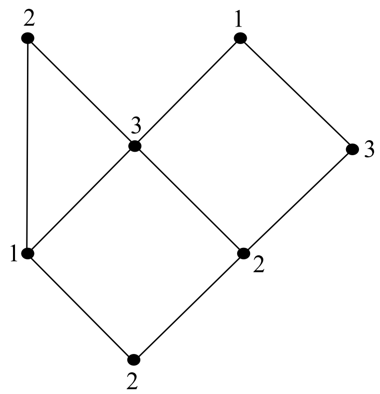

3-colourable graph

Fig. 7.1

The minimum number k for which there is a k-colouring of the graph is called the *chromatic number* (chromatic index) of G and is denoted by $\chi(G)$. If $\chi(G) = k$, the graph G is said to be *k-chromatic*.

We observe that colouring any one of the components in a disconnected graph does not affect the colouring of its other components. Also, parallel edges can be replaced by single edges, since it does not affect the adjacencies of the vertices. Thus, for colouring considerations, we opt only for simple connected graphs.

The following observations are the immediate consequences of the definitions introduced above.

1. A graph is 1-chromatic if and only if it is totally disconnected.
2. A graph having at least one edge is at least 2-chromatic (bichromatic).
3. A graph G having n vertices has $\chi(G) \leq n$.
4. If H is subgraph of a graph G, then $\chi(H) \leq \chi(G)$.
5. A complete graph with n vertices is n-chromatic, because all its vertices are adjacent. So, $\chi(K_n) = n$ and $\chi(\overline{K}_n) = 1$. Therefore we see that a graph containing a complete graph of r vertices is at least r-chromatic. For example, every graph containing a triangle is at least 3-chromatic.
6. A cycle of length $n \geq 3$ is 2-chromatic if n is even and 3-chromatic if n is odd. To see this, let the vertices of the cycle be labelled $1,\ 2,\ \ldots,\ n$, and assign one colour to odd vertices and another to even. If n is even, no adjacent vertices get the same colour, if n is odd, the nth vertex and the first vertex are adjacent and have the same colour, therefore need the third colour for colouring.
7. If $G_1, G_2, \ldots, G_r$ are the components of a disconnected graph G, then

   $$\chi(G) = \max_{1 \leq i \leq r} \chi(G_i).$$

We note that trees with greater or equal to two vertices are bichromatic as is seen in the following result.

Theorem 7.1 Every tree with $n \geq 2$ vertices is 2-chromatic.

Proof Let T be a tree with $n \geq 2$ vertices. Consider any vertex v of T and assume T to be rooted at vertex v (Fig. 7.2). Assign colour 1 to v. Then assign colour 2 to all vertices which are adjacent to v. Let $v_1, v_2, \ldots, v_r$ be the vertices which have been assigned colour 2. Now assign colour 1 to all the vertices which are adjacent to $v_1, v_2, \ldots, v_r$. Continue this process till every vertex in T has been assigned the colour. We observe that in T all vertices at odd distances from v have colour 2, and v and vertices at even distances from v have colour 1. Therefore along any path in T, the vertices are of alternating colours. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same colour. Thus T is coloured with two colours. Hence T is 2-chromatic. ❑

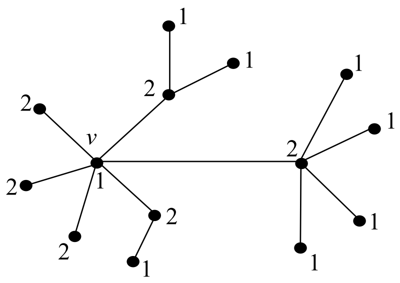


Fig. 7.2

The converse of the above theorem is not true, i. e., every 2-chromatic graph need not be a tree. To see this, consider the graph shown in Figure 7.3. Clearly, G is 2-chromatic, but is not a tree.

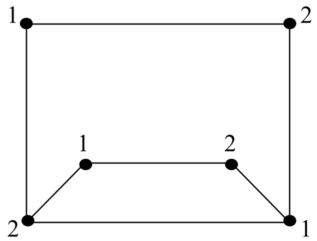


Fig. 7.3

The next result due to Konig [134] characterises 2-chromatic graphs.

Theorem 7.2 (Konig) A graph is bicolourable (2-chromatic) if and only if it has no odd cycles.

Proof Let G be a connected graph with cycles of only even length and let T be a spanning tree in G. Then, by Theorem 7.1, T can be coloured with two colours. Now add the chords to T one by one. As G contains cycles of even length only, the end vertices of every chord get different colours of T. Thus G is coloured with two colours and hence is 2-chromatic. Conversely, let G be bicolourable, that is, 2-chromatic. We prove G has even cycles only. Assume to the contrary that G has an odd cycle. Then by observation (6), G is 3-chromatic, a contradiction. Hence G has no odd cycles. ❑

Corollary 7.1 For a graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle.

The following result is yet another characterisation of 2-chromatic graphs.

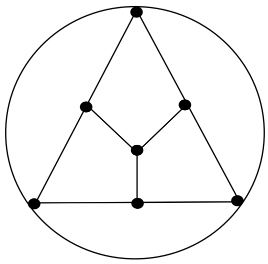
Theorem 7.3 A nonempty graph G is bicolourable if and only if G is bipartite.

Proof Let G be a bipartite graph. Then its vertex set V can be partitioned into two nonempty disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$. Now assigning colour 1 to all vertices in V_1 and colour 2 to all vertices in V_2 gives a 2-colouring of G. Since G is nonempty, $\chi(G) = 2$.

Conversely, let G be bicolourable, that is, G has a 2-colouring. Denote by V_1 the set of all those vertices coloured 1 and by V_2 the set of all those vertices coloured 2. Then no two vertices in V_1 are adjacent and no two vertices in V_2 are adjacent. Thus any edge in G joins a vertex in V_1 and a vertex in V_2. Hence G is bipartite with bipartition $V = V_1 \cup V_2$. ❑

7.2 Critical Graphs

If G is a k-chromatic graph and $\chi(G-v) = k-1$ for every vertex v in G, then G is called a k-critical graph. A 4-critical graph is shown in Figure 7.4. If G is k-chromatic, but $\chi(G-e) = k-1$ for each edge e of G, then G is called *k-edge-critical graph*, or k-minimal. A graph G is said to be *contraction critical* or con-critical if $\chi(H) < \chi(G)$ for every proper contraction H of G. A graph G is said to be *critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G.



A 4 – Critical Graph

Fig. 7.4

We have the following observations.

1. Every critical or minimal graph is connected.
2. Every connected k-chromatic graph contains a critical or minimal k-chromatic graph.
3. $\chi(G) = \max\{\chi(B) : B \text{ is a block of } G\}$.
4. The only 1-critical or 1-minimal graph is K_1, the only 2-critical or 2-minimal graph is K_2 and the only 3-critical or 3-minimal graphs are $C_{2n+1}, n \geq 1$, that is, odd cycles.

The following result due to Dirac [66] describes some of the important properties of a k-critical graph.

Theorem 7.4 If G is a k-critical graph, then

a. G is connected,

b. $\delta(G) \geq k-1$,

c. G has no pair of subgraphs G_1 and G_2 for which $G = G_1 \cup G_2$, and $G_1 \cap G_2$ is a complete graph,

d. $G-v$ is connected for every vertex v of G, provided $k > 1$.

Proof

a. Assume G is not connected. Since $\chi(G) = k$, by observation 7, there is a component G_1 of G such that $\chi(G_1) = k$. If v is any vertex of G which is not in G_1, then G_1 is a component of the subgraph $G-v$. Therefore, $\chi(G-v) = \chi(G_1) = k$. This contradicts the fact that G is k-critical. Hence G is connected.

b. Let v be a vertex of G so that $d(v) < k-1$. Since G is k-critical, the subgraph $G-v$ has a $(k-1)$-colouring. As v has at most $k-2$ neighbours, these neighbours use at most $k-2$ colours in this $(k-1)$-colouring of $G-v$. Now, colour v with the unused colour and this gives a $(k-1)$-colouring of G. This contradicts the given assumption that $\chi(G) = k$. Hence every vertex v has degree at least $k-1$.

c. Let $G = G_1 \cup G_2$, where G_1 and G_2 are subgraphs with $G_1 \cap G_2 = K_t$. Since G is k-critical, therefore G_1 and G_2 both have chromatic number at most $k-1$. Consider a $(k-1)$-colouring of G_1 and a $(k-1)$-colouring of G_2. As $G_1 \cap G_2$ is complete, in the overlap, every vertex in $G_1 \cap G_2$ has a different colour (in each of the $(k-1)$-colourings). This implies that colours in the $(k-1)$-colouring of G_2 can be rearranged such that it assigns the same colour to each vertex in $G_1 \cap G_2$, as is given by the colouring of G_1. Combining the two colourings then produces a $(k-1)$-colouring of all of G. This is impossible, since $\chi(G) = k$. Thus no subgraphs of the type G_1 and G_2 exist.

d. Assume $G-v$ is disconnected, for some vertex v of G. Then $G-v$ has a subgraph H_1 and H_2 with $H_1 \cup H_2 = G-v$ and $H_1 \cap H_2 = \Phi$. Let G_1 and G_2 be the subgraphs of G, where G_1 is induced by H_1 and v, while G_2 is induced by H_2 and v. Then $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_1$ (with K_1 as a single vertex). This contradicts (c) and thus $G-v$ is connected. ❑

Let $S = \{u, v\}$ be a 2-vertex cut of a critical k-chromatic graph G. Since no separating set of a critical graph is a complete graph, therefore uv is not an edge of G. Let G_i be the S-component of G. G_i is said to be of type 1 if every $(k-1)$ colouring of G_i assigns the same colour to u and v, G_i is of type 2 if every $(k-1)$-colouring of G_i assigns different colours to u and v, and G_i is of type 3 if some $(k-1)$-colouring of G_i assigns same colour to u and v, while some other $(k-1)$-colouring assigns different colours to u and v.

The following characterisation of k-critical graphs with a 2-vertex cut is due to Dirac.

Theorem 7.5 If G is a minimal k-chromatic graph with a 2-vertex cut $S = \{u, v\}$, then (i) $G = G_1 \cup G_2$, where G_i is the S-component of type i, $i = 1, 2$ and (ii) both $G + uv$ and $G : uv$ are k-minimal.

Proof Let G be a minimal k-chromatic graph with a 2-vertex cut $S = \{u, v\}$. So the S-components of G are $(k-1)$-colourable. If there is no set of $(k-1)$-colourings of the S-components all of which agree on S, then G is $(k-1)$-colourable. Therefore there is a type 1 S-component G_1 and a type 2 S-component G_2. Then $G_1 \cup G_2$ is not $(k-1)$-colourable. Since G is k-critical, there is no third S-component G_3. Hence, $G = G_1 \cup G_2$.

Let $H_1 = G_1 + uv$ and $H_2 = G_2 : uv$. We prove that H_2 is k-minimal. Since G_2 is of type 2, therefore every $(k-1)$-colouring of G_2 assigns different colours to u and v. As u and v are identified to a single vertex, say w in H_2, so a k-colouring is necessary to colour H_2, that is, H_2 is k-chromatic. We further prove that $\chi(H_2 - e) = k - 1$ for any edge of H_2. Any such edge e can be considered to belong to G and in the $(k-1)$-colouring of $G-e$, u and v get the same colour, since they can be considered to belong to G_1 which is a subgraph of $G-e$. The restriction of such a colouring of $G-e$ to $H_2 - e$ (with u and v identified as w with the common colour of u and v) is a $(k-1)$-colouring of $H_2 - e$. This proves the result.

That H_1 is minimal, can be proved in a similar manner. ❑

Theorem 7.6 Every k-chromatic graph can be contracted into a con-critical chromatic graph.

Proof Let G be a k-chromatic graph and let the edge e of G be contracted. Then a colouring of G can be used to give a colouring of $G|e$ except that, possibly the vertex formed by the contraction may be assigned an extra colour. Thus, $\chi(G|e) \le \chi(G) + 1$. On the other hand, a colouring of $G|e$ can be used to get a colouring for G by using an extra colour for one of the end vertices of e. Therefore, $\chi(G) \le \chi(G|e) + 1$. Thus the contraction of an edge changes the chromatic number by at most one. Sometimes contraction of an edge may increase the chromatic number, but by repeated contractions, the number of edges and therefore the chromatic number gets reduced. Clearly, the connected graph can be contracted to a single vertex whose chromatic number is one. In between, a stage arises where the chromatic number of the graph is the same as the original, but the contraction of any edge reduces the chromatic number by one. ❑

As noted earlier, every connected $k-$ chromatic graph contains a critical or minimal k-chromatic graph. To see this, we observe that if G is not k-critical, then $\chi(G - v) = k$, for some vertex v of G. If $G - v$ is $k-$critical, then this is the required subgraph. If not, then $G - \{v, w\} = (G - v) - w$ has chromatic number k, for some vertex w in $G - v$. If this new subgraph is k-critical, then again this is the required subgraph. If not, we continue this vertex deletion procedure, and we will clearly get a k-critical subgraph.

We have the following immediate observation.

Theorem 7.7 Any k-chromatic graph has at least k vertices of degree at least $k-1$ each.

Proof Let G be a k-chromatic graph and let H be a k-critical subgraph of G. Then, by Theorem 7.4 (b), every vertex of H has degree at least $k-1$ in H and hence in G. Since H is k-chromatic, H has at least k vertices. This completes the proof. ❑

We note that there is no easy characterisation of graphs with chromatic number greater or equal to three. The graph vertex - colouring problem is a standard NP-complete problem and no good algorithm for finding $\chi(G)$ has been discovered for the class of all graphs, though for some special classes of graphs polynomial time algorithms have been found. There are various results which give upper bounds for the chromatic number of an arbitrary graph G, provided the degrees of all the vertices of G are known. The first of these is due to Szekeres and Wilf [237].

Theorem 7.8 Let G be a graph and $k = \max\{\delta(G') : G'$ is a subgraph of $G\}$. Then $\chi(G) = k-1$.

Proof Let H be a k-minimal subgraph of G. Then H is a subgraph of G and therefore $\delta(H) \leq k$. Using Theorem 7.4, we have, $\delta(H) \geq \chi(H) - 1 = \chi(G) - 1$. Thus, $\chi(G) \leq \delta(H) + 1 = k+1$. ❑

The next result is due to Welsh and Powell [262] and its proof is due to Bondy [32].

Theorem 7.9 Let G b a graph with degree sequence $[d_i]_1^n$ such that $d_1 \geq d_2 \geq \ldots \geq d_n$. Then, $\chi(G) \leq \max\{\min\{i,\ d_i\ +1\}\}$.

Proof Let G be k-chromatic. Then, by Theorem 7.8, G has at least k vertices of degree at least $k-1$. Therefore, $d_k \geq k-1$ and max $\{$min $\{i,\ d_i + 1\}\} \geq$ min $\{k,\ d_k + 1\} = k = \chi(G)$. ❑

We have the following upper bounds for chromatic number.

Theorem 7.10 For any graph G, $\chi(G) \leq \triangle(G) + 1$.

Proof Let G be any graph with n vertices. To prove the result, we induct on n. For $n = 1, G = K_1$ and $\chi(G) = 1$ and $\triangle(G) = 0$. Therefore the result is true for $n = 1$.

Assume that the result is true for all graphs with $n-1$ vertices and therefore by induction hypothesis, $\chi(G) \leq \triangle(G-v)+1$. This shows that $G-v$ can be coloured by using $\triangle(G-v)+1$ colours. Since $\triangle(G)$ is the maximum degree of a vertex in G, vertex v has at most $\triangle(G)$ neighbours in G. Thus these neighbours use up at most $\triangle(G)$ colours in the colouring of $G-v$.

If $\triangle(G) = \triangle(G-v)$, then there is at least one colour not used by v's neighbours and that can be used to colour v giving a $\triangle(G)+1$ colouring for G.

In case $\triangle(G) \neq \triangle(G-v)$, then $\triangle(G-v) < \triangle(G)$. Therefore, using a new colour for v, we have a $\triangle(G-v)+2$ colouring of G and clearly, $\triangle(G-v)+2 \leq \triangle(G)+1$. Hence in both cases, it follows that $\chi(G) \leq \triangle(G)+1$. ❑

Remarks

1. Clearly, Theorem 7.10 is a simple consequence of Theorem 7.7. This is because if G is $k-$ chromatic, then Theorem 7.4 gives $\triangle \geq k-1$, that is, $\chi \leq \triangle + 1$.
2. The equality in Theorem 7.10 holds if $G = C_{2n+1}$, $n \geq 1$ and if $G = K_m$.

7.3 Brook's Theorem

Greedy colouring algorithm: The greedy colouring with respect to a vertex ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ is obtained by colouring vertices in the order $v_1, v_2, \ldots, v_n$ assigning to v_i the smallest – indexed colour not already used on its lower – indexed neighbours. This is reported in West [263].

The following recolouring technique as noted in Clark and Holton [60] is due to Kempe [128].

Kempe Chain argument: Let G be a graph with a colouring using at least two different colours represented by i and j. Let $H(i, j)$ denote the subgraph of G induced by all the vertices of G coloured either i or j and let K be a connected component of the subgraph $H(i, j)$. If we interchange the colours i and j on the vertices of K and keep the colours of all other vertices of G unchanged, then we get a new colouring of G, which uses the same colours with which we started. This subgraph K is called a Kempe chain and the recolouring technique is called the Kempe chain argument (Fig. 7.5).

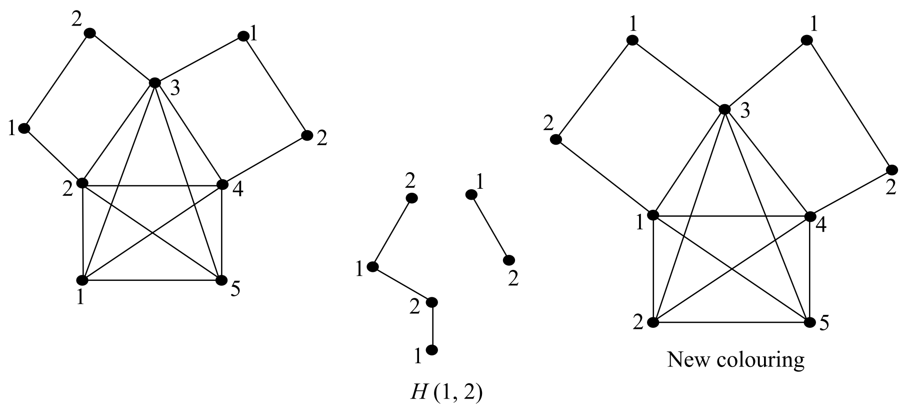


Fig. 7.5

The following result due to Brooks [39] is an improvement of the bounds obtained in Theorem 7.10. We give two proofs of Brooks theorem, the first given by Lovasz [150] uses greedy colouring, and the second proof uses Kempe chain argument.

Theorem 7.11 (Brooks) If G is a connected graph which is neither complete nor an odd cycle, then $\chi(G) \leq \triangle(G)$.

Proof Let G be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ which is neither a complete graph, nor an odd cycle and let $\triangle = k$. Since G is a complete graph for $k = 1$ and G is an odd cycle or a bipartite graph for $k = 2$, let $k \geq 3$.

Assume G is not k-regular. Then there exists a vertex say $v = v_n$ such that $d(v) < k$. Since G is connected, we form a spanning tree of G starting from v_n and whose vertices are arranged in the order $v_n, v_{n-1}, \ldots, v_1$ (Fig. 7.6).

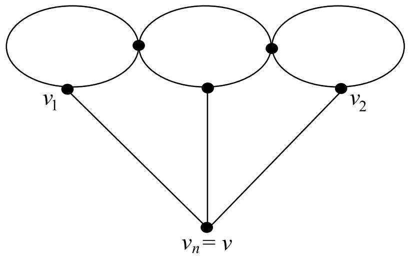


Fig. 7.6

Clearly, each vertex v_i other than v_n in the resulting order $v_n, v_{n-1}, \ldots, v_1$ has a higher indexed neighbour along the path to v_n in the tree. Therefore each vertex v_i has atmost $k-1$ lower indexed neighbours and the greedy colouring needs at most k colours (Fig. 7.7).

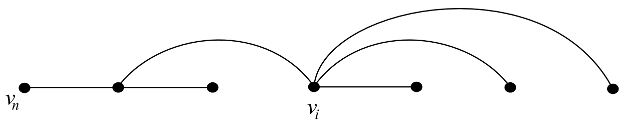


Fig. 7.7

Now, let G be k-regular. Assume G has a cut vertex say x and let G' be a subgraph containing a component of $G - x$ together with the edges of $G - x$ to x. Clearly, $d(x|G') < k$. Therefore, by using the above argument, we have a k-colouring of G'. By making use of the permutations of the colours, it can be seen that this is true for all such subgraphs. Thus G is k-colourable.

Now, let G be 2-connected. We claim that G has an induced 3-vertex path, with vertices say v_1, v_n, v_2 in order, such that $G - \{v_1, v_2\}$ is connected.

To prove the claim, let x be any vertex of G. If $k(G-x) \geq 2$, let v_1 be x and let v_2 be a vertex with distance two from x, which clearly exists, as G is regular and not a complete graph. If $k(G-x) = 1$, then x has a neighbour in every end block of $G - x$, since G has no cut vertex. Let v_1 and v_2 be the neighbours of x in two such blocks. Clearly v_1 and v_2 are non adjacent. Also, since blocks have no cut vertices, $G - \{x, v_1, v_2\}$ is connected. As $k \geq 3$, so $G - \{v_1, v_2\}$ is connected and we let $x = v_n$, proving the claim.

Now arrange the vertices of a spanning tree of $G - \{v_1, v_2\}$ as $v_3, v_4, \ldots, v_n$. As before, each vertex before n has atmost $k-1$ lower indexed neighbours. The greedy colouring uses at most $k-1$ colours on neighbours of v_n, since v_1 and v_2 get the same colour.

Second Proof (Using Kempe chain argument) Let G be a connected graph with n vertices which is neither complete nor an odd cycle. Let $\triangle(G) = k$. For $k = 1$, G is complete and for $k = 2$, G is an odd cycle or a bipartite graph. Therefore, assume $k \geq 3$.

We induct on n. Since $k \geq 3$, the induction starts from $n = 4$. As G is not complete, then for $n = 4$, G is one of the graphs given in Figure 7.8.

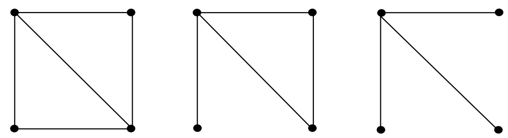


Fig. 7.8

Clearly, for each such graph, the chromatic index is at most three.

Now, let $n \geq 5$, and assume the result to be true for all graphs with fewer than n vertices.

If G has a vertex v of degree less than k, then it follows from Theorem 7.10 that G can be coloured by k colours, since the neighbours of v use up atmost $k-1$ colours. Therefore the result is true in this case.

Now assume that degree of every vertex of G is k, that is, G is k-regular. We show that G has a k-colouring.

Let v be any vertex of G. Then by induction hypothesis, the subgraph $G-v$ has a k-colouring. If the neighbours of v in G do not use all the k colours in the k-colouring of $G-v$, then any unused colour is assigned to v giving a k-colouring of G. Assume that the k neighbours of v are assigned all the k colours in the k-colouring of $G-v$. Let the neighbours of v be $v_1, v_2, \ldots, v_k$ which are coloured by the colours $1, 2, \ldots, k$ respectively.

Let the Kempe chains $H_{v_i}(i, j)$ and $H_{v_j}(i, j)$ containing the neighbours v_i and v_j be different. That is, v_i and v_j are in different components of the subgraph $H(i, j)$ induced by the colours i and j. Therefore, using Kempe chain argument, the colours in $H_{v_i}(i, j)$ are interchanged to give a k-colouring of $G-v$, where now v_i has been assigned the colour j. This implies that the neighbours of v use less than k colours and the unused colour assigned to v gives a k-colouring of G.

Now assume that for each i and j, the neighbours v_i and v_j are in the same Kempe chain, which is briefly denoted by H. If the degree of v_i in H is greater than one, then v_i is adjacent to at least two vertices coloured j. Therefore there is a third colour, say ℓ, not used in colouring the neighbours of v_i. Recolour v_i by ℓ and then colour v by i, giving the k-colouring of G. Assume that v_i and v_j both are of degree one in H. Let P be a path from v_i to v_j in H and let there be a vertex in P with degree at least three in H (Fig. 7.9). Let u be the first such vertex and coloured i. (If u is coloured j, the same argument is used as in case of i). Then at least three neighbours of u are coloured j and therefore there is a colour, say ℓ, not used by these neighbours. Recolour u by ℓ and interchange colours i and j on the vertices of P from v_i upto u, excluding u. So we get a colouring of $G-v$, where v_i and v_j are now both coloured j. This allows v to be coloured by i.

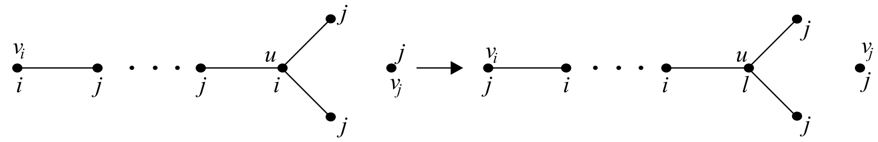


Fig. 7.9

Let all the vertices on a path v_i to v_j, excluding the end vertices v_i and v_j, be of degree two in H. Clearly H contains a single path from v_i to v_j.

Let all the Kempe chains be paths. Let H and K be such chains corresponding to v_i, v_j and v_i, v_ℓ respectively, with $j \neq \ell$. Let $w \neq v_i$ be a vertex present in both the chains (Fig. 7.10). Then w is coloured i, has two neighbours coloured j and two neighbours coloured ℓ. Therefore there is a fourth colour, say s, not used by the neighbours of w. Now colour w by s, and interchange colours ℓ and i on the vertices of K beyond w upto and including v_ℓ, we get a colouring of $G - v$, where v_i and v_ℓ are now both coloured i. This allows v to be coloured by ℓ.

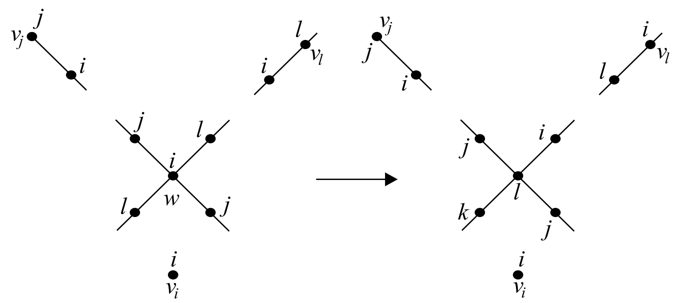


Fig. 7.10

Thus assume that two such Kempe chains meet only at their common end vertex v_i.

Let v_i and v_j be two neighbours of v which are nonadjacent and let x be the vertex coloured j, adjacent to v_i on the Kempe chain H from v_i to v_j. With $\ell \neq j$, let K denote the Kempe chain from v_i to v_j. Then by the Kempe chain argument, we interchange the colours in K, without changing the colours of the other vertices. This results in v_i coloured ℓ, and v_ℓ coloured i. Since x is adjacent to v_i, it is in the Kempe chain for colours ℓ and j. However,it is also the Kempe chain for colours i and j. This contradicts the assumption that Kempe chains have at most one vertex in common, the end vertex. This contradiction implies that any two v_i and v_j are adjacent. In other words, all neighbours of v are also neighbours of each other. This shows that G is the complete graph K_k, a contradiction to the hypothesis of G. ❑

Definition: In the depth first search tree (DFS), a search tree T is used to represent the edge examination process. In DFS a new adjacent vertex is selected, which is incident with the first edge incident with v. In other words, in DFS we leave$v$ as quickly as possible, examining only one of its incident edges and replacing v by a new vertex, which is adjacent to v.

The following result is due to Chartrand and Kronk [51].

Theorem 7.12 Let G be a connected graph every depth-first search tree of which is a Hamiltonian path. Then G is a cycle, a complete graph, or a complete bipartite graph $K_{n,n}$.

Proof Let P be a Hamiltonian path of G, with origin u. Because the path $P-u$ extends to a Hamiltonian path of G, the path P extends to a Hamiltonian cycle C of G.

When C has no chord, $G=C$ is a cycle. So let uv be a chord of C. Then u^-v^- is one too, because $u^-CvuC^{-1}v^-$ is a Hamiltonian path of G, likewise, u^-v^- is a chord of C (where u^- denotes the successor of u on C and u^{--} is the successor of u^-). And if the length of uCv is at least four, uv and u^-v^- are also chords of C, in view of the Hamiltonian path $u^{--}Cv^-u^-C^{-1}v^-u^-uv$ and the fact that $u^-v^-=(u^-)^-v^-$.

When C has a chord uw of length two, let $v=u^-(=w^-)$. Then $vw^-\in E$. Moreover, if $vw^-\in E$, then $vw^{-(-1)}\in E$ in view of the Hamiltonian path $w^{-(-1)}CuwCw^-v$. It follows that v is adjacent to every vertex of G. But then G is complete, because u^-w^- is a chord of length two for all i. If C has no chord of length two, every chord of C is odd, moreover, every odd chord must be present. Thus, $G=K_{n,\,n}$, where $|V(G)|=2n$. ❑

The following is the third proof of Brook's theorem which is due to Bondy [35].

Bondy's Proof Suppose first that G is not regular. Let u be a vertex of degree δ and let T be a search tree of G rooted at u. Colour the vertices with the colours $1,\,2,\,\ldots,\,\triangle$ according to the greedy heuristic, selecting at each step a pendent vertex of the subtree of T induced by the vertices not yet coloured, assigning to it the smallest available colour and ending with the root u of T. When each vertex v different from x is coloured, it is adjacent (in T) to at least one uncoloured vertex and so is adjacent to at most $d(v)-1<\triangle-1$ coloured vertices. It is therefore assigning one of the colours $1,\,2,\,\ldots,\,\triangle$, because $d(u)=\delta\le\triangle-1$. The greedy heuristic therefore produces a $\triangle$-colouring of G.

Now, let G be regular. If G has a cut vertex u, then $G=G_1\cup G_2$, where G_1 and G_2 are connected and $G_1\cap G_2=\{u\}$. Because the degree of u in G is less than $\triangle(G)$, neither subgraph of G is regular, so $\chi(G)\le\triangle(G_i)=\triangle(G)$, $i=1,2$ and $\chi(G)=\max\{\chi(G_1),\chi(G_2)\}\le\triangle(G)$. Therefore we assume that G is 2-connected.

If every depth- first search tree of G is a Hamiltonian path, then G is a cycle, a complete graph, or a complete bipartite graph $K_{n,\,n}$, by Theorem 7.12. Since by hypothesis, G is neither an odd cycle nor a complete graph, $\chi(G)=2\le\triangle(G)$. Suppose then, that T is a depth-first search tree of G, but not a path. Let u be a vertex of T with at least two children, v and w. Because G is 2-connected, both $G-v$ and $G-w$ are connected. Thus there are proper descendants of v and w, each of which is joined to an ancestor of u, and it follows that $G-\{v,\,w\}$ is connected. Consider a search tree T with root u in G. By colouring v and w with colour 1 and then the vertices of T by the greedy heuristic as above, ending with the root u, we obtain a $\triangle$-colouring of G. ❑

Brooks theorem and the observation that in a graph G containing K_n as a subgraph, $\chi(G)\ge n$, provide estimates for the chromatic number. For instance, in Figure 7.11(a) for the graph G_1, $\triangle(G_1)=8$ and G_1 has K_4 as a subgraph. Therefore, $4\le\chi(G_1)\le 8$. It can be easily seen that $\chi(G_1)=4$. Similarly for G_2 in Fig. 7.11(b) known as the Birkhoff diamond, $\triangle(G_2)=5$ and G_2 has K_3 as a subgraph. So, $3\le\chi(G_2)=5$. In fact, $\chi(G_2)=4$.

Independent set: A set of vertices in a graph G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence numberindependence number of G and is denoted by $\alpha_0(G)$ or α_0. Analogously, an independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the edge independence number $\alpha_1(G)$ or α_1. For the complete graph K_n, $\alpha_0 = 1$, $\alpha_1 = \left[\frac{n}{2}\right]$. In the graph of Figure 7.12, $\alpha_0(G) = 2$ and $\alpha_1(G) = 3$.

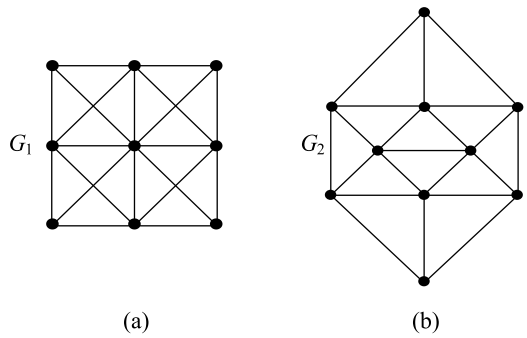


Fig. 7.11

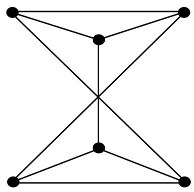


Fig. 7.12

A lower bound, noted in Berge [22] and Ore [178] and an upper bound by Harary and Hedetniemi [106] involve the vertex independent number α_0 of a graph.

Theorem 7.13 For any graph G, $\dfrac{n}{\chi(\overline{G})} \le \dfrac{n}{\alpha_o} \le \chi(G) \le n - \alpha_o + 1$.

Proof If $\chi(G) = k$, then V can be partitioned into k colour classes $V_1, V_2, \ldots, V_k$, each of which is an independent set of vertices.

If $|V_i| = n_i$, then every $n_i \le \alpha_o$, so that $n = \sum n_i \le k\alpha_o$. This proves the middle inequality.

Now, let S be a maximal independent set containing α_o vertices. Clearly, $\chi(G-S) \ge \chi(G) - 1$.

As $G-S$ has $n-\alpha_o$ vertices, $\chi(G-S) \le n-\alpha_o$, and $\chi(G) \le \chi(G-S)+1 \le n-\alpha_o+1$, proving the last inequality.

As $\overline{G}$ has a complete subgraph of order $\alpha_o(G)$,

$\chi(\overline{G}) \ge \alpha_o(G)$, or $\dfrac{n}{\chi(\overline{G})} \le \dfrac{n}{\alpha_o(G)}$, proving the first inequality. ❑

The following result due to Nordhaus and Gaddum [172] gives the bounds on the sum and product of the chromatic numbers of a graph and its complement.

Theorem 7.14 For any graph of order n,

$$2\sqrt{n} \le \chi + \overline{\chi} \le n+1, \tag{7.14.1}$$

$$\text{and } n \le \chi\overline{\chi} \le \left(\frac{n+1}{2}\right)^2, \tag{7.14.2}$$

where $\chi = \chi(G)$ and $\overline{\chi} = \chi(\overline{G})$.

Proof Evidently from Theorem 7.13, we have

$$\chi\overline{\chi} \ge n. \tag{7.14.3}$$

Since the arithmetic mean is greater than or equal to the geometric mean,

$$\frac{\chi+\overline{\chi}}{2} \ge \sqrt{\chi\overline{\chi}}. \tag{7.14.4}$$

Combining (7.14.3) and (7.14.4), we get $\dfrac{\chi+\overline{\chi}}{2} \ge n.$

Therefore the left inequalities of (7.14.1) and (7.14.2) are proved.

Now, let $d_1 \ge d_2 \ge \ldots \ge d_n$ be the degree sequence of G. Then $\overline{d}_1 \ge \overline{d}_2 \ge \ldots \ge \overline{d}_n$, where $\overline{d}_i = n-1-d_{n+1-i}$, is the degree sequence of $\overline{G}$. Then by using Theorem 7.9, we have

$$\begin{aligned}\chi(G)+\chi(\overline{G}) &\le \max_i \min\{d_i+1, i\} + \max_i \min\{n-d_{n+1-i}, i\}\\ &= \max_i \min\{d_i+1, i\} + (n+1) - \min_i \max\{d_{n+1-i}+1, n+1-i\}\\ &= \max_i \min\{d_i+1, i\} + (n+1) - \min_j \max\{d_j+1, j\}.\end{aligned}$$

Thus, $\chi(G)+\chi(\overline{G}) \le n+1$.

Also, $\dfrac{\chi(G)+\chi(\overline{G})}{2} \ge \sqrt{\chi\overline{\chi}}$. Therefore, $\sqrt{\chi\overline{\chi}} \le \dfrac{\chi+\overline{\chi}}{2} \le \dfrac{n+1}{2}$. Thus, $\chi\overline{\chi} \le \left(\dfrac{n+1}{2}\right)^2$.

Second Proof Let G be k-chromatic and let $v_1, v_2, \ldots, v_k$ be the colour classes of G, where $|V_i| = n_i$ Then $\Sigma n_i = n$ and max $n_i \ge n/k$. Since each V_i induces a complete subgraph of $\overline{G}, \overline{\chi} \ge \max n_i \ge n/k$, so that $\chi\overline{\chi} \ge n$. As the geometric mean of two positive numbers is always less or equal to their arithmetic mean, it follows that $\chi+\overline{\chi} \ge 2\sqrt{n}$.

To prove $\chi + \overline{\chi} \le n+1$, we induct on n. Clearly, the equality holds for $n = 1$. We assume that $\chi(G) + \overline{\chi}(G) \le n$ for all graphs G having fewer than n vertices. Let H and $\overline{H}$ be complementary graphs with n vertices and let v be a vertex of H. Then $G = H - v$ and $\overline{G} = \overline{H} - v$ are complementary graphs with $n-1$ vertices. Let the degree of v in H be d, so that degree of v in $\overline{H}$ is $n-d-1$. Clearly, $\chi(H) \le \chi(G)+1$ and $\overline{\chi}(H) \le \overline{\chi}(G)+1$. If either $\chi(H) < \chi(G)+1$ or $\overline{\chi}(H) < \overline{\chi}$ (G) + 1, then $\chi(H)+\overline{\chi}(H) \le n+1$. If $\chi(H) = \chi(G)+1$ and $\overline{\chi}(H) = \overline{\chi}(G)+1$, then the removal of v from H, producing G, decreases the chromatic number, so that $d \ge \chi(G)$. Similarly, $n-d-1 \ge \overline{\chi}(G)$. Thus, $\chi(G)+\overline{\chi}(G) \le n-1$. Therefore we always have $\chi(H)+\overline{\chi}(H) \le n+1$. Applying now the inequality $4\chi\overline{\chi} \le (\chi+\overline{\chi})^2$, we get $\chi\overline{\chi} \le \left(\frac{n+1}{2}\right)^2$. ❑

We now have the following result.

Theorem 7.15 If a connected k-chromatic graph has exactly one vertex of degree exceeding $k-1$, then it is minimal.

Proof Let G be a connected k-chromatic graph having exactly one vertex of degree exceeding $k-1$. Let e be any edge of G. Then $\delta(G-e) \le k-2$ (otherwise, G will have at least two vertices of degree exceeding $k-1$).

For every induced subgraph H of $G-e$, we have $\delta(H) \le k-2$. Thus, by Theorem 7.7, $\chi(G-e) \le k-1$ and hence $\chi(G-e) \le k-1$. Since e is arbitrary, therefore G is minimal. ❑

We observe from Theorem 7.4(b) that the number of edges m of a k-critical graph is at least $n(k-1)/2$. Dirac extended this to the inequality $2m \ge n(k+1)-2$ for a $(k+1)$-critical graph, the proof of which can be found in Bollobas [29].

7.4 Edge colouring

An edge colouring of a nonempty graph G is a labelling $f: E(G) \to \{1, 2, \ldots\}$; the labels are called *colours*, such that adjacent edges are assigned different colours. A k-edge colouring of G is a colouring of G which consists of k different colours and in this case G is said to be *k-edge colourable*.

The definition implies that the k-edge colouring of a graph $G(V, E)$ partitions the edge set E into k independent sets $E_1, E_2, \ldots, E_k$ such that $E = E_1 \cup E_2 \cup \ldots \cup E_k$. The independent sets $E_i, 1 \le i \le k$ are called the *colour classes* and the function $f: E(G) \to \{1, 2, \ldots, k\}$ such that $f(e) = i$, for each $e \in E_i$, $1 \le i \le k$, is called the *colour function*. The minimum number k for which there is a k-colouring of G is called the *edge chromatic umber* (or edge chromatic index) and is denoted by $\chi'(G)$.

We have the following observations.

1. If H is a subgraph of a graph G, then $\chi'(H) \le \chi'(G)$.

2. For any graph $G, \chi'(G) \ge \triangle(G)$.

If v is any vertex of G with $d(v) = \triangle(G)$, then the $\triangle(G)$ edges incident with v have a different colour in any edge colouring of G.

3. $\chi'(C_n) = \begin{cases} 2, & \text{if n is even}, \\ 3, & \text{if n is odd}. \end{cases}$

The following is a recolouring technique for edge colouring, called *Kempe chain argument*.

Let G be a graph with an edge colouring using at least two different colours say i and j. Let $H(i,\ j)$ represent the subgraph of G induced by all the edges coloured either i or j. Let K be a connected component of the subgraph $H(i,\ j)$. It can be easily verified that K is a path whose edges are alternately coloured by i and j. If the colours on these edges are interchanged and the colours on all other edges of G are kept unchanged, the result is a new colouring of G, using the same initial colours. The component K is called Kempe chain and this recolouring method is called the Kempe chain argument.

Definition: Let i be a colour used in the edge colouring of a graph G. If there is an edge coloured i incident at the vertex v of G, we say i is *present* at v, and if there is no edge coloured i at v, we say i is *absent* from v.

The following result is due to Konig [136].

Theorem 7.16 (Konig) For a nonempty bipartite graph G, $\chi'(G) = \triangle(G)$.

Proof The proof is by induction on the number of edges of G. If G has only one edge, the result is trivial.

Let G have more than one edge and assume that the result is true for all nonempty bipartite graphs having fewer edges than G. Since $\triangle(G) \leq \chi'(G)$, it is enough to prove that G has a $\triangle(G)$-edge colouring. We let $\triangle(G) = k$. Let $e = uv$be an edge of G. Then $G-e$ is bipartite with less edges than G. Therefore, by induction hypothesis, $G-e$ has a $\triangle(G-e)$-edge colouring. Since $\triangle(G-e) \leq \triangle(G) = k, G-e$ has a k-colouring. We show that the same k colours are used to colour G.

As $d(u) \leq k$ in G and the edge e is uncoloured, there is at least one of the k colours absent from u. Similarly, at least one of these colours is absent from v.

If one of the colours absent at u and v is same, we use this to colour e and we get a k-edge colouring of G.

Now take the case of a colour i present at u, but absent from v and a colour j present at v, but absent from u.

Let K be the Kempe chain containing u in the subgraph $H(i, j)$ induced by the edges coloured i or j. We claim that v does not belong to the Kempe chain K.

For if v belongs to K, then there is a path P in K from u to v. Since u and v are adjacent, they do not belong to the same bipartition subset of the bipartite graph G and therefore the length of the path P is odd. As the colour i is present at u, the first edge of P is coloured i. Since the edges of P are alternately coloured i and j, and P is of odd length, therefore the last edge of P, which is incident at v, is also coloured i. This is a contradiction, as i is absent from v, proving our claim.

Using Kempe chain argument on K, the interchanging of colours now makes i absent from u and does not affect the colours of the edges incident at v. Therefore i is absent from both u and v in this new edge colouring and colouring edge e by i gives a k-edge colouring of G. ❑

The next result gives edge chromatic number of complete graphs.

Theorem 7.17 If $G = K_n$ is a complete graph with n vertices, $n \geq 2$, then

$$\chi'(G) = \begin{cases} n-1, & if\ n\ is\ even, \\ n, & if\ n\ is\ odd. \end{cases}$$

Proof Let $G = K_n$ be a complete graph with n vertices.

Assume n is odd. Draw G so that its vertices form a regular polygon. Clearly, there are n edges of equal length on the boundary of the polygon. Colour the edges along the boundary using a different colour for each edge. Now, each of the remaining internal edges of G is parallel to exactly one edge on the boundary. Each such edge is coloured with the same colour as the boundary edge. So two edges have the same colour if they are parallel and therefore we have the edge colouring of G. Since it uses n colours, we have shown that $\chi'(G) \leq n$.

Let G have an $(n-1)$-colouring. From the definition of an edge colouring, the edges of one particular colour form a matching in G (set of independent edges). Since n is odd, therefore the maximum possible number of these is $(n-1)/2$. This implies that there are atmost $(n-1)(n-1)/2$ edges in G. This is a contradiction, as K_n has $n(n-1)/2$ edges. Thus G does not have an $(n-1)$ colouring. Hence, $\chi'(G) = n$.

Now, let n be even, and let v be any vertex of G. Clearly $G - v$ is complete with $n-1$ vertices. Since $n-1$ is odd, $G - v$ has an $(n-1)$-colouring. With this colouring, there is a colour absent from each vertex and different vertices having different absentees. Reform G from $G - v$ by joining each vertex w of $G - v$ to v by an edge and colour each such edge by the colour absent from w. This gives an $(n-1)$-colouring of G and therefore $\chi'(G) = n-1$. ❑

The above result is illustrated by taking K_5 and K_6 in Figure 7.13.

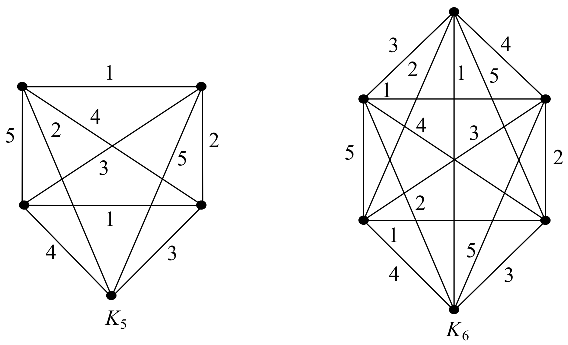


Fig. 7.13

Since $\triangle(G) = n-1$ in a complete graph, Theorem 7.17 shows that $\chi'(K_n)$ is either $\triangle(G)$ or $\triangle(G)+1$. The next result obtained by Vizing [258] and independently by Gupta [94] gives the tight bounds for edge chromatic number of a simple graph.

Theorem 7.18 (Vizing) For any graph $G, \triangle(G) \leq \chi'(G) \leq \triangle(G)+1$.

Proof Let G be a simple graph, we always have $\triangle(G) = \chi'(G)$.

To prove $\chi'(G) \leq \triangle(G)+1$, we use induction on the number of edges of G. Let $\triangle(G) = k$. If G has only one edge, then $k = 1 = \chi'(G)$. Therefore assume that G has more than one edge and that the result is true for all graphs having fewer edges than G.

Let $e = v_1v_2$ be an edge of G. Then by induction hypothesis the subgraph $G-e$ has $(k+1)$-edge colouring and let the colours used be 1, 2, $\ldots$, $k+1$.

Since $d(v_1) \leq k$ and $d(v_2) \leq k$, out of these $k+1$ colours at least one colour is absent from v_1 and at least one colour is absent from v_2. If there is a common colour absent from both v_1 and v_2, then we use this to colour e and get a $(k+1)$-colouring of G. Therefore in this case, $\chi'(G) \leq k+1$.

We now assume that there is a colour, say 1, absent from v_1 but present at v_2 and there is a colour, say 2, absent from v_2 but present at v_1. We start from v_1 and v_2 and construct a sequence of distinct vertices $v_1, v_2, \ldots, v_j$, where each v_i for $i \geq 2$ is adjacent to v_1. Let v_1v_3 be coloured 2. This v_3 exists, because 2 is present at v_1. We observe that not all the $k+1$ colours are present at v_3 and assume that the colour 3 is absent from v_3. But the colour 3 is present at v_1 and choose the vertex v_4 so that v_1v_4 is coloured 3. Continuing in this way, we choose a new colour i absent from v_i but present at v_1, so that v_1v_{i+1} is the edge coloured i. In this way, we get a sequence of vertices $v_1, v_2, v_3, \ldots, v_{j-1}, v_j$ such that

a. v_i is adjacent to v_1 for each $i > 1$,

b. the colour i is absent from each $i = 1, 2, \ldots, j-1$ and

c. the edge v_1v_{i+1} is coloured i for each $i = 1, 2, \ldots, j-1$.

This is illustrated in Figure 7.14.

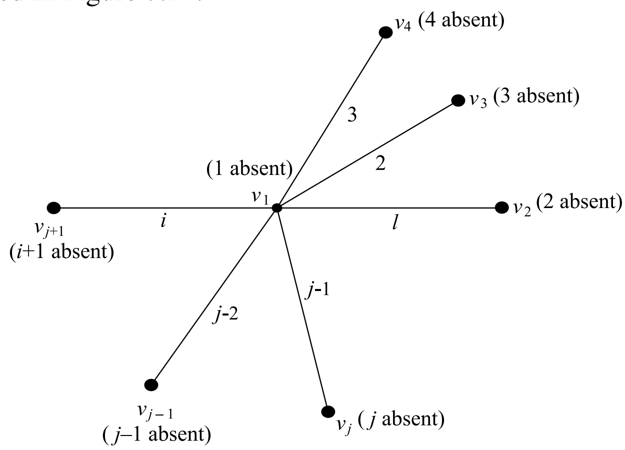


Fig. 7.14

As $d(v_1) \le k$, (a) implies that such a sequence has at most $k+1$ terms, that is, $j \le k+1$. Assume that $v_1, v_2, \ldots, v_j$ is a longest such sequence, that is, the sequence for which it is not possible to find a new colour j, absent from v_j, together with a new neighbour v_{j+1} of v_1 such that v_1v_{j+1} is coloured j.

We first assume that for some colour j absent from v_j there is no edge of that colour present at v_1. We colour the edge $e = v_1v_2$ by colour 2 and then recolour the edges v_1v_j by colour i, for $i = 3, \ldots, j-1$. Since i was absent from v_i, for each $i = 2, \ldots, j-1$, this gives a $(k+1)$-colouring of the subgraph $G - v_1v_j$. Now as the colour j is absent from both v_j and v_1, recolour v_1v_j by the colour j. This gives a $(k+1)$-colouring of G (Fig. 7.15).

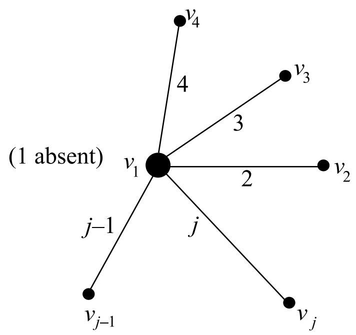


Fig. 7.15

Now assume that whenever j is absent from v_j, j is present at v_1. If v_{j+1} is a new neighbour of v_1 so that v_1v_{j+1} is coloured by j, then we have extended our sequence to $v_1, v_2, \ldots, v_j, v_{j+1}$ which is a contradiction to the assumption that $v_1, v_2, \ldots, v_j$ is the longest sequence. Thus one of the edges $v_1v_3, \ldots, v_1v_{j-1}$ is to be coloured by j, say v_1v_ℓ, with $3 \le \ell \le j-1$. Now colour $e = v_1v_2$ by 2, and for $i = 3, \ldots, \ell-1$, recolour each of the edges v_1v_i by i while unaltering the colours of the edges v_1v_i, for $i = \ell+1, \ldots, j$. Removing the colour j from v_1v_ℓ, we have a $(k+1)$-colouring of the edge deleted subgraph $G - v_1v_\ell$.

Let $H(1, j)$ represent the subgraph of G induced by the edges coloured 1 or j in this partial colouring of G. Since the degree of every vertex in $H(1, j)$ is either 1 or 2, each component of $H(1, j)$ is either a path or a cycle. As 1 is absent from v_1 and j is absent from both v_j and v_ℓ, it follows that all these three vertices do not belong to the same connected component of $H(1, j)$. Therefore, if K and L represent the corresponding Kempe chains containing v_j and v_ℓ respectively, then either $v_1 \notin K$ or $v_1 \notin L$. Let $v_1 \notin L$. Then interchanging the colours of L, the Kempe chain argument gives a $(k+1)$-colouring of $G - v_1v_\ell$ in which 1 is missing from both v_1 and v_ℓ. Colouring v_1v_ℓ by 1 gives a $(k+1)$-colouring of G.

Now, let $v_1 \notin K$. Colour the edge v_1v_ℓ by ℓ, recolour the edges v_1v_i by i, for $i = \ell, \ldots, j-1$, and remove the colour $j-1$ from v_1v_j. Then, from the definition of the sequence $v_1, v_2, \ldots, v_j$, we get a $(k+1)$-colouring of $G - v_1v_j$ without affecting two coloured subgraph $H(1, j)$. Using Kempe chain argument to interchange the colours of K, we obtain a $(k+1)$-colouring of $G - v_1v_j$ in which 1 is absent from both v_1 and v_j. Therefore, again colouring v_1v_j by 1 gives $(k+1)$-colouring of G. ❑

The following result is due to Vizing [259] and Alavi and Behzad [2].

Theorem 7.19 Let G be a graph of order n and let $\overline{G}$ be the complement of G. Then

a. $n-1 \leq \chi'(G)+\chi'(\overline{G}) \leq 2(n-1)$,

$0 \leq \chi'(G)\chi'(\overline{G}) \leq (n-1)^2$, for even n,

b. $n \leq \chi'(G)+\chi'(\overline{G}) \leq 2n-3$,

$0 \leq \chi'(G)\chi'(\overline{G}) \leq (n-1)(n-2)$, for odd n.

Further, the bounds are the best possible for every positive integer $n (n \neq 2)$.

Proof Let G be a graph of order n and let $\overline{G}$ be the complement of G. Clearly,

$\triangle(\overline{G}) \geq n-1-\triangle(G)$, so that $\triangle(G)+\triangle(\overline{G}) \geq n-1$.

Therefore, combining with $\chi'(G) \geq \triangle(G)$, we get

$\chi'(G)+\chi'(\overline{G}) \geq n-1$.

b. If n is odd, we have $\chi'(G)+\chi'(\overline{G}) \geq n$, since $\chi'(G)+\chi'(\overline{G}) < n$ implies $\chi'(K_n) < n$, which is a contradiction.

Obviously, $\chi'(G)\chi'(\overline{G}) \geq 0$. It can be seen that the lower bounds are attained in complete graphs K_n.

We now prove that $\chi'(G)+\chi'(\overline{G}) \leq 2n-3$ and $\chi'(G)\chi'(\overline{G}) \leq (n-1)(n-2)$.

Clearly, for $n = 1, 3$, the inequalities $\chi'(G)+\chi'(\overline{G}) \leq 2n-3$ are true. So, let $n \geq 5$. If $\triangle(G)+\triangle(\overline{G}) \leq 2n-5$, then by Vizing's theorem, we get $\chi'(G)+\chi'(\overline{G}) \leq 2n-3$.

Otherwise, we have the following cases.

i. $\triangle(G) = n-1$ and $\triangle(\overline{G}) = n-2$. So G has a pendant vertex v. Then $\triangle(G-v) \geq n-2$ and $\chi'(G-v) \geq n-2$. But $\chi'(G-v) \leq \chi'(K_{n-1}) = n-2$. Therefore, $\chi'(G-v) = n-2$. Thus, $\chi'(G) = n-1$.

   As $\overline{G}$ is the disjoint union of an isolated vertex and a subgraph of K_{n-1}, $\chi'(\overline{G}) = n-2$. Hence, $\chi'(G)+\chi'(\overline{G}) = 2n-3$.

ii. $\triangle(G) = n-2$ and $\triangle(\overline{G}) = n-2$. Again, G has a pendant vertex v and as before, $\chi(G-v) \leq n-2$ and $\chi(G) = n-2$. Similarly, $\chi(\overline{G}) = n-2$. Thus, $\chi'(G)+\chi'(\overline{G}) = 2n-4 < 2n-3$.

iii. $\triangle(G) = n-1$ and $\triangle(\overline{G}) = n-3$. In this case, G has a vertex v of degree two and so $\chi'(G-v) = n-2$.

Let vu and vw be the edges incident with v in G, with $d(u) = n-1$. In $(n-2)$-edge colouring of $G-v$, change the colour i of an edge $uu'(u' \neq w)$ to a new colour $n-1$, and now colour vu by i and vw by $n-1$. This gives an $(n-1)$-colouring of G. Now, by Vizing's theorem, $\chi'(\overline{G}) \leq n-2$.

Together, we get $\chi'(G)+\chi'(\overline{G}) \leq 2n-3$. Since $\chi'(G)+\chi'(\overline{G}) \leq 2n-3$, clearly we have $\chi'(G)\chi'(\overline{G}) \leq (n-1)(n-2)$. We observe that in graph $K_{1,n-1}$, the upper bounds are attained.

a. Let n be even. Then $\chi'(G)+\chi'(\overline{G}) \geq n-1$. Also, $\chi'(G)\chi'(\overline{G}) \geq 0$.
   The lower bounds are attained for complete graphs.
   To get the upper bounds, since G and $\overline{G}$ are subgraphs of K_n and $\chi'(K_n) = n-1$ for all even n, $\chi'(G)+\chi(\overline{G}) \leq 2(n-1)$ and $\chi'(G)\chi'(\overline{G}) \leq (n-1)^2$. These upper bounds are attained in the complete bipartite graphs $K_{1,n-1}, n \neq 2$. ❑

7.5 Region Colouring (Map Colouring)

A region colouring of a planar graph is a labeling of its regions $f : R(G) \to \{1, 2, \ldots\}$; the labels called *colours*, such that no two adjacent regions get the same colour. A k-region colouring of a planar graph G consists of k different colours and G is then called *k-region colourable*. From the definition, it follows that the k-region colouring of a planar graph G partitions the region set R into k independent sets $R_1, R_2, \ldots, R_k$, so that $R = R_1 \cup R_2 \cup \ldots \cup R_k$. The independent sets are called the *colour classes*, and the function $f : R(G) \to \{1, 2, \ldots, k\}$ such that $f(r) = i$, for each $r \in R_i$, $1 \leq i \leq k$, is called the *colour function*. The minimum number k for which there is a k-region colouring of the planar graph G is called the *region-chromatic number* of G, and is denoted by $\chi''(G)$. The colouring of regions is also called *map colouring*, because of the fact that in an atlas different countries are coloured such that countries with common boundaries are shown in different colours.

The four colour problem: Any map on a plane surface (or a sphere) can be coloured with at most four colours so that no two adjacent regions have the same colour.

Now coming to the origin of the four colour problem, there have been reports that Mobius was familiar with the problem in 1840. But the problem was introduced in 1852 by Francis Guthrie, student of Augustus DeMorgan and the problem first appeared in a letter (October 23, 1852) from DeMorgan to Sir William Hamilton. DeMorgan continued the discussion of the problem with other mathematicians and in the years that followed attempts were made to prove or disprove the problem by top mathematical minds of the world. In 1878, Cayley announced the problem to the London Mathematical Society, and in 1879, Alfred Kempe announced that he had found a proof. An error in Kempe's proof was discovered by P. J. Heawood in 1890. Kempe's idea was based on the alternating paths and Heawood used this idea to prove that five colours are sufficient. Kempe's argument did not prove the four colour problem, but did contain several ideas which formed the foundation for many later attempts at the proof, including the successful attempts by Appel and Haken. In 1976, K. Appel and W. Haken [5, 6, 7] with the help of J. Koch established what is now called four colour theorem. Their proof made use of large scale computers (using over

1000 hours of computer time) and this is the first time in the history of mathematics that a mathematical proof depended upon the external factor of the availability of a large scale computing facility. Though the Appel-Haken proof is accepted as valid, mathematicians are still in search of alternative proof. Robertson, Sanders, Seymour and Thomas [225] have given a short and clever proof, but their proof still requires a number of computer calculations. Saaty [230] presents thirteen colourful variations of four colour problem.

In the year 2000, Ashay Dharwadkar [64] has given a new proof of the four colour theorem, which will be discussed in details in Chapter 14.

The following observations are immediate from the definitions introduced above.

1. A planar graph is k-vertex colourable or k-region colourable if and only if its components have this property.

2. A planar graph is k-vertex colourable or k-region colourable if and only if its blocks have this property.

These observations imply that for studying vertex colourings or region colourings, it suffices to consider the graph to be a block.

Theorem 7.20

a. A planar graph G is k-region colourable if and only if its dual G is k-vertex colourable.

b. If G is a plane connected graph without loops, then G has a k-vertex colouring if and only if its dual G^* has a k-region colouring.

Proof

a. Let the regions and edges of G be respectively denoted by $r_1, \ldots, r_t$ and $e_1, \ldots, e_m$. Let the vertices of G^* be $r_1^*, \ldots, r_t^*$ and edges be $e_1^*, \ldots, e_m^*$. Then the vertices and edges of G^* are in one-to-one correspondence with the regions and edges of G, and two vertices r^* and s^* in G^* are joined by an edge e^* if and only if the corresponding regions r and s in G have the corresponding edge e as a common edge on their boundary.

   Let G be k-region colourable. We colour the vertices in G^* such that each vertex in G^* gets the same colour as assigned to the region r in G. Since the vertices r^* and s^* are only adjacent in G^* if the corresponding regions r and s are adjacent in G, G^* is k-vertex colourable.

   Conversely, let G^* be k-vertex colourable. Now colour the regions of G such that the region r in G gets the samecolour as the vertex r^* in G^*. This gives a k-region colouring of G, since the regions r and s are adjacent in G only if the corresponding vertices r^* and s^* are adjacent in G^*.

b. Since G has no loops its dual G^* has no bridges, and therefore G^* is planar. Thus by (a), G^* is k-region colourable if and only if the double dual G^{**} is k-vertex colourable. Since G is connected, G is isomorphic to G^{**} and hence the result follows. ❑

Remarks A graph has a dual if and only if it is planar and this implies that colouring the regions of a planar graph G is equivalent to colouring the vertices of its dual G^* and vice versa. It also follows from Theorem 7.17 that if G^* is the dual of the planar graph G, then $\chi(G) = \chi''(G^*)$ and $\chi(G^*) = \chi''(G)$. These observations give the dual form of the four colour problem which states that, every planar graph is 4-vertex colourable.

Since loops and multiple edges are not allowed in vertex colourings, it may be assumed that no two regions have more than one boundary edge in common, for region colouring of a planar graph

As every triangulation is a planar graph (in fact, a maximal planar graph) and every planar graph is a subgraph of a triangulation, the four colour problem is true if and only if every triangulation is 4-colourable.

The following result shows that a planar graph is 6-colourable.

Theorem 7.21 Every planar graph is 6-colourable.

Proof Let G be a planar graph and H be the dual of G. Then it is sufficient to prove that H has a vertex colouring of at most 6 colours. More generally, we prove that any graph H is 6-colourable.

To prove the result, we use induction on n, the order of H. The result is trivial if H has at most six vertices. So assume $n \geq 7$.

Let all planar graphs with fewer than n vertices be 6-colourable. Obviously, H has a vertex, say v, so that $d(v) \leq 5$. Therefore v has at most five neighbours in H and these neighbours evidently need at most five colours for colouring. The vertex deleted subgraph $H - v$ is planar with $n-1$ vertices and therefore by induction hypothesis is 6-colourable. Since at most five colours are used for colouring the neighbours of v, therefore assigning v the sixth colour not used by the its neighbours gives the 6-colouring of H. ❑

The following result is a consequence of Theorem 7.20.

Theorem 7.22 A planar graph G is 2-colourable if and only if it is an Euler graph.

Proof Let G be a planar graph which is 2-colourable. Then, if G^* is the geometric dual of G, we have $\chi(G^*) = 2$. Therefore G^* is bipartite and thus G^{**} (the dual of G^*) is an Euler graph. Since G and G^{**} are isomorphic, therefore G is an Euler graph.

Conversely, let G be an Euler graph. Then its double dual G^{**} is an Euler graph and thus G^* is bipartite. Therefore, $\chi(G^*) = 2$ and hence the planar graph G is 2-colourable. ❑

The next result due to Heawood [113] is called *five colour theorem* and Heawood used the Kempe chain argument in proving it.

Theorem 7.23 (Heawood) Every planar graph is 5-colourable.

Proof Let G be a planar graph with n vertices. We use induction on n, the order of G. The result is obvious for $n \leq 5$. So, let $n \geq 6$. Assume the result to be true for all planar graphs with fewer than n vertices.

Let G' be the graph obtained from G by deleting the vertex v and removing all the edges incident with v. The graph G' with order $n-1$ is clearly planar and by induction hypothesis is 5-colourable. Let the colours used to colour G' be c_1, c_2, c_3, c_4, c_5.

We know for a planar graph with $n \geq 6$ vertices, there exists a vertex, say v, such that $d(v) \leq 5$. Thus v has atmost five neighbours in G and all of these neighbours are the already coloured vertices in G'.

If in G' less than five colours are used to colour these neighbours, then the 5-colouring of G is obtained by using the colouring for G' on all vertices, and by colouring v with the colour not used to colour the neighbours of v (Fig. 7.16).

Let all the five colours be used in G', to colour the neighbours of v. This implies that there are exactly five neighbours of v, say u_1, u_2, u_3, u_4 and u_5. Assume without loss of generality that u_i is coloured with c_i, for each i, $1 \leq i \leq 5$.

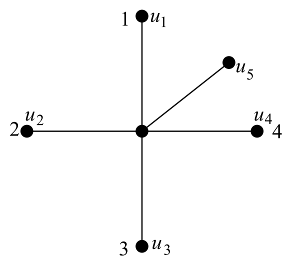


Fig. 7.16

Consider all the vertices of G' that are coloured with colour c_1 and c_3. If u_1 and u_3 are in different components of the Kempe subgraph $H(c_1, c_3)$ induced by those vertices coloured c_1 and c_3, then using the Kempe chain argument to interchange the colours c_1 and c_3 in the component containing u_1 leaves c_1 unused on the set $\{u_1, u_2, u_3, u_4, u_5\}$. We colour v by c_1 to get a 5-colouring of G.

Finally, if u_1 and u_3 are in the same component of $H(c_1, c_3)$, and u_2 and u_4 are in the same component of $H(c_2, c_4)$, then the $c_1 - c_3$ Kempe chain from u_1 to u_3 crosses the $c_2 - c_4$ Kempe chain from u_2 to u_4. This is impossible as the triangulation is a plane graph (Fig. 7.17). ❑

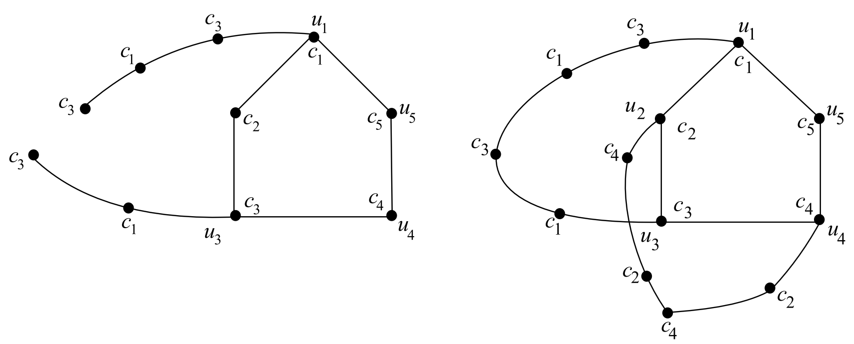


Fig. 7.17

The following result is due to Grunbaum [91].

Theorem 7.24 Every planar graph with fewer than four triangles is 3-colourable.

The next result due to Grotzsch [90] immediately follows from Theorem 7.24.

Theorem 7.25 Every planar graph without triangles is 3-colourable.

While making various attempts to solve the four colour problem, the problem got translated into several equivalent conjectures and sometimes conjectures were made which implied the four colour problem. The details of several such conjectures can be found in Saaty [229] and Saaty and Kainen [230]. Finch and Sachs [78] proved that every planar graph with at most 21 triangles is 4-colourable. Ore and Stemple [179] showed that all planar graphs with upto 39 regions are 4-colourable.

The following result is one such equivalence.

Theorem 7.26 Every planar graph is four colourable if and only if every cubic bridgeless planar graph is 4-colourable.

Proof We observe that every planar graph is four colourable if and only if every bridgeless planar graph (without cut edges) is four colourable. This is because if G has a bridge e and G' is obtained from G by contracting e, then $\chi''(G') = \chi''(G)$ (as the elementary contraction of identifying the end vertices of a bridge affects neither the number of regions in the planar graph nor the adjacency of any of the regions).

We now prove that every bridgeless planar graph is 4-colourable if and only if every cubic bridgeless planar graph is 4-colourable. If every bridgeless planar graph is 4-colourable, then evidently every cubic bridgeless planar graph is 4-colourable.

Conversely, let all cubic bridgeless planar graphs be 4-colourable. Let G be a bridgeless planar graph. We obtain G' from G by performing the following operations. In case G has a vertex v of degree two, let xv and yv be the edges incident with v. Subdivide xv at u and yv at w. Take two new vertices a and b and add the edges au, aw, bu, bw and ab. Now remove v (and uv and uw). In doing so, we have replaced v by a $K_4 - e$ and we see that each new vertex has degree three (Fig. 7.18).

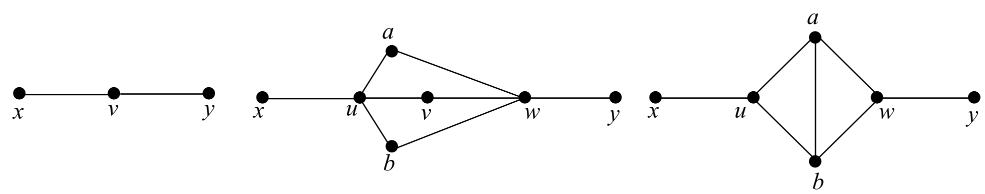


Fig. 7.18

If G has a vertex v so that $d(v) \geq 4$, let $vx_1, vx_2, \ldots, vx_d$ be the edges incident with v. Subdivide each vx_i producing a new vertex v_i, for $1 \leq i \leq d$. We then remove v and add the new edges v_1v_2, v_2v_3, $\ldots$, $v_{d-1}v_d$, v_dv_1. Here we have replaced v by C_d and again the degree of each new vertex is three (Fig. 7.19).

In both cases the graph G' formed is a bridgeless cubic graph. If these $K_4 - e$ and C_d introduced are contracted, we get the original graph G. Hence G is 4-colourable, since we have assumed that G' is 4-colourable. ❑

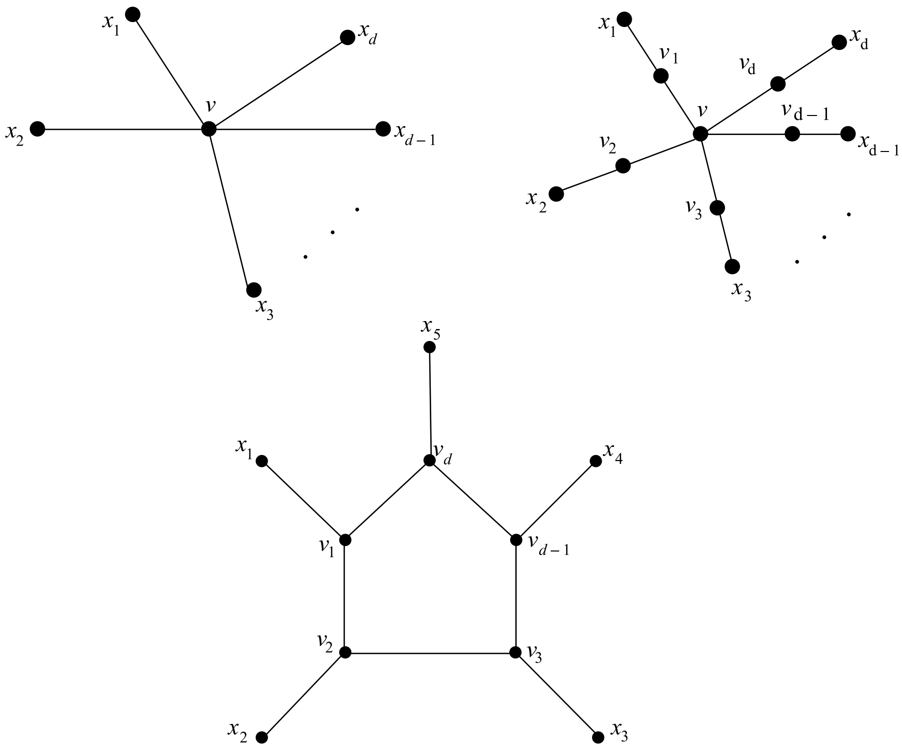


Fig. 7.19

The next result is due to Tait [238] and the details can also be found in Bondy and Chvatal [36] and Bollobas [29].

Theorem 7.27 (Tait) Every planar graph G is 4-colourable if and only if $\chi'(G) = 3$, for every bridgeless cubic planar graph G.

Proof As seen in Theorem 7.26, the statement that every planar graph is 4-colourable is equivalent to the statement that every cubic bridgeless planar graph is 4-colourable. Therefore, to prove the result, we prove that a cubic bridgeless planar graph G is 4-colourable if and only if $\chi'(G) = 3$.

Now assume that G is a bridgeless cubic planar graph which is 4-colourable. For the set of colours we choose the elements of the Klein four group $Q = \{c_0, c_1, c_2, c_3\}$, where addition in Q is defined by $c_i + c_i = c_0$ and $c_1 + c_2 = c_3$, with c_0 being the identity element.

Now, define the colour of an edge to be the sum of the colours of two distinct regions which are incident with that edge. We see that the edges are coloured with elements of the set $\{c_1, c_2, c_3\}$ and that no two adjacent edges get the same colour. Hence, $\chi'(G) = 3$.

Conversely, let G be a bridgeless cubic planar graph with $\chi'(G) = 3$ and colour its edges with the three non-zero elements of Q. Consider a region, say R_0, and give the colour c_0 to it. Let R be any other region of G and let C be any curve in the plane joining the interior of R_0 with the interior of R, so that C does not pass through a vertex of G. Then the colour of R is defined to be the sum of the colours of those edges which intersect C.

That the colours of the regions are well defined follows from the fact that the sum of the colours of the edges which intersect any simple closed curve not passing through a vertex of G is c_0. Suppose S is such a curve and assume $q_1, q_2, \ldots, q_n$ to be the colours of the edges which intersect S. Also assume $r_1, r_2, \ldots, r_m$ be the colours of those edges interior to S. If $c(v)$ denotes the sum of the colours of the three edges incident with v, then we see that $c(v) = c_0$. Thus, for all vertices v interior to S, we have $\sum c(v) = c_0$. While on the other hand, we have $\sum c(v) = q_1 + q_2 + \ldots + q_n + 2(r_1 + r_2 + \ldots + r_m) = q_1 + q_2 + \ldots + q_n$ as every element of Q is self-inverse. Thus, $q_1 + q_2 + \ldots. + q_n = c_o$. Hence we get the 4-colouring of the regions of G and the colours used are c_0, c_1, c_2, c_3. ❑

Remarks Because of Theorem 7.27, a three colouring of the edges of a cubic graph is called a *Tait colouring*.

In an attempt to solve the four colour problem, Tait considered edge colourings of bridgeless cubic planar graphs and proved that every such graph is 3-edge colourable. In 1880, Tait tried to give a proof of the four colour problem by using Theorem 7.27 and based on the wrong assumption that any bridgeless cubic planar graph is Hamiltonian. A counter example called the Tuttle graph to Tait's assumption was given by Tutte in 1946 and is shown in Figure 7.20.

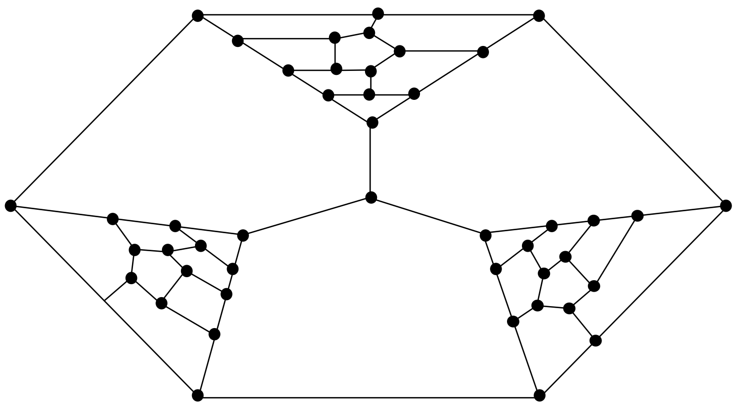


Fig. 7.20

7.6 Heawood Map-Colouring Theorem

Let S_p be the orientable surface of genus p, so that S_p is topologically equivalent to a sphere with p handles. The chromatic number of S_p, denoted by $\chi(S_p)$, is the maximum chromatic number among all graphs which can be embedded on S_p. The surface S_o is clearly the sphere. The four colour problem states that $\chi(S_0) = 4$.

Definition: The genus $\gamma(G)$ of a graph is the minimum number of handles which must be added to a sphere so that G can be imbedded on the resulting surface. Clearly, $\gamma(G) = 0$ if and only if G is planar. Further, for a polyhedron, $n - m + f = 2 - 2\gamma$.

For $n \geq 3$, the genus of the complete graph K_n is

$$\gamma(K_n) = \frac{(n-3)(n-4)}{12}.$$

The following inequality is due to Heawood [113].

Theorem 7.28 The chromatic number of the orientable surface of positive genus p has the upper bound

$$\chi(S_p) \leq \frac{7 + \sqrt{1 + 48p}}{2},\ p > 0.$$

Proof Let G be an $(n,\ m)$ graph embedded on S_p. Since any graph can be augmented to a triangulation of the same genus by adding edges without reducing χ, we assume G to be a triangulation. Let d' be the average degree of the vertices of G. Then n, m and f (the number of regions) are related by the equations

$$d'n = 2m = 3f.$$

Solving for m and f in terms of n and using Euler's equation $n - m + f = 2 - 2\gamma$, we get

$$d' = \frac{12(p-1)}{n} + 6. \qquad (7.28.1)$$

As $d' \leq n - 1$, this gives $n - 1 \geq \dfrac{12(p-1)}{n} + 6$.

Solving for n and taking the positive square root, we obtain $n \geq \dfrac{7 + \sqrt{1 + 48p}}{2}$.

Let $H(p) = \dfrac{7 + \sqrt{1 + 48p}}{2}$. Then we show that $H(p)$ colours are sufficient to colour the vertices of G. If $n = H(p)$, obviously we have sufficient colours. In case $n > H(p)$, we substitute $H(p)$ for n in (7.28.1) and obtain

$$d' < \frac{12(p-1)}{H(p)} + 6 = H(p) - 1.$$

Therefore, when $n > H(p)$, there is a vertex v of degree at most $H(p) - 2$. Identify v and any adjacent vertex by an elementary contraction to obtain a new graph G'. If $n' = n - 1 = H(p)$, then G' can be coloured in $H(p)$ colours. If $n' > H(p)$, repeat the argument and evidently we get an $H(p)$ colourable graph. It is then easy to see that the colouring of this graph induces a colouring of the preceding one in $H(p)$ colours, and so forth, so that G itself is $H(p)$−colourable. Hence the result follows. ❑

The next result is called Heawood map colouring theorem, the proof of which is due to Ringel and Youngs [223].

Theorem 7.29 (Heawood map-colouring theorem) For every positive integer p, the chromatic number of the orientable surface of genus p is given by

$$\chi(S_p) = \frac{7 + \sqrt{1+48p}}{2}, \; p > 0.$$

Proof Let G be an (n, m) graph embedded on S_p. Then,

$$\chi(S_p) \leq \frac{7 + \sqrt{1+48p}}{2}, p > 0.$$

Now, if the complete graph K_n is embedded in S_p, then

$$p \geq \gamma(K_n) = \frac{(n-3)(n-4)}{12}. \tag{7.29.1}$$

Setting n to be the largest integer satisfying (7.29.1), we have

$$\frac{(n-3)(n-4)}{12} \leq p \leq \left[\frac{(n-2)(n-3)}{12}\right] - 1 < \frac{(n-2)(n-3)}{12}.$$

Solving for n, we get

$$\frac{5 + \sqrt{1+48p}}{2} < n \leq \frac{7 + \sqrt{1+48p}}{2}$$

Thus, $n = \dfrac{7 + \sqrt{1+48p}}{2}$.

Since $\chi(K_n) = n$, we have found a graph with genus p and chromatic number equal to $H(p)$. This shows that $H(p)$ is a lower bound for $\chi(Sp)$, completing the proof. ❑

7.7 Uniquely Colourable Graph

A graph $G(V, E)$ is said to be *uniquely k-vertex-colourable* (or uniquely k-colourable) if there is a unique k-part partition of the vertex set V into independent subsets. That is, in the uniquely k-colouring of a graph G, every k-colouring of G induces the same partition of V. The graph of Figure 7.21(a) is uniquely 3-colourable, since every 3-colouring of G has the partition $\{v_1\}$, $\{v_2, v_4\}$, $\{v_3, v_5\}$. The graph of the Figure 7.21(b) is not uniquely 3-colourable. A pentagon is not uniquely 3-colourable, as five different partitions of its vertex set are possible.

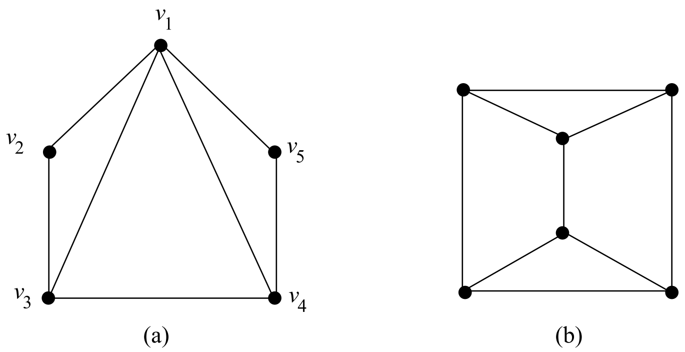


Fig. 7.21

We observe that the empty graphs $\overline{K_n}$ are the only uniquely 1-colourable graphs and the connected bipartite graphs are the only uniquely 2-colourable graphs. We note that unique k-colourability is not defined for $k > n$. Further, we see that K_n is uniquely n-colourable. These observations imply to assume $3 \le k \le n$, for studying uniquely k-colourable graphs.

We have the following observation.

Theorem 7.30 If G is uniquely k-colourable, then $\delta(G) \ge k-1$.

Proof Let G be a uniquely k-colourable graph. Then every vertex v of G is adjacent to at least one vertex of every colour different from that assigned to v. For otherwise, a different k-colouring of G is obtained by recolouring v. This implies that $d(v) \ge k-1$, for every v. ❑

Corollary 7.2 If G is a uniquely k-colourable graph with n vertices and m edges, then $2m \ge n(k-1)$.

The next result, due to Cartwright and Harary [45], gives a necessary condition for a graph to be uniquely colourable.

Theorem 7.31 The subgraph induced by the union of any two colour classes in a k-colouring of a uniquely k-colourable graph is connected.

Proof Let G be a uniquely k-colourable graph. Let C_1 and C_2 be two classes in the k-colouring of the graph G. Assume the subgraph S of G induced by $C_1 \cup C_2$ to be disconnected, and let S_1 and S_2 be components of S. Then each of S_1 and S_2 contain vertices of both C_1 and C_2. Now interchanging the colour of the vertices in $C_1 \cap S_1$ by the colour of the vertices in $C_2 \cap S_1$ gives a different k-colouring of G. This contradicts the hypothesis that G is uniquely k-colourable. Hence S is connected. ❑

The converse of Theorem 7.31 is not true. To see this, consider the 3-chromatic graph G of Fig. 7.21(b). The graph G has the property that in any 3-colouring, the subgraph induced by the union of any 2 colour classes is connected. But G is not uniquely colourable. It follows from Theorem 7.30 that every uniquely k-colourable graph, $k \geq 2$, is connected.

The following stronger result is due to Chartrand and Geller [49].

Theorem 7.32 Any uniquely k-colourable graph is $(k-1)$-connected.

Proof Let G be a uniquely k-colourable graph. In case G is K_k, then it is $(k-1)$-connected. Now assume G to be an incomplete graph which is not $(k-1)-$connected. Therefore there exists a set U of $k-2$ vertices whose removal disconnects G. Then in any k-colouring of G, there are at least two distinct colours, say c_1 and c_2, not assigned to any vertex of U. By Theorem 7.30, a vertex coloured c_1 is connected to any vertex coloured c_2 by a path all of whose vertices are coloured c_1 or c_2. Therefore the set of vertices of G coloured c_1 or c_2 lies within the same component of $G-U$, say G_1. Another k-colouring of G can thus be obtained by taking any vertex of $G-U$ which is not in G_1 and recolouring it with either c_1 or c_2. This contradicts the hypothesis that G is uniquely k-colourable. Hence G is $(k-1)$-connected. ❑

Corollary 7.3 In any k-colouring of uniquely k-colouring graph, the subgraph induced by the union of any h colour classes, $2 \leq h \leq k$, is $(h-1)$-connected.

The following result due to Bollobas [30] gives a sufficient lower bound for uniquely colourable graphs.

Theorem 7.33 If G is a k-colourable graph of order $n(k \geq 2)$ with $\delta(G) > n(3k-5)/(3k-2)$, then G is uniquely k-colourable.

The following result is due to Harary, Hedetniemi and Robinson [107].

Theorem 7.34 For all $k \geq 3$, there is a uniquely k-colourable graph which contains no subgraph isomorphic to K_k.

Clearly, a graph is uniquely 1-colourable if and only if it is 1-colourable, that is, totally disconnected. Also, a graph is uniquely 2-colourable if and only if G is 2-chromatic and connected.

The converse of Theorem 7.36 is not true. This is because a uniquely 3-colourable planar graph may have more than one region which is not a triangle, as shown in Figure 7.22.

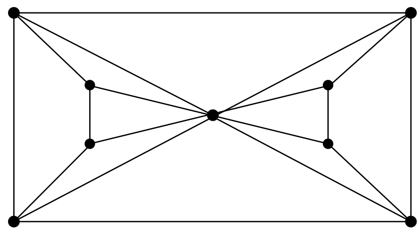


Fig. 7.22 *Uniquely 3-colourable planar graph*

Theorem 7.35 If G is a uniquely 3-colourable planar graph with at least four vertices, then G contains at least two triangles.

Theorem 7.36 Every uniquely 4-colourable planar graph is maximal planar.

Proof Let a 4-colouring be given to uniquely 4-colourable planar graph G with the colour classes denoted by $V_i, 1 \leq i \leq 4$, where $|V_i| = n_i$. Since the subgraph induced by $V_i \cup V_j, i \neq j$, is connected, G has at least $\sum(n_i + n_j - 1)$ edges, $1 \leq i < j < 4$. Clearly, $\sum(n_i + n_j - 1) = n_1 + n_2 - 1 + n_1 + n_3 - 1 + n_1 + n_4 - 1 + n_2 + n_3 - 1 + n_2 + n_4 - 1 + n_3 + n_4 - 1 = 3(n_1 + n_2 + n_3 + n_4) - 6 = 3n - 6$. Therefore, $m \geq 3n - 6$. Hence G is maximal planar. ❑

The next result for uniquely 5-colourable graphs is due to Hedetniemi [114].

Theorem 7.37 No planar graph is uniquely 5-colourable.

Theorem 7.38 A necessary and sufficient condition that a connected planar graph is 4-colourable is that G be the sum of three subgraphs G_1, G_2 and G_3 such that for each vertex v, the number of edges of each G_i incident with v are all even or odd.

The following equivalence is due to Whitney [264].

Theorem 7.39 The four colour problem holds if and only if every Hamiltonian planar graph is 4-colourable.

7.8 Hajos Conjecture

Hajos [97] made the following conjecture.

If a graph is k-chromatic, then it contains a subdivision of K_k.

When $k = 1$, or 2, the conjecture trivially holds. As for $k = 3$, every chromatic graph contains an odd cycle which is a subdivision of K_3, therefore proving the validity of the conjecture. The validity of the conjecture for $k = 4$, as noted in Parthasarthy [180] is due to Dirac [66].

Theorem 7.40 Hajos conjecture is true for $k = 4$.

Proof Assume without loss of generality that G is a 4-minimal graph. So G is a block and $\delta = 3$. In case $n = 4$, G is K_4 and the result is obvious. Therefore assume $n \geq 5$. We induct on n.

Let G have a 2-vertex cut $S = \{u, v\}$. Then by Theorem 7.5, $G = G_1 \cup G_2$, where G_1 is of type 1, and G_2 is of type 2 and $G_1 + uv$ is 4-minimal. By induction hypothesis, $G_1 + uv$ contains a subdivision of K_4. Here we replace uv by a $u - v$ path P in G_2 and so $G_1 \cup P$ contains a subdivision of K_4. Thus G also contains a subdivision of K_4. Now, let G be 3-connected. Since $\delta(G) \geq 3$, therefore it has a cycle C of length at least 4. If u and v are any two non-consecutive vertices on C, then $G - \{u, v\}$ is still connected and therefore there exist vertices x, y on C, and an $x - y$ path P in $G - \{u, v\}$. Similarly, there exists a $u - v$ path Q in $G - \{x, y\}$. If P and Q have no vertices in common, then $C \cup P \cup Q$ is a subdivision of K_4 in G. Otherwise, let w be the first vertex of P on Q, then $C \cup P_{xw} \cup Q$ is a subdivision of K_4 in G.

Hence in all cases G contains a subdivision of K_4. ❑

For $k \geq 5$, Hajos conjecture implies four colour problem. This is because if G is a planar graph which is not colourable by 4 colours, its chromatic number is at least 5 and thus contains a subdivision of K_5, and so cannot be planar, a contradiction. The four colour theorem implies that a 5-chromatic graph contains a homeomorph of K_5 or $K_{3,3}$. For $k = 5$, Hajos conjecture makes the stronger assertion that it contains a homeomorph of K_5. For $k = 5$, or 6, the conjecture has not been settled, but for $k = 7$, it is disproved by Catlin. The counter example of Catlin is the graph $H = L(3C_5) - \{v_1, v_2\}$, where $3C_5$ is the multigraph obtained from C_5 by replacing each edge by three edges, L represents the edge graph, and v_1 and v_2 are any two non-adjacent vertices of $L(3C_5)$ (Fig. 7.23).

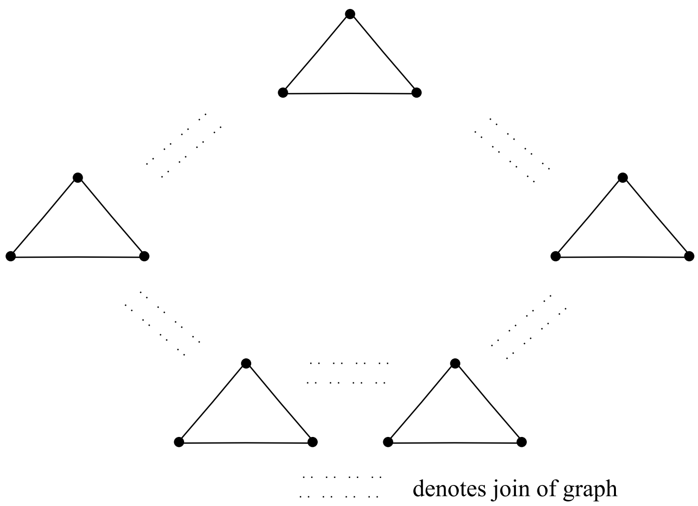


Fig. 7.23

The largest integer for which a given graph G contains a TK_n is called the subdivision number (or topological clique number or Hajos number) of G and is denoted by $tw(G)$. With this, Hajos conjecture is equivalent to $tw(G) \geq \chi(G)$. In the above example, we see that H contains a K_6 and from any vertex outside this K_6 there are no six internally disjoint paths to the vertices of the K_6. Therefore, $tw(H) = 6$. A maximum independent set of H has cardinality two, so that $\chi(H) \geq \left\lceil \frac{13}{2} \right\rceil = 7$ and a 7-colouring of H is shown in the Fig. 7.23. Thus, $\chi(H) = 7$ and hence H is a counter example for Hajos conjecture. We now observe that if G is a counter example for Hajos conjecture for k, then $G+v$ is a counter example for Hajos conjecture for $k+1$. This can be seen from the fact that $tw(G+v) = tw(G)+1$ and $\chi(G+v) = \chi(G)+1$. Hence Hajos conjecture is false for all $k = 7$. Erdos and Fajtlowicz [72] proved that almost every graph is a counter example to Hajos conjecture. Bollobas and Catlin [31] proved that $tw(G)$ is approximately $2\sqrt{n}$ for n-vertex graphs.

The following conjecture involving contractions is due to Hadwiger [95].

Hadwiger's conjecture [95] Every k-chromatic graph contains K_k as a subcontraction.

Hadwiger's conjecture is trivially true for $k = 1$. Since 2-chromatic graphs are the bipartite graphs and 3-chromatic graphs contain an odd cycle, contractible to K_3, the conjecture is true for $k = 2$ and 3. Dirac [66] proved the conjecture for $k = 4$. For $k = 5$, this conjecture states that every 5-chromatic graph is contractible to K_5 and therefore every such graph is non-planar. Thus Hadwiger's conjecture for $k = 5$ implies the four colour problem. The converse of this is given by Wagner.

7.9 Exercises

1. Prove that $\chi(G) = \triangle(G)+1$ if and only if G is either a complete graph or a cycle of odd length.
2. Show that if G contains exactly one odd cycle, then $\chi(G) = 3$.
3. If G is a graph in which any pair of odd cycles have a common vertex, then prove that $\chi(G) \leq 5$.
4. Find the chromatic number of the Peterson graph and the Birkhoff Diamond.
5. Determine the chromatic number of the graphs in Figure 7.24.

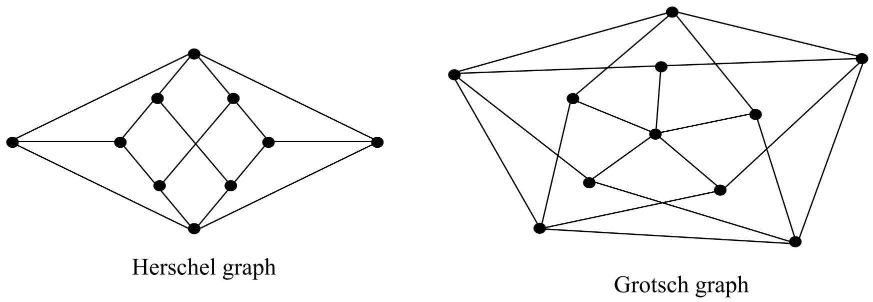


Fig. 7.24

6. If G is k-regular, prove that $\chi(G) \geq \dfrac{n}{n-k}$.
7. If G is connected and $m \leq n$, show that $\chi(G) \leq 3$.
8. Prove that the 3-critical graphs are the odd cycles C_{2n+1}.
9. Prove that the wheel W_{2n-1} is a 4-critical graph for each $n \geq 2$.
10. Prove that the wheel W_{2n} is a 4-critical graph for each $n \geq 2$.
11. Prove that the Peterson graph has edge chromatic number 4.
12. If $m(G)$ is the number of edges in a longest path of G, prove that $\chi(G) \leq 1 + m(G)$.
13. Show that if G is 3-regular Hamiltonian graph, then $\chi'(G) = 3$.
14. Show that a triangulation with a vertex of degree 2 or 3 can be coloured with five colours.
15. Prove that for every $k \geq 1$ there is a k-chromatic graph M_k with no triangle subgraphs. (Mycielski, 1955).
16. If G is a graph in which no set of four vertices induces P_4 as a subgraph, then prove that $\chi(G) = cl(G)$ (Seinsche, 1974).
17. Obtain proofs of Theorems 2.24 and 2.25.
18. Show that a uniquely 3-colourable graph contains three Hamiltonian cycles.