6. Planarity

Let $G(V,E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Let S be any surface (like the plane, sphere) and $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n distinct points of S, p_i corresponding to v_i, $1 \le i \le n$. If $e_i = v_j v_k$, draw a Jordan arc J_i on S from p_j to p_k such that J_i does not pass through any other p_i. Then $P \cup \{J_1, J_2, \ldots, J_m\}$ is called a *drawing* of G on S, or a diagram representing G on S. The p_i are called the *points* of the diagram and J_i, the *lines* of the diagram.

An *embedding* of a graph G on a surface S is a diagram of G drawn on the surface such that the Jordan arcs representing any two edges of G do not intersect except at a point representing a vertex of G.

A graph is *planar* if it has an embedding on the plane. A graph which has no embedding on the plane is *nonplanar*. That is, a graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect and a graph that cannot be drawn on a plane without a crossover between its edges is called nonplanar.

In order that a graph G is nonplanar, we have to show that of all possible geometric representations of G, none can be embedded in a plane. Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane.

An embedding of a planar graph G on a plane is called a *plane representation* of G. Figure 6.1 shows three diagrams of the same graph which is planar. The two graphs in Figure 6.2 represent the same planar graph.

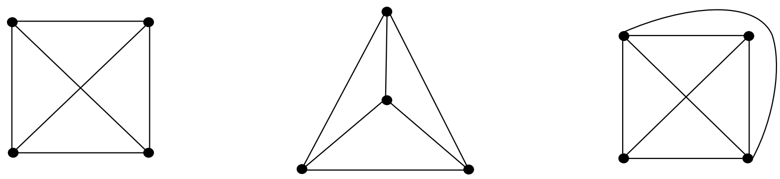


Fig. 6.1

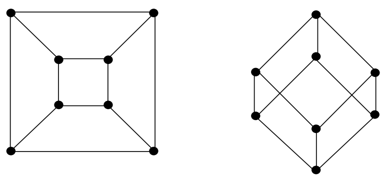


Fig. 6.2

6.1 Kuratowski's Two Graphs

The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called Kuratowski's graphs, after the polish mathematician Kasimir Kurtatowski, who found that K_5 and $K_{3,3}$ are nonplanar.

Theorem 6.1 The complete graph K_5 with five vertices is nonplanar.

Proof Let the five vertices in the complete graph be named v_1, v_2, v_3, v_4, v_5. Since in a complete graph every vertex is joined to every other vertex by means of an edge, there is a cycle $v_1v_2v_3v_4v_5v_1$ that is a pentagon. This pentagon divides the plane of the paper into two regions, one inside and the other outside, Figure 6.3(a).

Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose we choose to draw the line from v_1 to v_3 inside the pentagon, Figure 6.3(b). In case we choose outside, we end with the same argument. Now we have to draw an edge from v_2 to v_4 and another from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon, Figure 6.3(c). The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4. Therefore v_3 and v_5 have to be connected with an edge inside the pentagon, Figure 6.3(d). Now, we have to draw an edge between v_1 and v_4 and this cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane. ❑

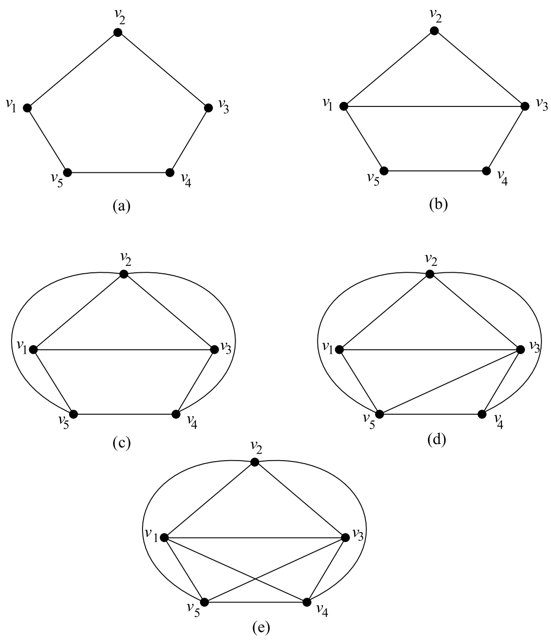
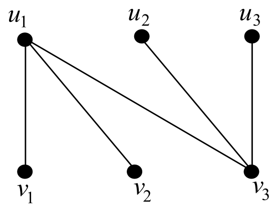


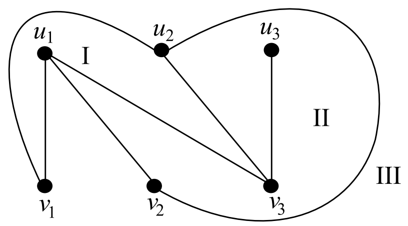
Fig. 6.3

Theorem 6.2 The complete bipartite graph $K_{3,3}$ is nonplanar.

Proof The complete bipartite graph has six vertices and nine edges. Let the vertices be u_1, u_2, u_3, v_1, v_2, v_3. We have edges from every u_i to each v_i, $1 \le i \le 3$. First we take the edges from u_1 to each v_1, v_2 and v_3. Then we take the edges between u_2 to each v_1, v_2 and v_3, Figure 6.4(a). Thus we get three regions namely I, II and III. Finally we have to draw the edges between u_3 to each v_1, v_2 and v_3. We can draw the edge between u_3 and v_3 inside the region II without any crossover, Figure 6.4(b). But the edges between u_3 and v_1, and u_3 and v_2 drawn in any region have a crossover with the previous edges. Thus the graph cannot be embedded in a plane. Hence $K_{3,3}$ is nonplanar. ❑



(a)



(b)

Fig. 6.4

We observe that the two graphs K_5 and $K_{3,3}$ have the following common properties.

1. Both are regular.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. K_5 is a nonplanar graph with the smallest number of vertices, and $K_{3,3}$ is the nonplanar graph with smallest number of edges.

Thus both are the simplest nonplanar graphs.

The following result given independently by Fary [77] and Wagner [260] implies that there is no need to bend edges in drawing a planar graph to avoid edge intersections.

Theorem 6.3 (Fary) Every triangulated planar graph has a straight line representation.

Proof The proof is by induction on the number of vertices. The result is obvious for $n = 4$. So, let $n \ge 5$ and assume that the result is true for all planar graphs with fewer than n vertices. Let G be a plane graph with n vertices.

First, we show that G has an edge e belonging to just two triangles. For this, let x be any vertex in the interior of a triangle T and choose x and T such that the number of regions inside T is minimal.

Let y be a neighbour of x, and the edge xy lies inside T, and let xy belong to three triangles xyz_1, xyz_2 and xyz_3. Then one of these triangles lies completely inside another. Assume that z_3 lies inside xyz_1. Then z_3 and xyz_1 contradict the choice of x and T (Fig. 6.5).

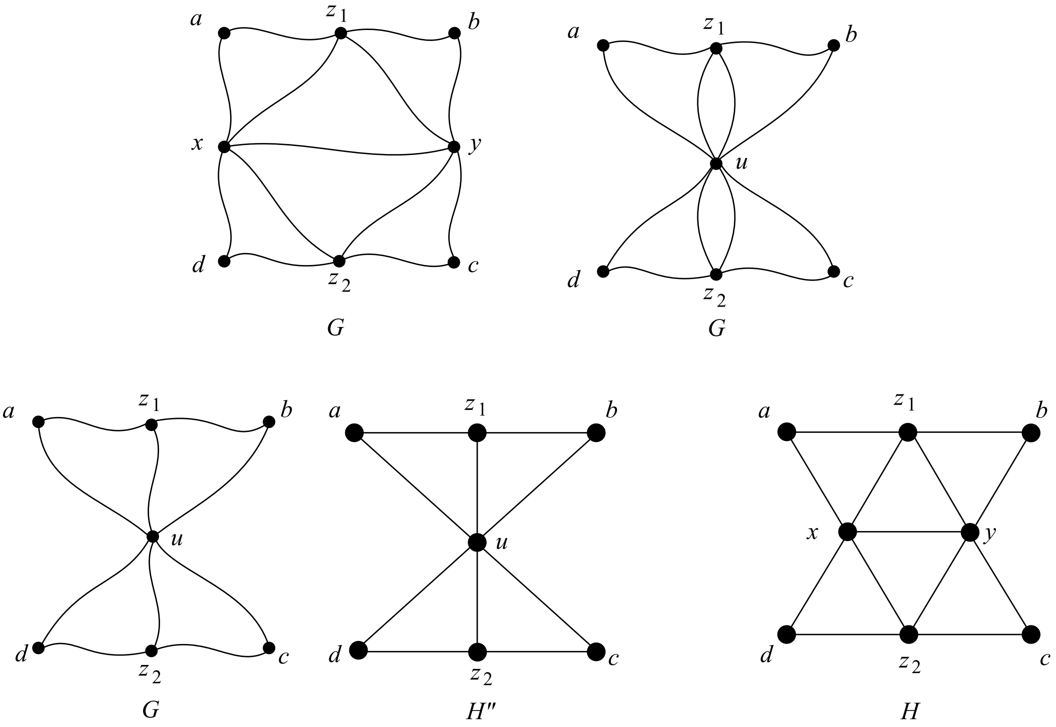


Fig. 6.5

Thus there is an edge $e = xy$ lying in just two triangles xyz_1 and xyz_2. Contracting xy to a vertex u, we get a new graph G' with a pair of double edges between u and z_1, and u and z_2. Remove one each of this pair of double edges to get a graph G'' which is a triangulated graph with $n-1$ vertices. By the induction hypothesis, it has a straight line representation H''. The edges of G''correspond to uz_1, uz_2 in H''. Divide the angle around u into two parts in one of which the pre-images of the edges adjacent to x in G lie, and in the other, the pre-images of the edges adjacent to y in G. Thus u can be pulled apart to x and y, and the edge xy is restored by a straight line to get a straight line representation of G. ❑

6.2 Region

A plane representation of a graph divides the plane into regions (also called windows, faces or meshes). A region is characterised by the set of edges (or the set of vertices) forming its boundary. We note that a region is not defined in a nonplanar graph, or even in a planar graph not embedded in a plane. Thus a region is a property of the specific plane representation of a graph and not of an abstract graph.

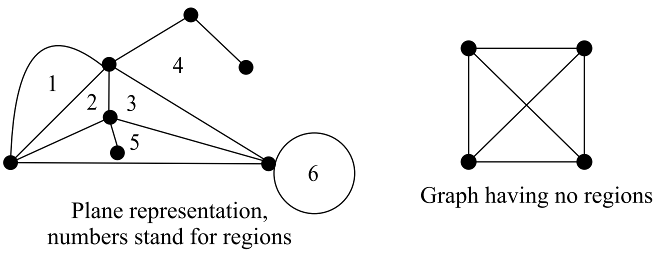


Fig. 6.6

Infinite region: The portion of the plane lying outside a graph embedded in a plane such as region 4 in the graph given in Figure 6.6, is infinite in its extent. Such a region is called an infinite, unbounded, outer or exterior region for that particular plane representation. Like other regions, the infinite region is also characterised by a set of edges (or vertices). Clearly, by changing the embedding of a given planar graph, we can change the infinite region. Consider the graphs of Figure 6.7. In the two embeddings of the same graph, finite region $v_1v_3v_5$ in (a) is infinite in (b).

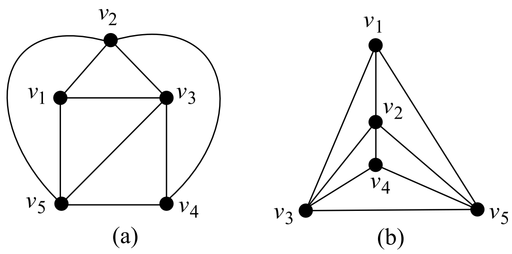


Fig. 6.7

Embedding on a sphere: A graph is spherical if it can be embedded on the surface of a sphere.

The following result implies that a planar graph can often be embedded on the surface of a sphere, so that there is no distinction between finite and infinite region.

Theorem 6.4 A graph can be embedded on the surface of a sphere if it can be embedded in a plane.

Proof Consider the stereographic projection of a sphere on the plane. Put the sphere on the plane and call the point of contact as SP (south-pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north-pole).

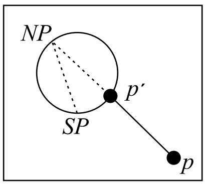


Fig. 6.8

Now, corresponding to any point p on the plane, there exists a unique point p' on the sphere, and vice versa, where p' is the point where the straight line from point p to point NP intersects the surface of the sphere. Thus there is a one-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane corresponding to the point NP on the sphere.

Therefore from this construction, it is clear that any graph that can be embedded in a plane can also be embedded on the surface of the sphere, and vice versa. ❑

We now have the following result.

Theorem 6.5 A planar graph can be embedded in a plane such that any specified region, specified by the edges forming it, can be made the infinite region.

Proof A planar graph embedded on the surface of the sphere divides the surface of the sphere into different regions. Each region of the sphere is finite, the infinite region on the plane having been mapped onto the region containing the point NP. Clearly, by suitably rotating the sphere, we can make any specified region map onto the infinite region on the plane. Hence the result. ❑

Remark Thinking in terms of the regions on the sphere, we observe that there is no real difference between the infinite region and the finite regions on the plane. Therefore, when we discuss the regions in a plane representation of a graph, we include the infinite region. Also, since there is no essential difference between an embedding of a planar graph on a plane, or on a sphere (a plane can be regarded as the surface of the sphere of infinitely large radius), the term plane representation of a graph is often used to include spherical as well as plane embedding.

6.3 Euler's Theorem

The following important result due to Euler gives a relation between the number of vertices, edges, regions and the components of a planar graph.

Theorem 6.6 If G is a planar graph with n vertices, m edges, f regions and k components, then

$$n-m+f=k+1. \tag{6.6.1}$$

Proof We construct the graph G by the addition of successive edges starting from the null graph $\overline{K_n}$. For this starting graph, $k = n$, $m = 0$, $f = 1$, so that (6.6.1) is true.

Let G_{i-1} be the graph at the start of ith stage and G_i be the graph obtained from G_{i-1} by addition of the ith edge e. If e connects two components of G_{i-1}, then f is not altered, m is increased by 1 and k is reduced by 1, so that (6.6.1) holds for G_i as it holds for G_{i-1}. If e joins two vertices of the same components of G_{i-1}, k is unaltered, m is increased by 1 and f is increased by 1, so that again (6.6.1) holds for G_i. ❑

The following relation between the number of vertices, edges and regions is the discovery of Euler [75] and is also called as Euler's formula for planar graphs.

Theorem 6.7 (Euler) In a connected planar graph with n vertices, m edges and f regions (faces), $n - m + f = 2$.

(The proof can also be deduced from Theorem 6.6 by taking $k = 1$, as the connected graph has one component.)

Proof Without loss of generality, assume that the planar graph is simple. Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon (a polygon net). Let the polygon net representing the given graph consist of f regions. Let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3k_3 + 4k_4 + 5k_5 + \ldots + rk_r = 2m, \tag{6.7.1}$$

where k_r is the number of polygons with r edges.

Also, $k_3 + k_4 + k_5 + \ldots + k_r = f$. (6.7.2)

The sum of all angles subtended at each vertex in the polygon net is $2\pi n$. (6.7.3.)

Now, the sum of all interior angles of a p-sided polygon is $\pi(p-2)$ and the sum of the exterior angles is $\pi(p+2)$. The expression in (6.7.3) is the total sum of all interior angles of $f - 1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\begin{aligned}
&\pi(3-2)k_3 + \pi(4-2)k_4 + \ldots + \pi(r-2)k_r + 4\pi \\
&\quad = \pi[3k_3 + 4k_4 + \ldots + rk_r + 2(k_3 + k_4 + \ldots + k_r)] + 4\pi \\
&\quad = \pi(2m - 2f) + 4\pi = 2\pi(m - f + 2).
\end{aligned} \tag{6.7.4}$$

Equating (6.7.3) and (6.7.4) we get

$2\pi(m - f + 2) = 2n\pi$, so that $f = m - n - 2$. ❑

Definition: Let ϕ be a region of a planar graph G. We define the *degree* of ϕ, denoted by $d(\phi)$, as the number of edges on the boundary of ϕ.

Second proof of Theorem 6.7 We use induction on m, the number of edges. If $m = 0$, then G is K_1, a graph with 1 vertex and 1 region. So, $n - m + f = 1 - 0 + 1 = 2$, and the result is true. If $m = 1$, then the number of vertices in G is either one or two, the first possibility occurring when the edge is a loop. These two possibilities give rise respectively to two regions and one region.

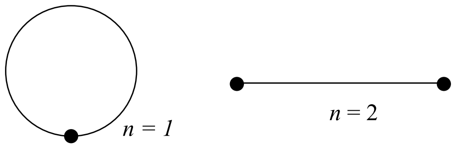


Fig. 6.9

Therefore, $n - m + f = \begin{cases} 1 - 1 + 2, & \textit{in the loop case}, \\ 2 - 1 + 1, & \textit{in the non loop case}. \end{cases}$

Thus, $n - m + f = 2$, and again the result is true.

Assume that the result is true for all connected planar graphs with fewer than m edges. Let G have m edges.

Case I Suppose G is a tree. Then $m = n - 1$ and $f = 1$, because a planar representation of a tree has only one region. Thus, $n - m + f = n - (n - 1) + 1 = 2$, and the result holds.

Case II Suppose G is not a tree. Then G has cycles. Let C be a cycle in G. Let e be an edge of C. The graph $G - e$ has one edge less than the graph G. Also the number of vertices in $G - e$ and G are same. Since removing e coalesces two regions in G into one in $G - e$, therefore $G - e$ has one region less than that in G. Thus by induction hypothesis in $G - e$, we have $n - (m - 1) + (f - 1) = 2$, so that $n - m + f = 2$. Hence the result. ❑

The next result gives an upper bound for the edges of a simple planar graph.

Theorem 6.8 Let G be a simple planar graph with n vertices and m edges, where $n \geq 3$. Then $m \leq 3n - 6$.

Proof First assume that the planar graph G is connected. If $n = 3$, then since G is simple, therefore G has at most three edges. Thus $m \leq 3$, that is, $m \leq 3 \times 3 - 6$, and the result is true.

Now, let $n = 4$. First let G be a tree so that $m = n - 1$. Since $n \geq 4$, obviously we have $n - 1 \leq 3n - 6$.

Now, let G be not a tree. Then G has cycles. Clearly, there is a cycle in G, all of whose edges lie on the boundary of the exterior region of G. Then, since G is simple, we have $d(\phi) \geq 3$, for each region ϕ of G.

Let $b = \sum_{\phi \in F} d(\phi)$, where F denotes the set of all regions of G.

Since each region has at least three edges on its boundary, therefore we have $b \geq 3f$, where f is the number of regions of G. However, when we sum to get b, each edge of G is counted either once or twice (twice when it occurs as a boundary edge for two regions). Therefore, $b \leq 2m$ so that $3f \leq b \leq 2m$. In particular, we have $3f \leq 2m$.

Using Euler's formula $n - m + f = 2$, we get $m \leq 3n - 6$.

Let G be not connected and let $G_1, G_2, \ldots, G_t$ be its connected components. For each i, $1 \leq i \leq t$, let n_i and m_i denote the number of vertices and edges in G_i. Since each G_i is a planar simple graph, by the above argument, we have $m_i \leq 3n_i - 6$, for each i, $1 \leq i \leq t$.

Also, $n = \sum_{i=1}^{t} n_i$ and $m = \sum_{i=1}^{t} m_i$.

Hence, $m = \sum_{i=1}^{t} m_i \leq \sum_{i=1}^{t} (3n_i - 6) = 3 \sum_{i=1}^{t} n_i - 6t \leq 3n - 6$. ❑

The following result gives the existence of a vertex of degree less than six in a simple planar graph.

Theorem 6.9 If G is a simple planar graph, then G has a vertex v of degree less than 6.

Proof If G has only one vertex, then this vertex has degree zero. If G has only two vertices, then both vertices have degree at most one.

Let $n \geq 3$. Assume degree of every vertex in G is at least six.

Then, $\sum_{v \in V(G)} d(v) \geq 6n$.

We know $\sum_{v \in V(G)} d(v) \geq 2m$.

Thus, $2m \geq 6n$ so that $m \geq 3n$.

This is not possible because, by Theorem 6.8, we have $m \leq 3n - 6$. Thus we get a contradiction. Hence G has at least one vertex of degree less than 6. ❑

Corollary 6.1 K_5 is nonplanar.

Proof Here $n = 5$ and $m = 10$. So, $3n - 6 = 15 - 6 = 9$. Thus $m > 3n - 6$. Therefore K_5 is nonplanar.

Corollary 6.2 $K_{3,3}$ is nonplanar.

Proof Since $K_{3,3}$ is bipartite, it contains no odd cycles, and so no cycle of length three. It follows that every region of a plane drawing of $K_{3,3}$ if it exists, has at least four boundary edges. We have $d(\phi) \geq 4$, for each region ϕ of G.

Let $b = \sum\limits_{\phi \in F} d(\phi)$, where F denotes the set of all regions of G. Since each region has at least four edges on its boundary, we have $b \geq 4f$, where f is the number of regions of G.

Now, when we sum up to get b, each edge of G is counted either once or twice, and so $b \leq 2m$. Thus, $4f \leq b \leq 2m$, so that 2f $\leq m$.

For $K_{3,3}$, we have $m = 9$, and so $2f \leq 9$ giving $f \leq \frac{9}{2}$. But by Euler's formula, $f = m - n + 2 = 9 - 6 + 2 = 5$, a contradiction. Hence $K_{3,3}$ is nonplanar. ❑

Euler's formula gives a necessary condition for connected planar graphs.

Theorem 6.10 For a connected simple planar graph with n vertices and m edges, and girth g, we have $m \leq \frac{g(n-2)}{g-2}$.

Proof Let G be a connected simple planar graph with n vertices, m edges and girth g. Let F be the set of regions and $d(\phi)$ be the degree of the region ϕ. Then $\sum\limits_{\phi \in F} d(\phi) = 2m$. Let f be the number of regions, that is $|F| = f$. As g is the length of the smallest cycle in G, therefore g is the smallest degree of a region. So, $\sum\limits_{\phi \in F} d(\phi) \geq gf$. Thus, $2m \geq gf$. Now, using Euler's formula, $f = m - n + 2$, we have $2m \geq g(m - n + 2)$, so that $m \leq \frac{g(n-2)}{g-2}$.

Corollary 6.3 The Peterson graph P is nonplanar.

Proof The girth of the Perterson graph P is 5, $n(P) = 10$, and $m(P) = 15$. Thus, if P were planar, $15 = 5\left(\frac{10-2}{5-2}\right)$, which is not true. Hence P is nonplanar. ❑

Corollary 6.4 Any graph containing a homeomorph of a K_5 or $K_{3,3}$ is nonplanar. This follows from the fact that any homeomorph of a nonplanar graph is nonplanar.

The following result characterises nonplanar 3-connected graphs with minimum edges.

Theorem 6.11 A nonplanar connected graph G with minimum number of edges that contains no subdivision of K_5 or $K_{3,3}$ is simple and 3-connected.

Proof Since G has minimum number of edges, G is minimal with respect to these properties. The minimality of G ensures that it is simple. Also, a graph is planar if and only if each of its blocks is planar. This implies that a minimal nonplanar graph is a block.

Suppose such a graph has a two vertex cut $S = \{u, v\}$. Then $G - S$ is disconnected. Let G_1 be one of its components and G_2 be the union of all of its other components.

Let $H_1 = < V(G_1) \cup S > + e$ and $H_2 = < V(G_2) \cup S > + e$, where $e = uv$.

Clearly, these are nonempty graphs with fewer edges than G. Since G is nonplanar, at least one of H_1 and H_2 is nonplanar. If both are planar, let $\tilde{H}_1$ be a plane embedding of H_1 and let ϕ be a region of $\tilde{H}_1$ containing the edge e.

A plane embedding $\tilde{H}_2$ of H_2 inside ϕ can then be obtained with the edge e of both $\tilde{H}_1$ and $\tilde{H}_2$ coinciding. Then $\tilde{H}_1 \cup \tilde{H}_2 - e$ is a plane embedding of G, contradicting our hypothesis.

Suppose H_1 is nonplanar. Since $m(H_1) < m(G)$, therefore H_1 contains a subdivision K of either K_5 or $K_{3,3}$ according to the hypothesis. Clearly $e \in E$, since otherwise $K \subset G$, contradicting the assumption. Now, replacing e of K by a $u - v$ path in H_2, we get a homeomorph of K contained in G. This is again a contradiction and shows that G has no 2-vertex cut. Hence G is 3-connected. ❑

The following result is used in proving Kuratowski's theorem.

Theorem 6.12

a. If $G|e$ contains a subdivision of K_5, then G contains a subdivision of K_5 or $K_{3,3}$.

b. If $G|e$ contains a subdivision of $K_{3,3}$, then G contains a subdivision of $K_{3,3}$.

Proof Let $G' = G|e$ be a graph obtained by contracting the edge $e = xy$ of G. Let w be the vertex of G' obtained by contracting $e = xy$.

a. Let $G|e$ contains a subdivision of K_5, say H. If w is not a branch vertex of H, then G also contains a subdivision of K_5, obtained by expanding w back into the edge xy, if necessary (Fig. 6.10).

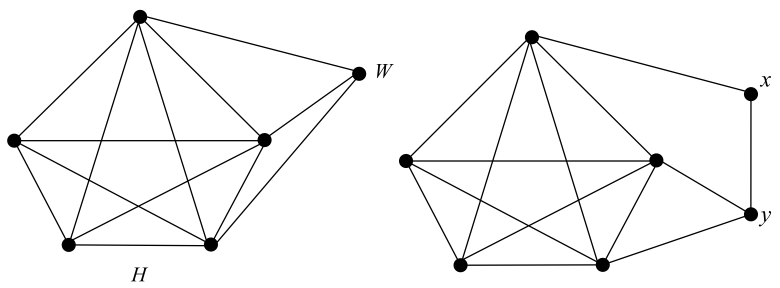


Fig. 6.10

Assume w is a branch vertex of H and each of x, y is incident in G to two of the four edges incident to w in H. Let u_1 and u_2 be the branch vertices of H that are at the other ends of the paths leaving w on edges incident to x in G. Let v_1, v_2 be the branch vertices of H that are at the other ends of the paths leaving w on edges incident to y in G (Fig. 6.11).

By deleting the $u_1 - u_2$ path and $v_1 - v_2$ path from H, we obtain a subdivision of $K_{3,3}$ in G, in which y, u_1, u_2 are branch vertices for one partite set, and x, v_1, v_2 are branch vertices of the other.

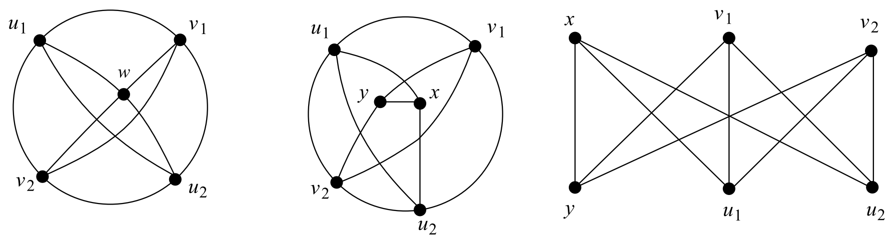


Fig. 6.11

b. Let $G|e$ contain a subdivision of $K_{3,\,3}$, say H. If w is not a branch vertex of H, then G also contains a subdivision of $K_{3,\,3}$, obtained by expanding w back into the edge xy, if necessary (Fig. 6.12).

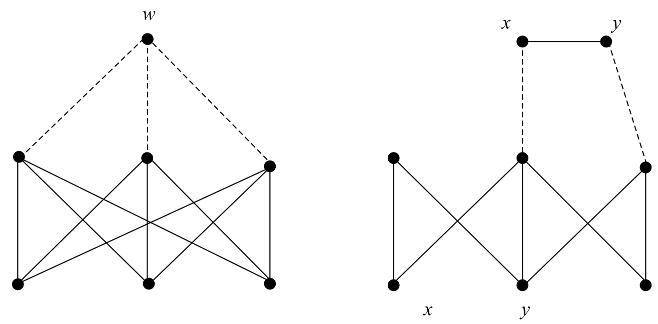


Fig. 6.12

Now, assume that w is a branch vertex in H and at most one of the edges incident to w in H is incident to x in G. Then w can be expanded into xy to lengthen that path and y becomes the corresponding branch vertex of $K_{3,\,3}$ in G (Fig. 6.13). ❑

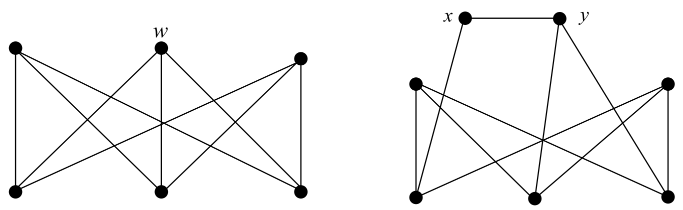


Fig. 6.13

6.4 Kuratowski's Theorem

This theorem was independently given by Kuratowski [144] and Frink and Smith. In 1954, Dirac and Schuster [69] found a proof that was slightly shorter than the original proof. The proof given here is due to Thomassen [241].

Theorem 6.13 (Kuratowski) A graph is planar if and only if it does not have any subdivision of K_5 or $K_{3,3}$.

Proof

Necessity Let G be a planar graph. Then any of its subgraphs is neither K_5 nor $K_{3,3}$ nor does it contain any subdivision of K_5 or $K_{3,3}$.

Sufficiency It is enough to prove sufficiency for 3-connected graphs. Let G be a 3-connected graph with n vertices. We prove that the 3-connected graph G either contains a subdivision of K_5 or $K_{3,3}$ or has a convex plane representation. This we prove by using induction on n. Since G is 3-connected, therefore $n \geq 4$. For $n = 4$, $G = K_4$ and clearly has a plane representation.

Now, let $n \geq 5$. Assume the result to be true for all 3-connected graphs with fewer than n vertices. Since G is 3-connected, G has an edge e such that $G|e$ is 3-connected. Let $e = xy$. If $G|e$ contains a subdivision of K_5 or $K_{3,3}$, then G also contains a subdivision of K_5 or $K_{3,3}$. Therefore, let $G|e = H$ have a convex plane representation. Let z be the vertex obtained by contraction of $e = xy$. The plane graph obtained by deleting the edges incident to z has a region containing z (this may be the exterior region). Let C be the cycle of $H - z$ bounding this region.

Since we started with a convex plane representation of H, we have straight segments from z to all its neighbours. Let $x_1, x_2, \ldots, x_k$ be the neighbours of x in that order on C.

If all the neighbours of y belong to a single segment from x_i to x_{i+1} on C, then we obtain a convex plane representation of G by putting x at z in H, and putting y at a point close to z in the wedge formed by $x\,x_i$ and $x\,x_{i+1}$.

If all the neighbours of y do not belong to any single segment $x_i\,x_{i+1}$ on C ($1 \leq i \leq k$, $x_{k+1} = x_1$), then we have the following cases (Fig. 6.14).

a. y shares three neighbours with x. In this case C together with these six edges involving x and y form a subdivision of K_5.

b. y has two u, v in C that are in different components of the subgraph of C obtained by deleting x_i and x_{i+1}, for some i. In this case, C together with the paths uyv, $x_i\,xx_{i+1}$ and xy form a subdivision of $K_{3,3}$. ❑

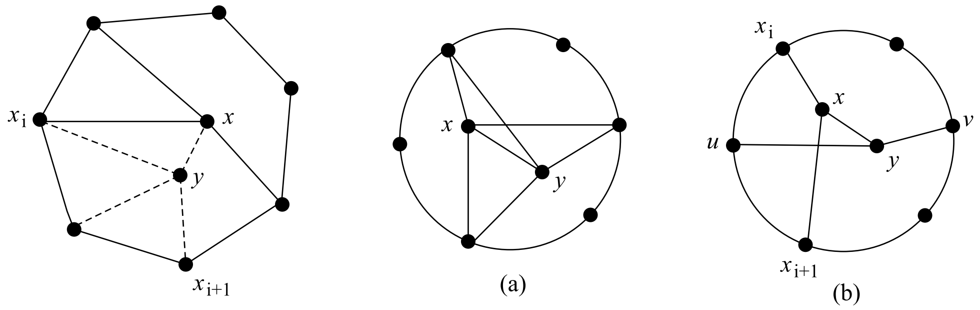


Fig. 6.14

6.5 Geometric Dual

Let G be a plane graph. The dual of G is defined to be the graph G^* constructed as follows. To each region f of G there is a corresponding vertex f^* of G^* and to each edge e of G there is corresponding edge e^* in G^* such that if the edge e occurs on the boundary of the two regions f and g, then the edge e^* joins the corresponding vertices f^* and g^* in G^*. If the edge e is a bridge, i.e., the edge e lies entirely in one region f, then the corresponding edge e^* is a loop incident with the vertex f^* in G^*. For example, consider the graph shown in Figure 6.15.

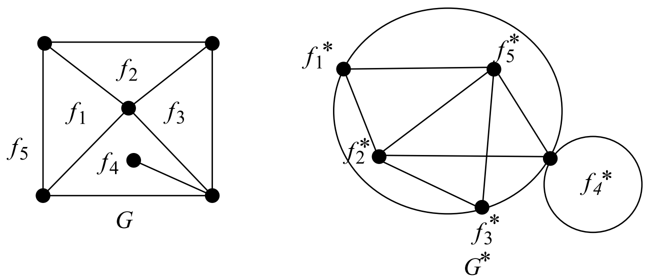


Fig. 6.15

Theorem 6.14 The dual G^* of a plane graph is planar.

Proof Let G be a plane graph and let G^* be the dual of G. The following construction of G^* shows that G^* is planar.

Place each vertex f_k^* of G^* inside its corresponding region f_i. If the edge e_i lies on the boundary of two regions f_j and f_k of G, join the two vertices f_j^* and f_k^* by the edge e_i^*, drawing so that it crosses the edge e exactly once and crosses no other edge of G (Fig. 6.16). ❑

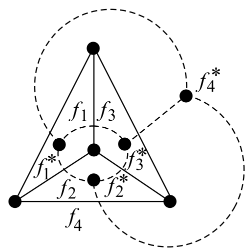


Fig. 6.16

Remarks Clearly, there is one-one correspondence between the edges of plane graph G and its dual G^* with one edge of G^* intersecting one edge of G.

1. An edge forming a self-loop in G gives a pendant edge in G^* (An edge incident on a pendant vertex is called a pendant edge).

2. A pendant edge in G gives a self loop in G^*.

3. Edges that are in series in G produce parallel edges in G^*.

4. Parallel edges in G produce edges in series in G^*.

5. The number of edges forming the boundary of a region f_i in G is equal to the degree of the corresponding vertex f_i^* in G^*.

6. Considering the process of drawing a dual G^* from G, it is evident that G is a dual of G^*. Therefore, instead of calling G^* a dual of G, we usually say that G and G^* are dual graphs.

7. Let n, m, f, r and μ denote the number of vertices, edges, regions, rank and nullity of a connected plane graph G and let n^*, m^*, f^*, r^* and μ^* be the corresponding numbers in G^*. Then $n^* = f$, $m^* = m$, $f^* = n$. We have $r^* = n^* - 1$, $\mu^* = m^* - n^* + 1$, $r = n - 1$, $\mu = m - n + 1$. So, $r^* = f - 1$, $\mu^* = m - f + 1$, $r = n - 1$, $\mu^* = m - n + 1$.

Using Euler's formula, $n - m + f = 2$ or $f = m - n + 2$, we have $r^* = m - n - 2 - 1 = m - n + 1 = \mu$ and $\mu^* = m - f + 1 = n + f - 2 - f + 1 = n - 1 = r$.

We now have the following result.

Theorem 6.15 The edge e is a loop in G if and only if e^* is a bridge in G^*.

Proof Let the edge e be a loop in a plane graph G. Then it is the edge on the common boundary of two regions on which, say f, lies within the area of the plane surrounded by e with the other, say g, lying outside the area. Thus, from definition of G^*, e^* is the only path from f^* to g^* in G^*. Thus $e^*$is a bridge in G^*.

Conversely, let e^* be a bridge in G^*, joining vertices f^* and g^*. Thus e^* is the only path in G^* from f^* to g^*. This implies, again from the definition of G^*, that the edge e in G completely encloses one of the regions f and g. So e is a loop in G. ❑

Remark The occurrence of parallel edges in G^* is easily described. Given two regions f and g of G, there are k parallel edges between f^* and g^* if and only if f and g have k edges on their common boundary.

Note We have defined the dual of a plane graph instead of a planar graph. The reason for this is that different plane drawings G_1 and G_2 of the same planar graph G may result in non-isomorphic duals G_1^* and G_2^*.

2-isomorphism Two graphs $G(V, E)$and $H(W, F)$ are said to be 2-isomorphic if there exists a bijection ϕ: $E \to F$ such that both ϕ and ϕ^{-1} preserve cycles.

Two graphs G and H are said to be 2-isomorphic if there exists a one-one correspondence between their edge sets such that the edges of a cycle in G correspond to the edges of a cycle in G_2 and vice versa.

Example Figure 6.17(a) shows 2-isomorphic graphs.

1-isomorphism Graphs which become isomorphic after the splitting of all of their cut vertices are said to be 1-isomorphic.

Example Figure 6.17(b) shows 1-isomorphic graphs.

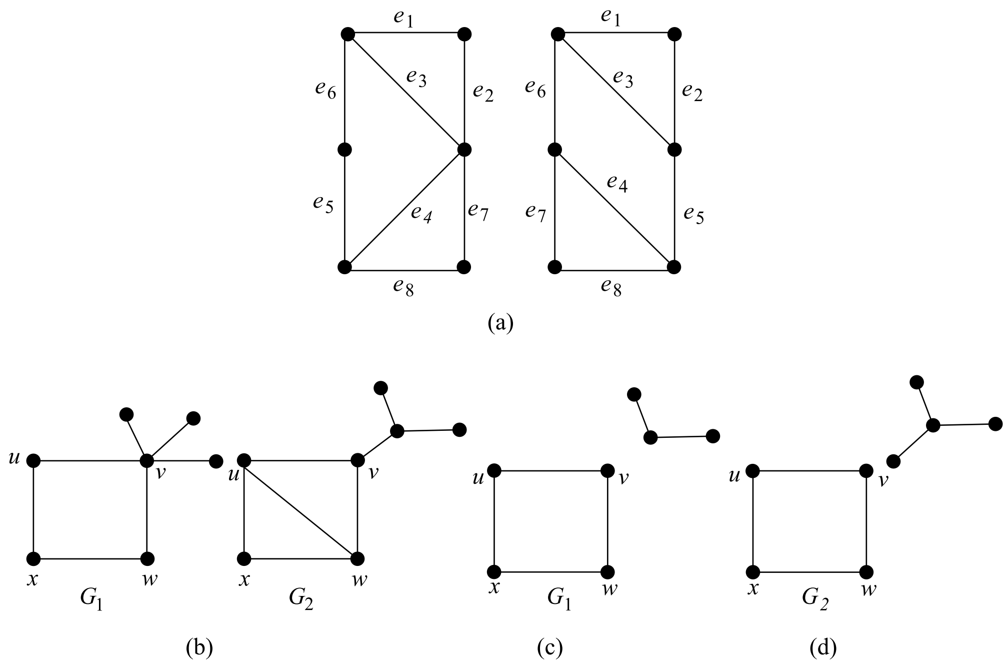


Fig. 6.17

Consider the two non-isomorphic graphs G_1 and G_2 shown in Fig. 6.17(a) and (b). They both have a single cut vertex, namely v. If this cut vertex is split into two vertices in each of G_1 and G_2, we obtain the edge disjoint graphs G_1' and G_2' as shown in (c) and (d). Clearly, G_1' and G_2' are isomorphic. Thus G_1 and G_2 are 1-isomorphic.

The next result gives a correspondence between the edges of a graph with the edges of its dual.

Theorem 6.16 Edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

Proof Let $D \subseteq E(G)$. It is sufficient to prove that D contains a cycle if and only if the set D^* of dual edges contains a bond of G^*. Now, let D contain a cycle C. Then by the Jordan Curve theorem, some region of G lies inside C and some region lies outside C. These regions correspond to vertices v^*, w^* in G^*, one drawn inside C and one outside C. A $v^* - w^*$ path in G^* crosses C and hence uses an edge of G^*, that is dual to an edge of C. Thus D^* disconnects v^* from w^* and hence D^* contains a bond (Fig. 6.18(a)).

Conversely, let D^* contain a bond. We show that D contains a cycle. Assume that D does not contain a cycle. Then D encloses no region. It remains possible to reach each region of G from every other without crossing D. Hence $G^* - D^*$ is connected, and so D^* contains no bond. This is a contradiction. Hence D contains a cycle. ❑

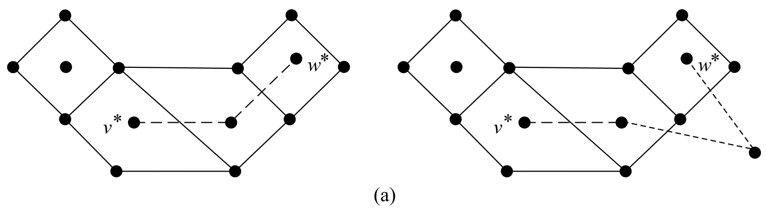


Fig. 6.18(a)

Theorem 6.17 All duals of a planar graph G are 2-isomorphic and any graph H 2-isomorphic to a dual G^* of G is itself a dual of G.

Proof Let $G^*(V^*, E^*)$ be a dual of G. Then there is an edge bijection $\phi : E \to E^*$ such that cycles in G^* correspond to bonds in G. Let $H(W, F)$ be any other dual of G. Then there is an edge bijection ψ: $E \to E^*$ such that bonds in G^* correspond to cycles in H. Therefore $\phi\psi : E^* \to F$ is an edge bijection which preserves cycles between G^* and H. Hence G^* and H are 2-isomorphic.

Conversely, let H be a graph which is 2-isomorphic to a dual G^* of G. Then there is an edge bijection $\chi : E^* \to F$ which preserves cycles between G^* and H. Also, there is an edge bijection $\phi : E \to E^*$ such that cycles in G^* correspond to bonds in G. Thus we have an edge bijection $\chi\phi : F \to E$ such that cycles in F correspond to bonds in G. Hence H is a dual of G. ❑

The following result provides the condition for two given graphs to be duals of each other.

Theorem 6.18 A necessary and sufficient condition for two planar graphs G_1 and G_2 to be the duals of each other is that there should be a one-one correspondence of the edges in G_1 with the edges in G_2 such that the subset of edges in G_1 forms a cycle if and only if the corresponding set of edges in G_2 forms a bond.

Proof

Necessity Consider a plane representation of a planar graph G_1 and let G_1^* be the dual of G_1. Let C be an arbitrary cycle in G_1.

Clearly, C will form some closed simple curve in the plane representation of G_1, dividing the plane into two areas. Thus the vertices of G_1^* are partitioned into two non-empty, mutually exclusive subsets, one inside C and the other outside. In other words, the set of edges C^* in G_1^* corresponding to the set C in G_1 is a bond in G_1^*.

Similarly, corresponding to a bond S^* in G_1^*, there is a unique cycle consisting of the corresponding edge set S in G_1 such that S is a cycle.

Sufficiency Let G be a planar graph, and let G' be a graph for which there is a one-one correspondence between the bonds of G and the cycles of G' and vice versa. Let G^* be the dual graph of G. Therefore there is a one-one correspondence between the bonds of G and cycles of G^*. So there is a one-one correspondence between the cycles of G' and G^*. Thus G' and G^* are 2-isomorphic. Hence G' is the dual of G. ❑

Let G be a plane graph and G^* be the dual of G. So to every vertex of G we have a region of G^*. Let the region ϕ of the plane graph G^* corresponding to the vertex v of G has $e_1^*, e_2^*, \ldots, e_k^*$ as its boundary edges. Then each of these edges e_i^* crosses the corresponding edge e_i of G, and these edges are all incident at the vertex v. It follows that ϕ contains the vertex v. As G^* is a plane graph, we can construct the dual of G^*, called the double dual of G and is denoted by G^{**}.

We now have the following observation.

Theorem 6.19 Let G be a plane connected graph. Then G is isomorphic to its double dual G^{**}.

Proof Let G be a plane connected graph and G^* be the dual of G. Now, any region ϕ of dual G^* contains exactly one vertex of G, namely its corresponding vertex v of G. This is because the number of the regions of G^* is same as the number of vertices of G.

Thus in the construction of G^{**}, we take the vertex v to be the vertex in G^{**} corresponding to the region ϕ of G^*. This choice gives the required isomorphism. ❑

Definition: A connected graph G is called *self-dual* if it is isomorphic to its dual G^*. For example, K_4 is self-dual.

The next result is a characterisation of bipartite graphs in terms of duality.

Theorem 6.20 A connected plane graph G is bipartite if and only if its dual graph G^* is Eulerian.

Proof Let G be a bipartite graph. Then G does not contain odd cycles. Since G is a plane, all the regions of G are of even length. So degree of every region in G is even. Let G^* be the dual graph of G. Since the vertex degrees of G^* are same as the corresponding degrees of the regions of G, degree of every vertex in G^* is even. Hence G^* is Eulerian.

Conversely, assume G^* is Eulerian. Then degree of every vertex in G^* is even. Since the degree of a vertex in G^* corresponds to the degree of the region in G, degree of every region in G is even. This shows that G contains only even cycles. Hence G is bipartite. ❑

Dual of a subgraph: Let G be a plane graph and let G^* be its dual. Let e be an edge in G and let e^* be the corresponding edge in G^*. We delete the edge e and find dual of $G-e$. If edge e is on the boundary of two regions, then removal of e merges these two regions into one.

Thus the dual $(G-e)^*$ is obtained from G^* by deleting the corresponding edge e^* and then fusing the two end vertices of e^* in G^*-e^*. On the other hand, if edge e is not on the boundary, then e^* forms a self-loop. In that case, G^*-e^* is the same as $(G-e)^*$.

Thus, if a graph G has a dual G^*, then the dual of any subgraph of G can be obtained by successive application of this procedure.

Dual of a homeomorphic graph: Let G be a plane graph and G^* be its dual. Let e be an edge in G and let e^* be the corresponding edge in G^*. We create an additional vertex in G by introducing a vertex of degree two in edge e (e now becomes two edges in series). This simply adds an edge parallel to e^* in G^*. Likewise, the reverse process of merging two edges in series will simply eliminate one of the corresponding parallel edges in G^*. Thus, by this procedure, if the graph G has dual G^*, the dual of any graph homeomorphic to G can be obtained from G^*. These are illustrated in Figure 6.18(b).

The following important result is due to Whitney [267].

Theorem 6.21 (Whitney) A graph has a dual if and only if it is planar.

Proof We have only to prove that a nonplanar graph does not have a dual. Let G be a nonplanar graph. Then according to Kuratowski's theorem, G contains K_5 or $K_{3,3}$, or a sub-division of K_5 or $K_{3,3}$.

We know that a graph G has a dual only if every subgraph H of G and every subdivision of G has a dual. Thus to prove the result, we show that neither K_5 nor $K_{3,3}$ has a dual.

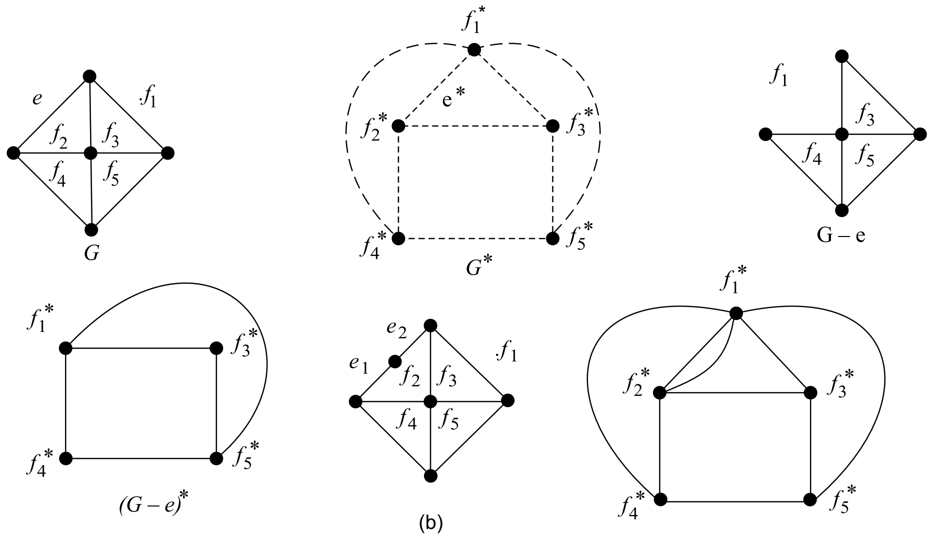


Fig. 6.18(b)

a. Suppose that $K_{3,3}$ has a dual D. Then the bonds in $K_{3,3}$ correspond to cycles in D and vice-versa. Since $K_{3,3}$ has no bond consisting of two edges, D has no cycle consisting of two edges. That is, D contains no pair of parallel edges. Since every cycle in $K_{3,3}$ is of length four or six, D has no bond with less than four edges. Therefore the degree of every vertex in D is at least four. As D has no parallel edges and the degree of every vertex is at least four, D has at least five vertices each of degree four or more (≥ 4). That is, D has at least $\frac{5\times 4}{2} = 10$ edges. This is a contradiction to the fact that $K_{3,3}$ has nine edges. Hence $K_{3,3}$ has no dual.

b. Suppose K_5 has a dual H. We note that K_5 has

   i. 10 edges,
   ii. no pair of parallel edges,
   iii. no bond with two edges, and
   iv. bonds with only four or six edges.

   Therefore graph H has

   i. 10 edges,
   ii. no vertex with degree less than three,
   iii. no bond with two edges, and
   iv. cycles of length four or six only.

Now graph H contains a hexagon (a cycle of length six) and no more than three edges can be added to a hexagon without creating a cycle of length three, or a pair of parallel edges. Since both of these are not present in H, and H has 10 edges, there must be at least seven vertices in H. The degree of each of these vertices is at least three. This implies that H has at least 11 edges, which is a contradiction. Hence K_5 has no dual. ❑

Definition: Two plane graphs G and G^* are said to be *duals* (or combinatorial duals) of each other if there is a one-one correspondence between the edges of G and G^* such that if H is any subgraph of G, and H^* is the corresponding subgraphs of G^*, then rank $(G^* - H^*) =$ rank $G^* -$ nullity H.

6.6 Polyhedron

A polyhedron is a solid bounded by surfaces, called *faces*, each of which is a plane. That is, a solid bounded by plane surfaces is called a polyhedron. For example, brick, window frame, tetrahedron.

A polyhedron is said to be *convex* if any two of its interior points can be joined by a straight line lying entirely within the interior. For example, a brick is convex, but a window frame is not convex.

The vertices and edges of a polyhedron, which form a skeleton of the solid, give a simple graph in three dimensional space. It can be shown that for a convex polyhedron this graph is planar.

To see this, imagine the faces of the convex polyhedron to be made of rubber, with one face, say at the base, missing. Then taking hold of the edges at the missing face, we are able to stretch out the rubber to form one plane sheet. The vertices and edges in this transformation now form a plane graph. Moreover, each face of this plane graph corresponds to a face of a solid, with the exterior face of the graph corresponding to the missing face we used in stretching process.

Clearly, the plane graph is also connected, the degree of every vertex is at least 3 and the degree of every face is also at least 3. Also, the graph is simple.

A simple connected plane graph G is called *polyhedral* if $d(v) \geq 3$, for each vertex v of G, and $d(\phi) \geq 3$, for every face ϕ of G.

The following is a simple but useful property of polyhedral graphs.

Theorem 6.22 Let P be a convex polyhedron and G be its polyhedral graph. For each $n \geq 3$, let n_i denote the number of vertices of G of degree i and let f_i denote the number of faces of G of degree i. Then

a. $\sum\limits_{i \geq 3} in_i = \sum\limits_{i \geq 3} if_i = 2m$, where m is the number of edges of G.

b. The polyhedron P, and so the graph G, has at least one face bounded by a cycle of length n for either $n = 3$, 4 or 5.

Proof

a. The expression $\sum\limits_{i \geq 3} in_i$ is simply $\sum\limits_{v \in V(G)} d(v)$, since $d(v) \geq 3$ for each vertex v.

We know, $\sum\limits_{v \in V(G)} d(v) = 2m$, so that $\sum\limits_{i \geq 3} in_i = 2m$.

For each $n \geq 3$, the expression if_i is obtained by going round the boundary of each face having a cycle of length i as its boundary, counting up the edges as we go. If we do this over all possible i, then we count each edge twice, and so $\sum\limits_{i \geq 3} if_i = 2m$.

b. Assume to the contrary that P has no faces bounded by cycles of length 3, 4 or 5. Then $f_3 = f_4 = f_5 = 0$.

So by (a), $2m = \sum\limits_{i \geq 6} if_i \geq \sum\limits_{i \geq 6} 6f_i = 6 \sum\limits_{i \geq 6} f_i = 6f$, where f is the total number of faces of P. Thus, $f \leq \frac{1}{3}m$.

Also by (a), $2m = \sum\limits_{i \geq 3} in_i \geq \sum\limits_{i \geq 3} 3n_i = 3 \sum\limits_{i \geq 3} n_i = 3n$, where n is the total number of vertices of P. Thus, $n \leq \frac{2}{3}m$.

Now, by Euler's formula, $m = n + f - 2$ and we have

$$m \leq \frac{2}{3}m + \frac{1}{3}m - 2 = m - 2 \text{, and this implies that } 2 \leq 0,$$

which is impossible. Hence P must have a face of length 3, 4 or 5. ❑

Regular polyhedra: A polyhedron is called regular if it is convex and its faces are congruent regular polygons (so that the polyhedral angles are all equal). The regular polyhedra are also called Platonic bodies or Platonic solids.

A fact that has been known for at least 2000 years is that there are only five regular polyhedra, which is proved next by using graph theoretic argument.

Theorem 6.23 The only regular polyhedra are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

Proof Let P be a regular polyhedron and let G be its corresponding polyhedral graph. Let n, m and f denote the number of vertices, edges and faces of G. Since the faces of G are congruent to each other, each one is bounded by the same number of edges. Thus, $d(\phi) = k, k \geq 3$ for each face ϕ of G. Also, we know that there is at least one cycle of length 3, 4 or 5, therefore, $3 \leq k \leq 5$. Similarly, since the polyhedral angles are all equal to each other, the graph G is regular, say r-regular, where $r \geq 3$. Thus we have

$rn = 2m$ and $kf = 2m$, so that

$$2m = rn = kf. \tag{6.23.1}$$

Using this in Euler's formula $n - m + f = 2$, we have

$$8 = 4n - 4m + 4f = 4n - 2m + 4f - 2m = 4n - rn + 4f - kf.$$

Therefore, $8 = (4 - r)n + (4 - k)f$. (6.23.2)

Since n and f are both positive and also $3 \leq k \leq 5$ and $r = 3$, there are only five possible cases (Fig. 6.19).

Case 1 Let $r = 3$ and $k = 3$. Then $8 = (4 - 3)n + (4 - 3)f$, so that $8 = n + f$.

From (6.23.1.), $n = f$. So, $n + f = 8$ gives $n = 4$, $f = 4$.

Thus, $n = 4$, $m = 6$, $f = 4$, which is a tetrahedron.

Case 2 Let $r = 3$, $k = 4$. Then from (6.23.2), $8 = (4 - 3)n + (4 - 4)f$.

So, $n = 8$. Then, (6.23.1) gives $3n = 4f$, so that $f = 6$. Also, $m = 12$.

So, $n = 8$, $m = 12$, $f = 6$, and thus is a cube.

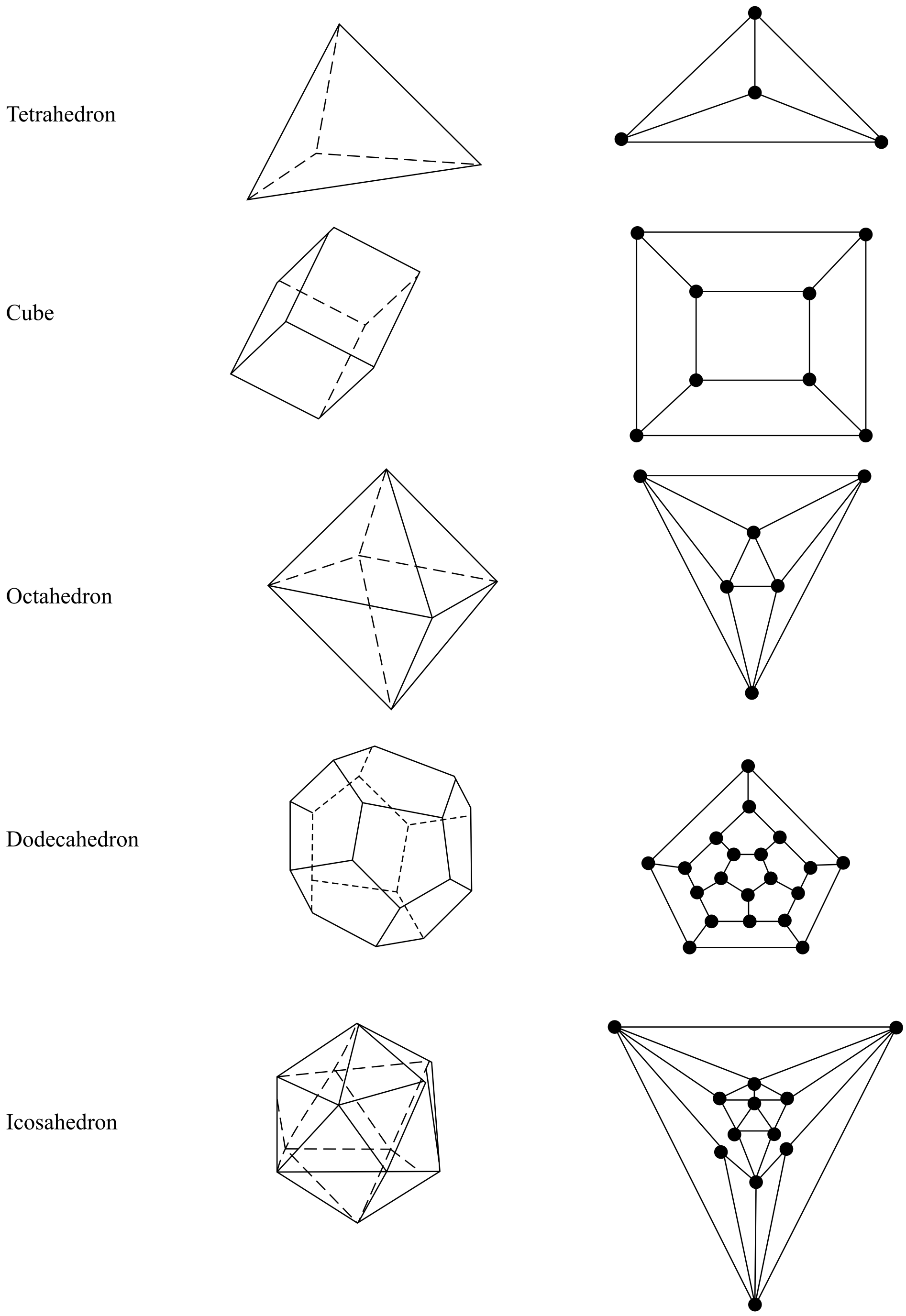


Fig. 6.19

Case 3 Let $r = 3$, $k = 5$. Then (6.23.2) gives $8 = (4-3)n + (4-5)f$, so that $8 = n - f$.

From (6.23.1), $3n = 5f$. Solving $8 = n - f$ and $3n = 5f$, we get $n = 20$, $f = 12$.

Thus, $n = 20$, $m = 30$, $f = 12$, which is a dodecahedron.

Case 4 Let $r = 4$, $k = 3$. Then we get $n = 6$, $f = 8$.

So, $n = 6$, $m = 12$, $f = 8$, which is an octahedron.

Case 5 Let $r = 5$, $k = 3$. Then we have $n = 12$, $f = 20$.

Thus, $n = 12$, $m = 30$, $f = 20$, which is an icosahedron. ❑

The following is an elegant necessary condition for a plane graph to be Hamiltonian and is due to Grinberg [86].

Theorem 6.24 If G is a loopless plane graph having a Hamiltonian cycle C, then $\sum\limits_{i=2}^{n}(i-2)(\phi_i' - \phi_i'') = 0$, where ϕ_i' and ϕ_i'' are the number of regions of G of degree i contained in interior C and exterior C respectively.

Proof Let E' and E''denote the sets of edges of G contained in int $C$and ext C, respectively, and let $|E'| = m'$ and $|E''| = m''$. Then int C contains exactly $m'+1$ regions (Fig. 6.20) and so

$$\sum_{i=2}^{n} \phi_i' = m' + 1, \tag{6.24.1}$$

Since G is loopless, $\phi_1' = \phi_1''= 0$.

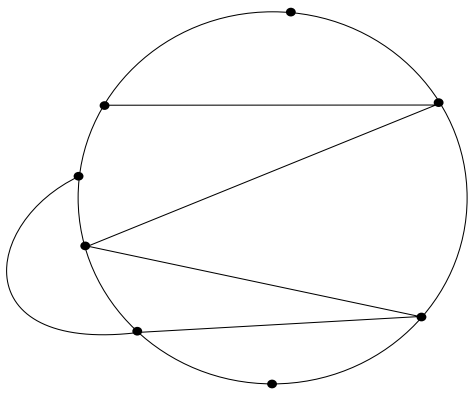


Fig. 6.20

Also, each edge in int C is on the boundary of exactly two regions in int C and each edge of C is on the boundary of exactly one region in int C. Thus, counting the edges of all the regions in int C, we obtain

$$\sum_{i=2}^{n} i\phi_i'' = 2m' + n. \tag{6.24.2}$$

Eliminating m' between (6.24.1) and (6.24.2), we get

$$\sum_{i=2}^{n} (i-2)\phi_i'' = n-2. \tag{6.24.3}$$

Similarly, $\sum_{i=2}^{n} (i-2)\phi_i'' = n-2.$ (6.24.4)

The required result follows from equation (6.24.3) and (6.24.4). ❑

Example Consider the Herschel graph G shown in Figure 6.21. Clearly, G has 9 regions and all the regions are of degree 4. Thus, if G were Hamiltonian, then $2(\phi_4' - \phi_4'') = 0$. This implies $\phi_4' = \phi_4''$, which is impossible, since $\phi_4' + \phi_4''$ = number of regions of degree 4 in $G = 9$ is odd. Hence G is non-Hamiltonian.

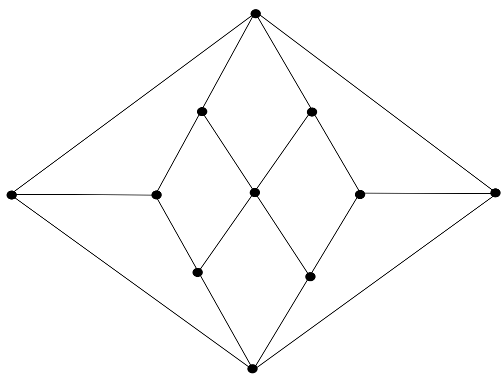


Fig. 6.21

Konig observed that every graph is embeddable on some orientable surface. This is seen by drawing an arbitrary graph G on the plane, possibly with edges that cross over each other and then attaching a handle to the plane at each crossing and allowing one edge to go over the handle and the other under it. Konig further showed that any embedding of a graph on an orientable surface with a minimum number of handles has all its regions simply connected.

Obviously, planar graphs can be embedded on a sphere. A toroidal graph is a graph embedded on a torus, for instance, K_5, $K_{3,3}$, K_7 and $K_{4,4}$.

Definition: The *genus* $\gamma(G)$of a graph G is the minimum number of handles which are added to the sphere so that G can be embedded on the resulting surface. Obviously $\gamma(G) = 0$ if and only if G is planar. The homeomorphic graphs have the same genus.

The genus of a polyhedron is the number of handles required on the sphere for a surface to contain the polyhedron.

Definition: The least number of planar subgraphs whose union is a given graph G is called the *thickness* of G. Clearly the thickness of any planar graph is 1. Also, K_5 and $K_{3,3}$ have thickness 2, while the thickness of K_9 is 3.

The *crossing number* of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane. Clearly, crossing number of a graph is zero if and only if G is planar.

The following result is due Euler and the proof can be found in Courant and Robbins [61].

Theorem 6.25 For a polyhedron of genus γ with n vertices, m edges and f regions

$$n - m + f = 2 - 2\gamma.$$

We have the following observations.

If G is a connected graph of genus γ, then

$$m = \begin{cases} 3(n-2+2\gamma), & if\ every\ region\ in\ G\ is\ a\ triangle, \\ 2(n-2+2\gamma), & if\ every\ region\ in\ G\ is\ a\ quadrilateral. \end{cases}$$

From this, we have the following.

If G is a connected graph of genus γ, then $\gamma \geq \frac{1}{6}m - \frac{1}{2}(n-2)$, and if G has no triangles, then $\gamma \geq \frac{1}{4}m - \frac{1}{2}(n-2)$.

The proof of the following result is due to Ringel and Youngs [223] and some details can be found in Harary [104].

Theorem 6.26 For $n \geq 3$, the genus of the complete graph is

$$\gamma(K_n) = \frac{(n-3)(n-4)}{12}.$$

6.7 Decomposition of Some Planar Graphs

Schnyder [232] proved that each triangulated planar graph G can be decomposed into three edge disjoint trees. If instead triangles all faces (or regions) of G are quadrilaterals, and G is without loops and multiple edges, Petrovic [183] proved that two trees are sufficient.

The following two lemmas are used in proving Theorem 6.27.

Lemma 6.1 If G is a planar graph on $n(n \geq 4)$ vertices whose all regions are quadrilaterals, then $|E(G)| = 2n-4$.

Lemma 6.2 If G is a planar graph on $n(n \geq 4)$ vertices whose all regions are quadrilaterals, then G contains at least three vertices of degree ≤ 3.

The following result is due to Petrovic [183].

Theorem 6.27 (Petrovic) Let G be a planar graph on $n(n \geq 4)$ vertices whose all faces are quadrilaterals, and v_r and v_b any two non-adjacent vertices of G (we assume v_r is coloured red and v_b blue). Then the edges of G can be partitioned into red and blue ones so that red ones form a spanning tree T_r of $G-v_b$, and blue ones a spanning tree T_b of $G-v_r$.

Proof We call T_r the red, and T_b the blue tree. A vertex of G is red (respectively blue) if it belongs to T_r (respectively T_b). A vertex is red-blue if it belongs to both T_r and T_b. Thus v_r is the only red, and v_b is only blue vertex in G. We proceed by induction on n. For $n = 4$, the assertion is obvious. Assume that it holds for each graph on less than $n(n > 4)$ vertices, and consider a graph G on n vertices. Let u $(u \neq v_r,\, v_b)$ be a vertex of G of degree ≤ 3 existing by Lemma 6.2.

a. $d(u) = 2$. Let uv_1, uv_2 be the edges, and uv_1xv_2u, uv_1yv_2u the faces incident with u. Since $G \neq C_4$, each of vertices v_1 and v_2 has degree ≥ 3. It implies that all regions of $G-u$ are quadrilaterals. By induction hypothesis $G-u$ can be partitioned into trees T_r' and T_b'. Now, independent of the kind of vertices v_1 and v_2, (red, blue or red-blue) we can always colour the edges uv_1 and uv_2 so that the obtained trees T_r and T_b are as desired.

b. $d(u) = 3$. Let uv_1, uv_2, uv_3 be the edges and uv_1xv_2u, uv_2yv_3u, uv_3zv_1u be the faces incident with u.

First assume that all vertices x, y, z are distinct. Since no triangle can be partitioned into disjoint quadrilaterals whose vertices are vertices of the triangle and some its inner points, $xy,\, yz,\, zx \notin E(G)$. Suppose one of vertices x, y, z, is blue and there is no edge connecting it with the opposite vertex of the hexagon $v_1xv_2yv_3z$. Without loss of generality, we may assume it is x. Then $xv_3 \notin E(G)$. Denote by G'' the graph obtained from G by deleting the edges xv_1 and xv_2 and identifying vertices u and x. Since each face of G'' is a quadrilateral it can be partitioned into two trees T_r'' and T_b'', where $V(T_r'')= V(G''-v_b)$ and $V(T_b'') = V(G''-v_r)$. If edges uv_1 and uv_2 are coloured differently in G''', say uv_1 red and uv_2 blue, we colour xv_1 red and xv_2 blue obtaining the required trees $T_r = T_r' + xv_1$ and $T_b = T_b'' + xv_2$ in G. Therefore assume that uv_1 and uv_2 are monocoloured, say red. Then one of vertices v_1 and v_2, say v_1, is red–blue. As u is also a red-blue vertex, there is the unique blue $u-v_1$ path P in G''. If P contains uv_3, we colour xv_1 and xv_2 blue and red, respectively. If not, we colour both edges xv_1 and xv_2 red, and recolour uv_1 blue. It is routine to check that obtained trees $T_r = T_r'' + xv_1$, $T_b = T_b'' + xv_2$ in the first case, and $T_r = T_r'' - uv_1 + xv_1 + xv_2$, $T_b = T_b'' + uv_1$ in the second, are as required.

Now, assume none of the vertices x, y, z satisfy the condition above. Since at most one of edges xv_3, yv_1, zv_2 can exist, we may assume without loss of generality that x is red-blue, y is the red, z is the blue vertex, $xv_3 \in E(G)$ and $yv_1 \notin E(G)$. Then each region of the graph G''' obtained from G by deleting edges yv_2 and yv_3 and identifying y and u is a quadrilateral. By induction ypothesis, G''' can be composed into trees T_r''' and T_b''', where $V(T''') = V(G''' - z)$ and $V(T_b''') = V(G''' - y)$. Colouring all edges incident with y red, and recolouring the edge uv_3 blue, we obtain the decomposition of G into trees T_r and T_b, where $V(T_r) = V(G-z)$ and $V(T_b) = V(G-y)$.

Suppose two of vertices x, y, z, say y and z, coincide. Then $d(v_3) = 2$. If v_3 is red-blue, we finish the proof as in case (a). If x is red-blue we proceed as above. So we may assume that $\{v_3, x\} = \{v_r, v_b)$. Let $v_3 = v_r$, $x = v_b$. By induction hypothesis the graph $G^{IV} = G - v_3$ can be decomposed into trees T_r^{IV} and T_b^{IV}, where $V(T_r^{IV}) = V(G^{IV} - x)$ and $V(T_b^{IV}) = V(G^{IV} - u)$. Then edges uv_1, uv_2 are coloured red and edges xv_1, xv_2, blue. It implies that edges yv_1, yv_2 are coloured differently, yv_1, red and yv_2, blue. Now, colouring v_3u and v_3y red and recolouring uv_1 blue, we get from T_r^{IV} and T_b^{IV} the desired trees T_r and T_b, where $V(T_r) = V(T_r) = V(G-x)$ and $V(T_b) = V(G-v_3)$.

If $x = y = z$, then $v(G) = \{u, x, v_1, v_2, v_3\}$, $E(G) = \{uv_1, uv_2, uv_3, xv_1, xv_2, xv_3\}$ and the statement holds trivially. ❑

6.8 Exercises

1. If G is a connected planar graph of order less than 12, prove that $\delta(G) = 4$.

2. If G is a planar graph of order 24 and is regular of degree 3, then what is the number of regions in a planar representation of G?

3. Prove that Euler's formula fails for disconnected graphs.

4. Show that every graph with at most three cycles is planar.

5. Find a simple graph G with degree sequence [4, 4, 3, 3, 3, 3] such that

   a. G is planar,

   b. G is nonplanar.

6. Show that every simple bipartite cubic planar graph contains a C_4.

7. Prove that a simple planar graph has at least 4 vertices of degree 5 at most.

8. Let G be a planar, triangle free graph of order n. Prove that G has no more than $2n-4$ edges.

9. Let G be of order $n = 11$. Show that at least one of G or $\overline{G}$ is non planar.

10. Show that the average degree of a planar graph is less than 6.

11. Use Kuratowski's theorem to prove that the Peterson graph is nonplanar.

12. Show that the edges forming a spanning tree in a planar graph G correspond to the edges forming a set of chords in the dual G^*.

13. Prove that the complete graph K_4 is self-dual.