# 5. Connectivity

In a connected graph there is at least one path between every pair of its vertices. If in a graph, it happens that by deleting a vertex, or by removing an edge, or performing both, the graph becomes disconnected, we can say that such vertices or edges hold the whole graph, or in other words have the property of destroying the connectedness of a graph. For example, consider a communication network which is modelled as the graph $G$ shown in Figure 5.1, where vertices correspond to communication centers and the edges represent communication channels. Clearly, deletion of vertex $v$ results in the breakdown of the communication. This implies that in the above communication network, the center represented by vertex $v$ has the property of destroying the communication system and thus communication network depends on the connectivity. We start this chapter with the following definitions.

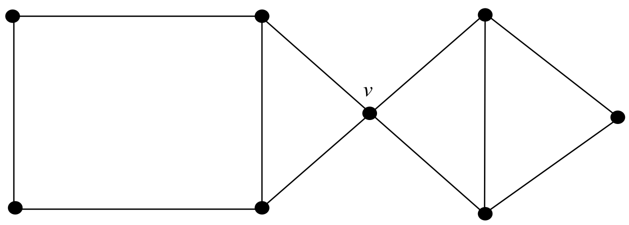



**Fig. 5.1**

## 5.1 Basic Concepts

**Cut vertex:** Let $G$ be a graph with $k(G)$ components. A vertex $v$ of $G$ is called a cut vertex of $G$ if $k(G-v) > k(G)$. For example, in the graph of Figure 5.2, the vertices $u$ and $v$ are cut vertices.

**Cut edge:** An edge $e$ of a graph $G$ is said to be a cut edge if $k(G-e) > k(G)$. In the graph of Figure 5.2, $e$ and $f$ are cut edges.

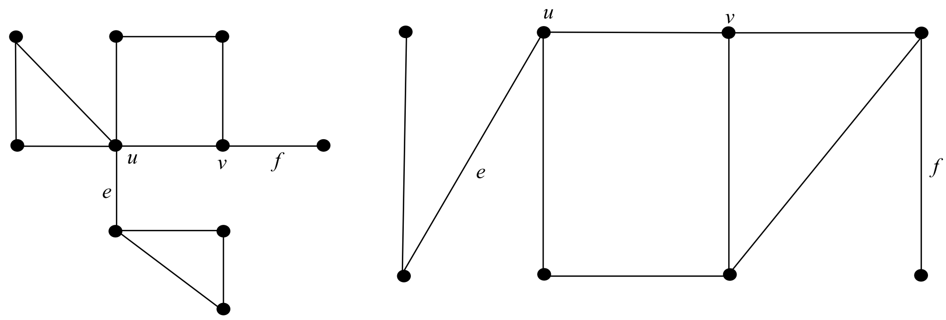



**Fig. 5.2**

The following observations are the immediate consequences of the definitions introduced above.

1. Removal of a vertex may increase the number of components in a graph by at least one, while removal of an edge may increase the number of components by at most one.
2. The end vertices of a cut edge are cut vertices if their degree is more than one.
3. Every non-pendant vertex of a tree is a cut vertex.

We now give the first result which characterises cut vertices.

**Theorem 5.1** If $G(V, E)$ is a connected graph, then $v$ is a cut vertex if there exist vertices $u, w \in V - \{v\}$ such that every $u-w$ path in $G$ passes through $v$.

**Proof** Let $G(V, E)$ be a connected graph and let $v$ be the cut vertex of $G$. Then $G-v$ is disconnected. Let $G_1, G_2, \ldots, G_k$ be the components of $G-v$. Let $U = V(G_1)$ and $W = \bigcup_{i=2}^{k} V(G_i)$. Also, let $u \in U$ and $w \in W$, and to be definite, let $w \in V(G_i)$, $i \neq 1$. If there is a $u-w$ path $P$ in $G$ not passing through $v$, then $P$ connects $u$ and $w$ in $G-v$ also. Therefore $G_1 \cup G_i$ is a single component in $G-v$, contradicting our assumption. Thus every $u-w$ path in $G$ passes through $v$.

Conversely, let there be vertices $u, w \in V - \{v\}$ such that every $u-w$ path in $G$ passes through $v$. Then there is no $u-w$ path in $G-v$. Therefore $u$ and $w$ belong to different components of $G-v$. Thus $G-v$ is disconnected and $v$ is a cut vertex of $G$. ❑

The following result characterises cut edges.

**Theorem 5.2** For a connected graph $G$, the following statements are equivalent.

i. $e$ is a cut edge of $G$.

ii. If $e = ab$, there is a partition of the edge subset $E - \{e\}$ as $E_1 \cup E_2$ with $a \in V(<E_1>)$ and $b \in V(<E_2>)$ such that for any $u \in V(<E_1>)$ and any $w \in V(<E_2>)$, every $u-w$ path contains $e$.

iii. There exist vertices $u$ and $w$ such that every $u-w$ path in $G$ contains $e$.

iv. $e$ is not a cycle edge of $G$.

**Proof** (i) $\Rightarrow$ (ii). Let $e$ be a cut edge of $G$. So $G-e$ is disconnected. Let $G_1$ and $G_2$ be two components of $G-e$ and $E_1 = E(G_1)$ and $E_2 = E(G_2)$. If $u \in V(G_1)$ and $w \in v(G_2)$ exist such that there is a $u-w$ path $P$ in $G$ which does not contain $e$, then $u$ and $w$ are connected in $G-e$ by the path $P$. This implies that $G_1 \cup G_2$, that is, $G-e$ is connected, contradicting the hypothesis. This proves (ii).
(ii) $\Rightarrow$ (iii). Obvious.
(iii) $\Rightarrow$ (iv). Suppose $e$ lies on a cycle $C$. Then $C-e$ gives an $a-b$ path $Q$ not containing $e$. With vertices $u$ and $w$ following the condition given in (iii), let $P$ be any $u-w$ path. Without loss of generality, assume that $a$ and $b$ occur in that order in $P$. Let $u_0$ and $w_0$ be the first and last vertices that $P$ has in common with $C$ (the possibility of these coinciding with $a$, $b$, $u$ or $w$ is not ruled out). Then $P_{u,\,u_o} \cup Q_{u_0,\,w_0} \cup P_{w_0,\,w}$ is a $u-w$ path $P'$ of $G$ which does not contain $e$, contradicting (iii). (See Figure 5.3(a), where broken curves represent path $Q$ and thick curves represent path $P$.)

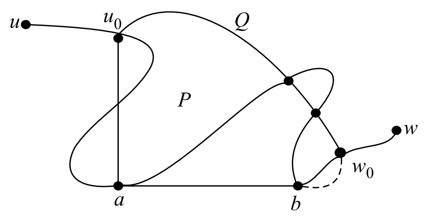



**Fig. 5.3(a)**

(iv) $\Rightarrow$ (i). Let $e$ be not a cyclic edge of $G$. We have to prove that $e$ is a cut edge of $G$, that is, $G-e$ is disconnected. Assume $G-e$ is connected. Then there is an $a-b$ path $P$ in $G-e$. But then $P \cup e$ is a cycle containing $e$, which contradicts (iv). ❑

**Block:** A block is a connected graph which does not have any cut edge. We observe that a block does not have any cut edge. The graph $K_2 = (\{a,\, b\},\, e)$ does not have a cut vertex and hence is a block. However, $e$ is a cut edge in this case. We call $K_2$ a *trivial block*. All other blocks are *non-trivial*.

**Separable graph:** A connected graph with at least one cut vertex is called a separable graph. A block of a graph $G$ is a maximal graph f$H$ of $G$ such that $H$ is a block. That is, $H$ has no cut vertex, but for any $v \in V(G) - V(H)$, $\langle V(H) \cup \{v\} \rangle$ is either a disconnected graph or a separable graph.

The next result characterises blocks.

**Theorem 5.3** For a connected graph $G$, the following are equivalent.

i. $G$ is a non-trivial block.

ii. Any two vertices of $G$ lie on a cycle.

iii. Given any vertex $u$ and any edge $vw$, there is a cycle of $G$ containing both.

iv. Given any pair of edges $e = uv$ and $e' = u'v'$, there is a cycle of $G$ containing both.

v. Given any pair of vertices $u$ and $u'$ and any edge $e = vw$, there is a $u - u'$ path of $G$ containing $e$.

**Proof**

a. (i) $\Rightarrow$ (ii). The proof is by induction on the distance between the vertices. If $d(u, v) = 1$, then $uv$ is an edge, and since $G$ is a block, $uv$ is not a cut edge. Hence $uv$ is a cyclic edge, and so $u$ and $v$ lie on a cycle. Now, for the induction hypothesis, we assume that if $u$ is any vertex, then any vertex $v'$ at a distance at most $k-1$ from $u$ lies on a cycle with $u$.

Let $v$ be a vertex at a distance $k$ from $u$. We prove that $u$ and $v$ lie on a cycle. Let $P$ be a shortest $u - v$ path and $v'$ the nearest vertex on $P$ from $u$. By induction hypothesis there is a cycle $C$ containing $u$ and $v'$. Since $v'$ is not a cut vertex of $G$, there is a $u - v$ path $Q$ not passing through $v'$. Let $z$ be the last vertex from $u$ that $Q$ has in common with $C$. Then $C_{uv'} \cup \{v'v\} \cup Q_{vz} \cup C_{zu}$ is a cycle of $G$ containing $u$ and $v$. (Here $C_{uv'}$ is the $uv'$ segment of $C$ not containing $z$, and $C_{zu}$ is the $zu$ segment of $C$ not containing $v'$) (Fig. 5.3(b)).

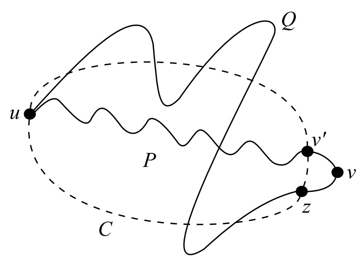



**Fig. 5.3(b)**

b. (ii) $\Rightarrow$ (i). Let any two vertices of $G$ lie on a cycle. We prove that $G$ is a non-trivial block, that is, $G$ has no cut vertex. Assume to the contrary that $G$ has a cut vertex $u$. Then there are vertices $v$ and $w$ such that every $v - w$ path passes through $u$. But then there is no cycle containing $v$ and $w$, which is a contradiction. Thus $G$ has no cut vertex.

c. (ii) $\Rightarrow$ (iii). Let any two vertices of $G$ lie on a cycle. Let vertex $u$ and edge $vw$ be given. So by (b), $G$ is a block and therefore $vw$ is not a cut edge.

Let $C$ be a cycle containing $vw$. If $C$ contains $u$, the proof is complete. If not, by (ii), there is a cycle $Z$ containing $u$ and $v$. Taking any orientation of $Z$, let $x$ and $y$ be the first and last vertices from $u$ that $Z$ has in common with $C$. Then the $u-x$ segment of $Z$, the $x-y$ segment of $C$ containing $vw$ and the $y-u$ segment of $Z$ constitute a cycle of $G$ containing $u$ and $vw$.

d. (iii) $\Rightarrow$ (ii). Let $u$ and $v$ be any two vertices. Since $v$ cannot be an isolated vertex, there is an edge $vw$. By (iii) there is a cycle containing $u$ and $vw$ (Fig. 5.4).

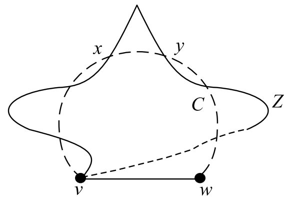



**Fig. 5.4**

e. (iii) $\Rightarrow$ (iv). Let $uv$ and $u'v'$ be the given edges. By (iii) there is a cycle $C$ through $u$ containing $u'v'$. If it passes through $v$, then the $u-v$ segment of $C$ containing $u'v'$ and the edge $vu$ constitute a cycle as required. If not, then as the earlier implications show that $G$ is a block, $u$ is not a cut vertex, and hence there is a $v-v'$ path $P$ in $G$ not passing through $u$ (Fig. 5.5).

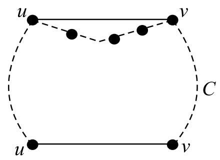



**Fig. 5.5**

Let $w$ be the first vertex from $v$ that $P$ has in common with $C$. Then $v-w$ segment of $P$, the $w-u$ segment of $C$ containing $u'v'$ and the edge $uv$ constitute a cycle of $G$ as desired (Fig. 5.6).

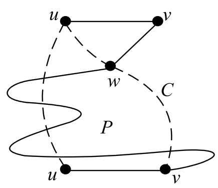



**Fig. 5.6**

f. (iv) $\Rightarrow$ (iii). This can be proved as in (d).

g. (iv) $\Rightarrow$ (v). Let $u$ and $u'$ be the vertices and $vw$ the given edge. If $uv'$ is also an edge, then there is nothing to prove. If not, since $u$ is not an isolated vertex, there is an edge

$uv_1$ and by (iv) there is a cycle $C$ containing $uv_1$ and $vw$. By previous implications, $G$ is a block and hence $u$ is not a cut vertex. Therefore there is a $u'w$ path $P$ not passing through $u$. Let $x$ be the first vertex from $u'$ that $P$ has in common with $C$. Then the $u'x$ segment of $P$ and the $xu$ segment of $C$ containing $vw$ constitute a path as desired (Fig. 5.7).

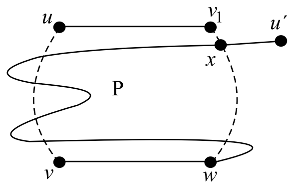



**Fig. 5.7**

h. (v) $\Rightarrow$ (iv). Obvious. ❑

**Remark** Property (iv) of the above theorem can be used to define an equivalence relation on the edge set $E$ of a graph $G$ by $e \sim f$ if and only if $e$ and $f$ lie on a common cycle in $G$. The equivalence classes are simply the blocks of $G$ and the edge set $E$ is partitioned into blocks. These blocks are joined at cut vertices, two blocks having at most one vertex in common.

## 5.2 Block–Cut Vertex Tree

Let $B$ be the set of blocks and $C$ be the set of cut vertices of a separable graph $G$. Construct a graph $H$ with vertex set $B \cup C$ in which adjacencies are defined as follows: $c_i \in C$ is adjacent to $b_j \in B$ if and only if the block $b_j$ of $G$ contains the cut vertex $c_i$ of $G$. The bipartite graph $H$ constructed above is called the *block-cut vertex tree* of $G$.

**Example** Consider the graph in Figure 5.8. The blocks are $b_1 = <1, 2>$, $b_2 = <2, 3, 4>$, $b_3 = <2, 5, 6, 7>$, $b_4 = <7, 8, 9, 10, 11>$, $b_5 = <8, 12, 13, 14, 15>$, $b_6 = <10, 16>$, $b_7 = <10, 17, 18>$ and cut vertices are $c_1 = 2$, $c_2 = 7$, $c_3 = 8$, $c_4 = 10$.

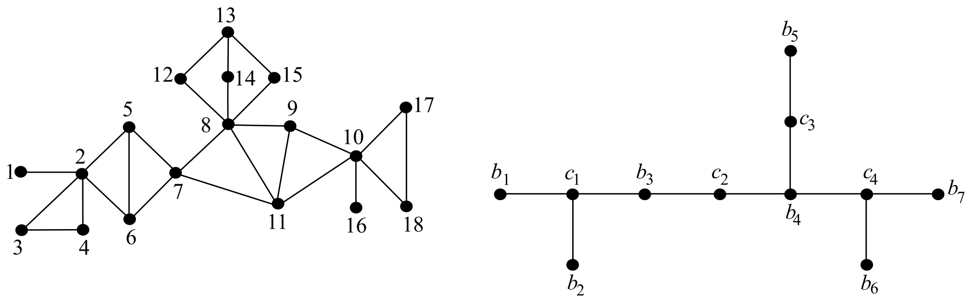



**Fig. 5.8**

**End block:** A block of a graph $G$ containing only one cut vertex is called an *end block* of $G$.

We have the following result on blocks.

**Theorem 5.4** Every separable graph has at least one cut vertex and therefore has at least two end blocks.

**Proof** A separable graph $G$ has at least one cut vertex and therefore has at least two blocks. Thus its block-cut vertex tree $T$ has at least three vertices. Now, for any separable graph the end blocks correspond to the pendant vertices of its block-cut vertex tree. Also, any tree with at least two vertices has two vertices of degree one. Thus the block-cut vertex tree $T$ has at least two pendant vertices. Hence $G$ has at least two end-blocks. ❑

The next result is due to Harary and Norman [109].

**Theorem 5.5** The center of any connected graph $G$ lies on a block of $G$.

**Proof** If not, let $B_1$, $B_2$ be blocks of $G$ containing central vertices. If $b_1$, $b_2$ are the vertices of the block-cut vertex tree $T$ of $G$ corresponding to $B_1$ and $B_2$, then there is at least one vertex $c$ in the unique $b_1 - b_2$ path of $T$, corresponding to a cut-vertex $c$ of $G$. So there are two components $G_1$ and $G_2$ of $G - c$ such that $B_1 - c \subseteq G_1$ and $B_2 - c \subseteq G_2$. Let $\bar{c}$ be an eccentric vertex of $c$ in $G$ and $P$ be a $c - \bar{c}$ path of $G$ having length $e(c)$. Then at least one of the components $G_1$ and $G_2$, say $G_2$, contains no vertex of $P$. Let $s$ be a central vertex in $G_2$ and $Q$ be a shortest $s - c$ path in $G$. Then $Q \cup P$ is clearly an $s - \bar{c}$ path in $G$ and thus $d(s, \bar{c}) = d(s, c) + e(c)$. Therefore $e(s) > e(c)$, contradicting the fact that $s$ is a central vertex. Thus the center of $G$ lies in a single block. Hence the center of any connected graph $G$ lies on a block of $G$. ❑

**Definition:** If $G$ is a separable graph and $c$ a cut-vertex of $G$, then a maximal connected subgraph of $G$ containing $c$ in which $c$ is not a cut-vertex is called a *branch* of $G$ at $c$. The induced subgraph $\langle C \rangle$ on the central vertices of $G$ is called the *central graph* of $G$. If $G$ has a unique central vertex $c$, then $G$ is said to be a *unicentric graph*. The unique block $B$ of $G$ to which the center $c$ of $G$ belongs is called the *central block* of $G$. This is unambiguously defined except when $G$ is unicentric and the unique central vertex is a cut-vertex of $G$. When $B = \langle C \rangle = G$, then $G$ is called a *self-centered graph*. If the unique central vertex $c$ of $G$ is a cut-vertex of $G$, the unique block of any of the branches of $G$ at $c$ in which $c$ has an eccentric vertex $\bar{c}$ may be taken as the chosen central block of $G$.

We note that the central graph of a tree is either $K_1$ or $K_2$. Buckley, Miller and Slater [53] have studied graphs with specified central graphs. The following result is atributed to Hedetniemi and is reported in Parthasarathy [180].

**Theorem 5.6** For any graph $H$ there exists a graph $G$ with 4 more vertices such that $H$ is the central graph of $G$.

**Proof** Take two new vertices $v$ and $w$, and join each to every vertex of $H$. Take two other vertices $x$ and $y$, join $x$ to $v$, and $y$ to $w$. Then in the resulting graph $G$, $e(x) = e(y) = 4$, $e(v) = e(w) = 3$ and $e(u) = 2$, for every vertex $u \in V(H)$. Thus $H$ is an induced subgraph of $G$ and the central graph of $G$. ❑

## 5.3 Connectivity Parameters

Assume that a graph does not get disconnected by deleting a single vertex, or by removing a single edge. A natural question then arises: what is the minimum number of vertices or edges required to disconnect a graph? This and other related questions are answered in this section. Before proceeding, we have the following definitions.

**Definition:** Let $G = (V, E)$ be a graph. A subset $S$ of $V \cup E$ is called a *disconnecting set* of the graph $G$ if $k(G-S) > k(G)$, or $G-S$ is the trivial graph.

If a disconnecting set $S$ is a subset of $V$, it is called a *vertex cut* of $G$, and if it is a subset of $E$ it is called an *edge cut* of $G$. If a disconnecting set $S$ contains vertices and edges it is called a *mixed cut*.

**Example** For the graph shown in Figure 5.9, $S = \{3, e_3\}$ is a mixed cut, $S = \{3\}$ is a vertex cut and $S = \{e_1, e_3\}$ is an edge cut.

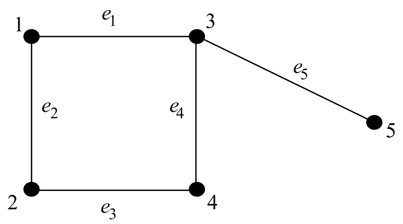



**Fig. 5.9**

A mixed cut/vertex cut/edge cut $S$ is *minimal* if no proper subset of $S$ has the same property as $S$. A mixed cut/vertex cut/edge cut $S$ is *minimum* if it has least cardinality among all such minimal sets. A minimal vertex cut is called a *knot* and a minimum vertex cut is called a *clot*. The cardinality of a clot is called the *vertex-connectivity* number, or clot number of the graph $G$ and is denoted by $\kappa(G)$.

A minimal edge cut is called a *bond* and a minimum edge cut is called a *band*. The cardinality of a band is called the *edge-connectivity* number, or band number of the graph $G$, and is denoted by $\lambda(G)$.

The minimum cardinality of a mixed set is denoted by $\sigma(G)$.

Let $S$ be a disconnected set of the graph $G(V, E)$. Let vertices $s$ and $t$ be in the same component of $G$, but in different components of $G-S$. Then $S$ is called an $s-t$ *separating* set in $G$. Minimal $s-t$ separating vertex cut is called an $s-t$ *knot*, and the minimum $s-t$ separating vertex cut is called an $s-t$ *clot*. Minimal $s-t$ separating edge cut is called an $s-t$ *bond*, and the minimum $s-t$ separating edge cut is called an $s-t$ *band*. The cardinality of an $s-t$ clot is called the $s-t$ *clot number* and is denoted by $\kappa(s, t)$, and the cardinality of an $s-t$ band, called the $s-t$ *band number*, is denoted by $\lambda(s, t)$. The cardinality of a minimum $s-t$ separating mixed cut is denoted by $\sigma(s, t)$.

The following result gives vertex connectivity of complete graphs and an upper bound for non-complete graphs.

**Theorem 5.7** $\kappa(K_n) = n-1$. If $G$ is incomplete, then $\kappa(G) \leq n-2$.

**Proof**

i. Clearly, $K_n$ is a connected graph with $n$ vertices. Deletion of a vertex $v_1$ keeps the graph $G-v_1$ connected. Clearly, $G-v_1$ has $n-1$ vertices. Deleting one more vertex, say $v_2$ from $G-v_1$, gives a graph $G-\{v_1, v_2\}$, which is again connected. Continuing this process, we observe that deleting any number of ertices $i$, $1 \leq i \leq n-1$ does not disconnect the graph, but deleting exactly $n-1$ vertices gives a trivial graph with one vertex. Thus, $\kappa(K_n) = n-1$.

ii. Let $G$ be an incomplete graph with $n$ vertices. Then there are at least two vertices, say $v_i$ and $v_j$ which are not adjacent. If there is exactly one edge $v_iv_j$ missing, then deleting the $n-2$ vertices other than $v_i$ and $v_j$ disconnects the graph. So in this case $\kappa(G) = n-2$. If there are more edges missing, then clearly $\kappa(G) < n-2$ (Fig. 5.10). Hence, $\kappa(G) \leq n-2$. ❑

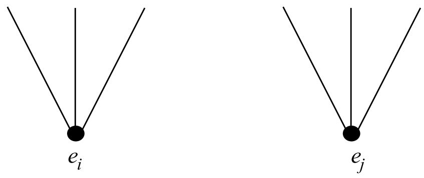



**Fig. 5.10**

The following result is obvious.

**Theorem 5.8** $\kappa(G) \min\limits_{\substack{s,\,t\in v \\ st\notin E}} \kappa(s, t)$, $\lambda(G) = \min\limits_{s,\,t\in v} \lambda(s, t)$, $\sigma(G) = \min\limits_{s,\,t\in v} \sigma(s, t)$.

**Cut of a graph:** Let $G(V, E)$ be a graph and let $A$ be any non-empty sub-set of the vertex set $V$. Let $\bar{A} = V - A$. The set of all edges with one end in $A$ and the other end in $\bar{A}$, denoted by $[A, \bar{A}]$ is called a *cut* of $G$. The concept of a cut of a graph is intermediate between that of an edge cut and a bond.

We note that every cut is an edge cut, but the converse is not true. Consider the graph in Figure 5.11. Here $F = \{e_1, e_2, e_3, e_4, e_5\}$ is an edge cut, but $F$ is not a cut.

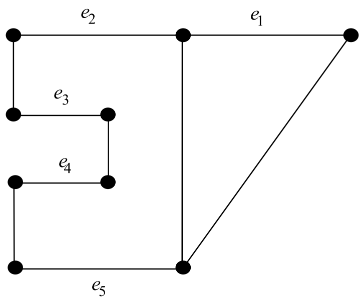



**Fig. 5.11**

Also, every bond is a cut, but the converse is not true. This is illustrated by the graph in Figure 5.12. Let $A = \{1, 2, 3, 4\}$. Then $\bar{A} = \{5, 6, 7, 8\}$. So $[A, \bar{A}] = \{e_1, e_2, e_3, e_4\}$. Here $[A, \bar{A}]$ is a cut but not a bond.

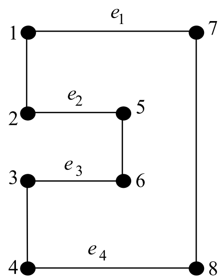



**Fig. 5.12**

**Theorem 5.9** Every minimal cut is a bond and every bond is a minimal cut.

***Proof*** Let $G(V, E)$ be a graph and let $C = [A, \bar{A}]$ be a cut of $G$. Assume $C$ to be a minimal cut. Then no subset of the edges of $C$ is a cut and this implies that $G - C$ has only two components $\langle A \rangle$ and $\langle \bar{A} \rangle$. Therefore $C$ is a bond.

Conversely, let $F$ be a bond. Then $G - F$ has only two components, say $C_1$ and $C_2$. Then $F = [V_1, V_2]$, with $V_2 = \bar{V}_1$. Thus $F$ is a cut and hence a minimal cut. ❑

**Theorem 5.10** Every cut is a disjoint union of minimal cuts.

***Proof*** Let $G(V, E)$ be a graph and let $C = [A, \bar{A}]$ be a cut of $G$. Let $C$ be not a minimal cut. Then at least one of $\langle A \rangle$, or $\langle \bar{A} \rangle$ has more than one component.

Assume $C_1, C_2, \ldots, C_r$ to be the components of $\langle A \rangle$ and $C'_1, C'_2, \ldots, C'_s$ be the components of $\langle \bar{A} \rangle$. (Clearly, at least one of $r$ and $s$ is greater than one.) Let $C_i$ be coalesced to vertices $c_i$, $1 \le i \le r$ and $C'_i$ be coalesced to vertices $c'_i$, $1 \le i \le s$, and let $H$ be the simple coalescence thus obtained. Obviously in $H$, there are no edges of the form $c_i c_j$ and $c'_i c'_j$, $i \ne j$. Thus $H$ is a bipartite graph (because there are edges $c_i c'_j$ in $H$) (Fig. 5.13).

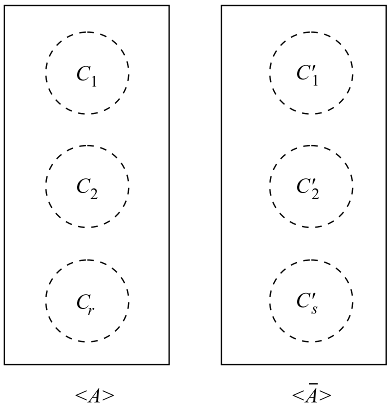



**Fig. 5.13**

If we can partition the edge set of $H$ into a disjoint union of bonds of $H$, the edges of $G$ corresponding to these bonds will be disjoint bonds of $G$ whose union is $C$. To achieve such a partition of $E(H)$, we first take the cut edges of $H$ as members of the partition and let $F$ be the set of such cut edges. For the remaining members of the partition, we take the stars at the remaining (non-isolated) vertices $c_i$ (or $c'_i$). This gives the required partition and hence the result follows. ❑

**Illustration** Consider the graph of Figure 5.14. Partition of $E(H)$ is $\{e_1\} \cup \{e_2\} \cup \{e_3, e_4\} \cup \{e_5, e_6\}$. $e_1$ is a cut edge, $e_2$ is a cut edge, $\{e_3, e_4\}$ form the star $K_{1,2}$ and $\{e_5, e_6\}$ form the star $K_{1,2}$.

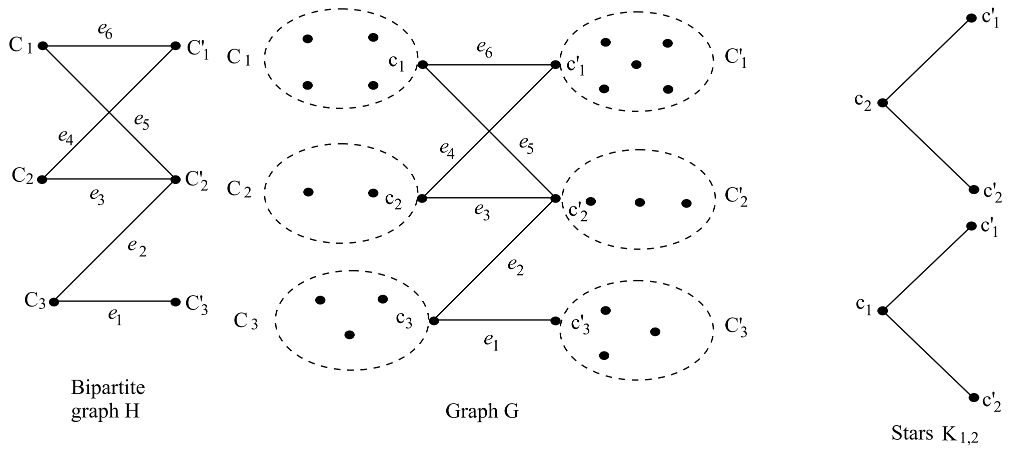



**Fig. 5.14**

**Remark** Though the equivalence of minimal edge cuts and minimal cuts is brought out by Theorem 5.10, there is an essential difference between edge cuts and cuts as already mentioned. To emphasise this, we observe that Theorem 5.10 cannot be generalised to state that every edge cut is a disjoint union of bonds. The example in Figure 5.11 illustrates this point.

Since it is enough to consider connected graphs for discussing connectivity concepts, in what follows, we shall assume that graphs are connected, unless stated otherwise.

The following results are reported by Harary and Frisck [105].

**Theorem 5.11** In a connected graph $G(V,\ E)$, if $st \notin E$, then $\kappa(s,\ t) \leq \sigma(s,\ t)$.

**Proof** Let $G(V,\ E)$ be a connected graph, and $s,\ t$ be vertices in $V$ such that $st \notin E$. Let $\kappa(s,\ t)$ be the cardinality of the minimum $s-t$ separating vertex cut ($s-t$ clot). Let $\sigma(s,\ t)$ be the cardinality of a minimum $s-t$ separating mixed cut. We prove that from any mixed $s-t$ separating set, we can get an $s-t$ separating vertex cut with no more elements.

Let $S$ be a minimum mixed $s-t$ separating set. If $ij$ is an edge in $S$, then both $i$ and $j$ cannot coincide with $s$ and $t$, since $st \notin E$.

If $i = s$, add $i$ to $S$, and remove from $S$ all edges with $i$ as an end vertex. If $i \neq s$, add $j$ to $S$ and remove from $S$ all edges with $j$ as an end vertex. The resulting, possibly mixed set is clearly an $s-t$ separating set with no more elements than $S$.

We repeat this process and remove all edges from $S$ and obtain a vertex cut $S'$ with atmost $|S|$ elements.

Since $\kappa(s,\ t) \leq |S'| \leq |S| = \sigma(s,\ t)$, we have $\kappa(s,\ t) \leq \sigma(s,\ t)$. ❑

**Corollary** In a connected graph $G(V,\ E)$, if $st \notin E$, then $\kappa(s,\ t) \leq \lambda(s,\ t)$.

**Note** If $st \in E$, then $\kappa(s,\ t)$ is not defined.

**Theorem 5.12** For any graph $G$, $\sigma(G) = \kappa(G)$.

**Proof**

**Case (i)** When $G = K_n$, then $\kappa(G) = n-1$ and $\lambda(G) = n-1$.

Let $S$ be a minimum mixed disconnecting set of $G$ and let $S = T \cup F$, where $T \subseteq V$, $F \subseteq E$, and $|T| = n_1$, $|F| = m_1$. Then $G-T$ is $K_{n-n_1}$. Therefore, $|F| \geq \lambda(K_{n-n_1-1}) = n-n_1-1$. Thus, $m_1 \geq n-n_1-1$. So, $\sigma = |S| = m_1+n_1 \geq n-n_1-1+n_1 = n-1$. Therefore, $\sigma \geq n-1 = \kappa$.

Also, $\sigma \leq \kappa$. Hence, $\sigma = \kappa$.

**Case (ii)** When $G$ is incomplete, then clearly $\sigma \leq \kappa$. We have to prove that $\sigma = \kappa$ when $G$ is complete. If possible, let there be a minimum $s-t$ separating mixed set $S = M \cup \{st\}$ with $\sigma = |S| < \kappa$. Now, $M$ can be replaced by a set of vertices $T$ (a subset of the vertex set of the induced subgraph $\langle M \rangle$) to provide a vertex cut of $G^* = G - st$ with cardinality at most $|M|$.

Let $C_1$ and $C_2$ be the components of $G-S$ to which $s$ and $t$ respectively belong. Let there be another component $C_3$ of $G-S$ and let $v$ be a vertex of $C_3$. Then $T \cup \{s\}$ is a $v-t$ separating vertex cut of $G$. But then $|T \cup \{s\}| \leq |S| < \kappa$, a contradiction (Fig. 5.15).

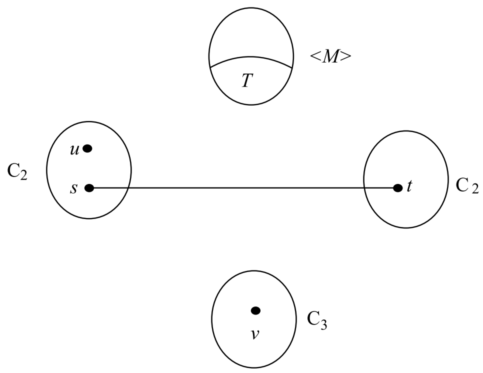



**Fig. 5.15**

Thus $C_1$ and $C_2$ are the only components of $G - S$. Also, if $u \in V(C_1)$, and $u \neq s$, then $T \cup \{s\}$ is a $u - t$ separating vertex cut of $G$, again leading to a contradiction. Thus $C_1 = \{s\}$, and similarly $C_2 = \{t\}$. So $G$ has $|V(M)| + 2$ vertices, and is incomplete. Therefore $\kappa(G) \leq n - 2 = |V(M)| + 2 - 2 = |V(M)| < \kappa$ implying $\kappa(G) < \kappa$, a contradiction. Thus $\dot{\sigma} \not< \kappa$. Hence $\sigma = \kappa$. ❑

The following inequalities are due to Whitney [265].

**Theorem 5.13 (Whitney)** For any graph $G$, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

***Proof*** We first prove $\lambda(G) \leq \delta(G)$.

If $G$ has no edges, then $\lambda = 0$ and $\delta = 0$. If $G$ has edges, then we get a disconnected graph, when all edges incident with a vertex of minimum degree are removed. Thus, in either case, $\lambda(G) \leq \delta(G)$.

We now prove $\kappa(G) \leq \lambda(G)$. For this, we consider the various cases. If $G = K_n$, then $\kappa(G) = \lambda(G) = n - 1$. Now let $G$ be an incomplete graph. In case $G$ is disconnected or trivial, then obviously $\kappa = \lambda = 0$.

If $G$ is disconnected and has a cut edge (bridge) $x$, then $\lambda = 1$. In this case, $\kappa = 1$, since either $G$ has a cut vertex incident with $x$, or $G$ is $K_2$.

Finally, let $G$ have $\lambda \geq 2$ edges whose removal disconnects it. Clearly, the removal of $\lambda - 1$ of these edges produces a graph with a cut edge (bridge) $x = uv$. For each of these $\lambda - 1$ edges, select an incident vertex different from $u$ or $v$. The removal of these vertices also removes the $\lambda - 1$ edges and quite possibly more. If the resulting graph is disconnected, then $\kappa < \lambda$. If not, $x$ is a cut edge (bridge) and hence the removal of $u$ or $v$ will result in either a disconnected or a trivial graph, so that $\kappa \leq \lambda$ in every case. ❑

**Illustration** We illustrate this by the graph shown in Figure 5.16. Here $\kappa = 2$, $\lambda = 3$ and $\delta = 4$.

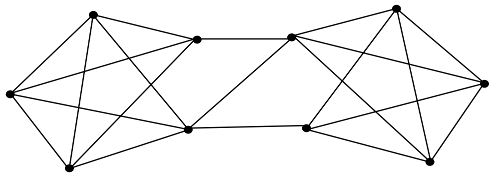


**Fig. 5.16**

**Theorem 5.14** For any $v \in V$ and any $e \in E$ of a graph $G(V, E)$, $\kappa(G) - 1 \leq \kappa(G - v)$ and $\lambda(G) - 1 < \lambda(G - e) \leq \lambda(G)$.

*Proof* We observe that the removal of a vertex or an edge from a graph can bring down $\kappa$ or $\lambda$ by at most one, and that while $\kappa$ may be increased by the removal of a vertex, $\lambda$ cannot be increased by the removal of an edge. ❑

**Theorem 5.15** For any three integers $r, s, t$ with $0 < r \leq s \leq t$, there is a graph $G$ with $\kappa = r$, $\lambda = s$ and $\delta = t$.

*Proof* Take two disjoint copies of $K_{t+1}$. Let $A$ be a set of $r$ vertices in one of them and $B$ be a set of $s$ vertices in the other. Join the vertices of $A$ and $B$ by $s$ edges utilising all the vertices of $B$ and all the vertices of $A$. Since $A$ is a vertex cut and the set of these $s$ edges is an edge cut of the resulting graph $G$, it is clear that $\kappa(G) = r$ and $\lambda(G) = s$. Also, there is at least one vertex which is not in $A \cup B$, and it has degree $t$, so that $\delta(G) = t$. ❑

**Illustration** Let $r = 1$, $s = 2$, $t = 3$. Take two copies of $K_4$. Here $\kappa(G) = 1$, $\lambda(G) = 2$, $\delta(G) = 3$ (Fig. 5.17).

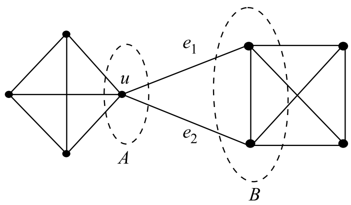



**Fig. 5.17**

**Theorem 5.16** For a graph, $\delta \geq \frac{n}{2}$ ensures $\lambda = \delta$.

*Proof* Let $G$ be a graph with $\delta \geq \frac{n}{2}$. Let $\lambda < \delta$. Let $F$ be a set of $\lambda$ edges disconnecting $G$. Let $C_1$ and $C_2$ be the components of $G - F$, and $A_1$ and $A_2$ be the end vertices of $F$ in $C_1$ and $C_2$, respectively.

Suppose $|A_1| = r$, $|A_2| = s$ and also $V(C_1) = A_1$. Then each vertex of $C_1$ is adjacent with at least one edge of $F$. So the number $m_1$ of edges in $C_1$ satisfies the inequality

$$m_1 \geq \frac{1}{2}(r\delta - \lambda) > \frac{1}{2}(r\delta - \delta), \text{ since } \lambda < \delta \text{ by assumption.}$$

Therefore, $m_1 > \frac{1}{2}(r-1)\delta > \frac{1}{2}(r-1)r$, since $r \leq |F| = \lambda < \delta$.

But a graph on $n$ vertices cannot have more than $\frac{1}{2}r(r-1)$ edges. Thus, $|V(C_1)| > |A_1|$. Similarly, $V(C_2) > |A_2|$. Thus each of $C_1$ and $C_2$ contains at least $\delta + 1$ vertices.

Therefore, $n = |V(G)| \geq 2(\delta+1) \geq 2(\frac{n}{2}+1) = n+2$ or $n \geq n+2$, which is a contradiction. Hence $\lambda < \delta$ is not possible. So $\lambda = \delta$. ❑

## 5.4 Menger's Theorem

Harary [104] listed eighteen variations of Menger's theorem including those for digraphs. Clearly, all these are equivalent and one can be obtained from the other. Several proofs of the various forms of Menger's theorems have appeared, for example, in Dirac [67], Ford and Fulkerson [81], Lovasz [150], McCuaig [156], Menger [158], Nash-Williams and Tutte [169], O'Neil [173], Pym [213] and Wilson [269].

Let $u$ and $v$ be two distinct vertices of a connected graph $G$. Two paths joining $u$ and $v$ are called disjoint (vertex disjoint) if they have no vertices other than $u$ and $v$ (and hence no edges) in common. The maximum number of such paths between $u$ and $v$ is denoted by $p(u, v)$. If the graph $G$ is to be specified, it is denoted by $p(u, v|G)$.

The following is the vertex form of Menger's theorem. The proof is due to Nash-Williams [9] and Tutte [169].

**Theorem 5.17 (Menger-vertex form)** The minimum number of vertices separating two non-adjacent vertices $s$ and $t$ is equal to the maximum number of disjoint $s-t$ paths, that is, for any pair of non-adjacent vertices $s$ and $t$, the clot number equals the maximum number of disjoint $s-t$ paths. That is, $\kappa(s, t) = p(s, t)$, for every pair $s, t \in V$ with $st \notin E$.

**Proof** Let $G(V, E)$ be a graph with $|E| = m$. We use induction on $m$, the number of edges. The result is obvious for a graph with $m = 1$ or $m = 2$. Assume that the result is true for all graphs with less than $m$ edges. Let the result be not true for the graph $G$ with $m$ edges. Then we have

$$p(s, t|G) < \kappa(s, t|G) = q \text{ (say)}, \qquad (5.17.1)$$

as for any graph, we obviously have $p(s, t) \leq \kappa(s, t)$.

Let $e = uv$ be an edge of $G$. The deletion graph $G_1 = G - e$, and the contraction graph $G_2 = G|e$ have less number of edges than $G$. Therefore, by induction hypothesis, we have

$$p(s, t|G_1) = \kappa(s, t|G_1) \text{ and } p(s, t|G_2) = \kappa(s, t|G_2). \qquad (5.17.2)$$

Let $I$ be an $(s, t)-$ clot in $G_1$ and $J'$ be an $(s, t)-$ clot in $G_2$. Then we have

$|I| = \kappa(s,\, t|G_1) = p(s,\, t|G_1) \leq p(s,\, t|G) < q$ and

$|J'| = \kappa(s,\, t|G_2) = p(s,\, t|G_2) \leq p(s,\, t|G) < q$, by using (5.17.2) and (5.17.1).

So $|J'| < q$ and therefore $|J'| \leq q-1$.

Now to $J'$ there corresponds an $(s-t)$ vertex cut $J$ of $G$ such that $|J| \leq |J'|+1$, since, by elementary contraction, $\kappa(s,\, t)$ can be decreased by at most one, and this decrease actually occurs when e $\in E(\langle J \rangle)$.

Thus, $|J| \leq |J'|+1 \leq q-1+1 = q$,

that is, $|J| \leq q$. (5.17.3)

Since $J$ is an $(s,\, t)$ vertex cut in $G$, $\kappa(s,\, t) \leq |J|$, $q \leq |J|$.

Thus, $q \leq |J| \leq q$, so that $|J| = q$.

Therefore, $|I| < q$ and $|J| = q$ and $u,\, v \in J$ by (5.17.3). (5.17.4)

Let

$H_s = \{w \in I \cup J$ : there exists an $s-w$ path in $G$, vertex-disjoint from $I \cup J - \{w\}\}$ and

$H_t = \{w \in I \cup J$ : there exists a $t-w$ path in $G$, vertex-disjoint from $I \cup J - \{w\}\}$.
Clearly, $H_s$ and $H_t$ are $(s-t)$ separating vertex cuts in $G$. Therefore,

$|H_s| \geq q$ and $|H_t| \geq q$. (5.17.5)

Obviously, $H_s \cup H_t \subseteq I \cup J$.
We claim that $H_s \cap H_t \subseteq I \cup J$. For this, let $w \in H_s \cap H_t$. Then there exists an $s-w$ path $P_1$ and $w-t$ path $P_2$ in $G$ vertex disjoint from $I \cup J - \{w\}$. So $P_1 \cup P_2$ contains a path, say $P$. If $e \in P$ then we have $u,\, v \in V(P) \cap J \subseteq \{w\}$, which is impossible. Therefore $e \notin P$ and so $P \subseteq G-e$. Since $I$ is an $(s,\, t)$ separator in $G-e$ and $J$ is an separator in $G$, $P$ has a vertex common with $I$ and also with $J$. So $w \in I \cap J$. Thus, $H_s \cap H_t \subseteq I \cap J$.
Combining (5.17.4) and (5.17.5), and the above observation, we have

$$q+q \leq |H_s|+|H_t| = |H_s \cup H_t|+|H_s \cap H_t| \leq |I \cup J|+|I \cap J|$$

$$= |I|+|J| < q+q,$$

which is a contradiction.

Thus (5.17.1) is not true, and therefore, we have

$$\kappa(s,\, t|G) = p(s,\, t|G).$$ ❑

**Definition:** Two paths joining $u$ and $v$ are said to be *edge-disjoint* if they have no edges in common. The maximum number of edge-disjoint paths between $u$ and $v$ is denoted by $l(u, v)$.

The following is the edge form of Menger's theorem and the proof is adopted from Wilson [196].

**Theorem 5.18 (Menger-edge form)** For any pair of vertices $s$ and $t$ of a graph $G$, the minimum number of edges separating $s$ and $t$ equals the maximum number of edge-disjoint paths joining $s$ and $t$, that is, $\lambda(s, t) = l(s, t)$ for every pair $s, t \in V$.

**Proof** Let $G(V, E)$ be a graph and let $|E| = m$. We use induction on the number of edges $m$ of $G$. For $m = 1, 2$, the result is obvious. Assume the result to be true for all graphs with fewer than $m$ edges. Let $\lambda(s, t) = k$. We have two cases to consider.

**Case (i)** Suppose $G$ has an $(s-t)$ band $F$ such that not all edges of $F$ are incident with $s$, nor all edges of $F$ are incident with $t$. Then $G-F$ consists of two non-trivial components $C_1$ and $C_2$ with $s \in C_1$ and $t \in C_2$. Let $G_1$ be the graph obtained from $G$ by contracting the edges of $C_1$, and $G_2$ be a graph obtained from $G$ by contracting the edges of $C_2$. Therefore,

$$G_1 = G||E(C_1) \text{ and } G_2 = G||E(C_2).$$

Since $G_1$ and $G_2$ have less edges than $G$, the induction hypothesis applies to them. Also, the edges corresponding to $F$ provide an $(s-t)$ band in $G_1$ and $G_2$, so that $\lambda(s, t|G_1) = k$ and $\lambda(s, t|G_2) = k$. Thus, by induction hypothesis, there are $k$ edge-disjoint paths joining $s$ and $t$ in $G_1$, and there are $k$ edge-disjoint paths joining $s$ and $t$ in $G_2$. Thus $l(s, t|G_1) = k$ and $l(s, t|G_2) = k$.

The section of the path of the $k$ edge-disjoint paths joining $s$ and $t$ in $G_2$ which are in $C_1$ and the section of the paths of the $k$ edge-disjoint paths joining $s$ and $t$ in $G_1$ which are in $C_2$ can now be combined to get $k-$ edge disjoint paths between $s$ and $t$ in $G$. Hence $l(s, t|G) = k$.

**Case (ii)** Every $(s-t)$ band of $G$ is such that either all its edges are incident with $s$, or all its edges are incident with $t$.

If $G$ has an edge $e$ which is not in any $(s-t)$ band of $G$, then $\lambda(s, t|G-e) = \lambda(s, t|G) = k$. Since the induction hypothesis is applicable to $G-e$, there are $k$ edge-disjoint paths between $s$ and $t$ in $G-e$ and thus in $G$. Hence $l(s, t|G) = k$.

Now, assume that every edge of $G$ is in at least one $(s-t)$ band of $G$. Then every $s-t$ path $P$ of $G$ is either a single edge or a pair of edges. Any such path $P$ can therefore contain at most one edge of any $(s-t)$ band. Then $G-E(P) = G_1$ is a graph with $\lambda(s, t|G_1) = \kappa - 1$.

Appling induction hypothesis, we have $l(s, t|G_1) = \kappa - 1$. Together with $P$, we get $l(s, t|G) = k$. ❑

**Definition:** A graph $G$ is said to be $n$-(*vertex*) connected if $\kappa(G) \geq n$ and $n$-(*edge*) connected if $\lambda(G) \geq n$. Thus a separable graph ($\kappa = 1$) is 1-connected and not 2-connected. A separable graph without cut edges is only 1-edge connected.

## 5.5 Some Properties of a Bond

We give some properties of a bond (bond is also called a cut-set). The first property follows.

**Theorem 5.19** Every bond in a connected graph $G$ connects at least one branch of every spanning tree of $G$.

*Proof* Let $G$ be a connected graph and $T$ be a spanning tree of $G$. Let $S$ be an arbitrary bond in $G$. Clearly, there are edges which are common in $S$ and $T$. For, if there is no edge of $S$ which is also in $T$, then removal of the bond $S$ from $G$ will not disconnect the graph, as $G-S$ contains $T$ and is therefore connected. Thus $S$ and $T$ have at least one common edge. ❑

**Theorem 5.20** In a connected graph $G$, any minimal set of edges containing at least one branch of every spanning tree of $G$ is a bond.

*Proof* Let $G$ be a connected graph and let $Q$ be a minimal set of edges containing at least one branch of every spanning tree of $G$.

Consider $G-Q$, the subgraph that remains after removing the edges in $Q$ from $G$. Since $G-Q$ contains no spanning tree of $G$, therefore $G-Q$ is disconnected (one component of which may just consist of an isolated vertex). Also, since $Q$ is a minimal set of edges with this property, therefore any edge $e$ from $Q$ returned to $G-Q$ creates at least one spanning tree. Thus the subgraph $G-Q+e$ is a connected graph. Therefore $Q$ is a minimal set of edges whose removal from $G$ disconnects $G$. This, by definition, is a bond. ❑

**Theorem 5.21** Every cycle has an even number of edges in common with any bond.

*Proof* Let $G$ be a graph and let $S$ be a bond of $G$. Let the removal of $S$ partition the vertices of $G$ into two mutually disjoint subsets $V_1$ and $V_2$. Consider a cycle $C$ in $G$ (Fig. 5.18).

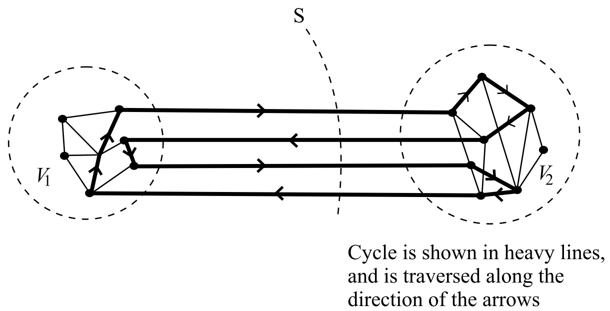



**Fig. 5.18**

If all the vertices in $C$ are entirely within vertex set $V_1$ (or $V_2$), then the number of edges common to $S$ and $C$ is zero, which is an even number. If, on the other hand, some vertices in $C$ are in $V_1$ and some in $V_2$, we traverse back and forth between the sets $V_1$ and $V_2$ as we traverse the cycle. Because of the closed nature of a cycle, the number of edges between $V_1$ and $V_2$ must be even. And, since every edge in $S$ has one end in $V_1$ and other in $V_2$, and no other edge in $G$ has the property of separating sets $V_1$ and $V_2$, the number of edges common to $S$ and $C$ is even. ❑

## 5.6 Fundamental Bonds

Consider a spanning tree $T$ of a connected graph $G$. Take any branch $b$ in $T$. Since $\{b\}$ is a bond in $T$, therefore $\{b\}$ partitions all vertices of $T$ into two disjoint sets, one at each end of $b$. Consider the same partition of vertices in $G$ and the bond $S$ in $G$ that corresponds to this partition. Bond $S$ will contain only one branch $b$ of $T$ and the rest (if any) of the edges in $S$ are chords with respect to $T$. Such a bond $S$ containing exactly one branch of a tree $T$ is called a *fundamental bond* with respect to $T$.

**Theorem 5.22** The ring sum of any two bonds is either a third bond, or an edge-disjoint union of bonds.

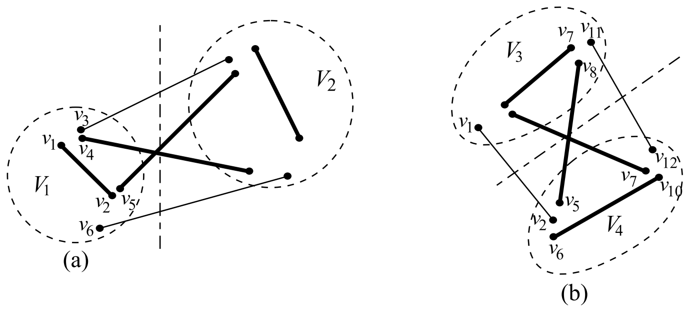
**Proof** Let $G$ be a connected graph, and $S_1$ and $S_2$ be two bonds. Let $V_1$ and $V_2$ be the unique and disjoint partitioning of the vertex set $V$ of $G$ corresponding to $S_1$. Let $V_3$ and $V_4$ be the partitioning corresponding to $S_2$.

Clearly, $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \varphi$, $V_3 \cup V_4 = V$ and $V_3 \cap V_4 = \varphi$ (Fig. 5.19(a) and (b)).

Now, let $(V_1 \cap V_4) \cup (V_2 \cap V_3) = V_5$ and $(V_1 \cap V_3) \cup (V_2 \cap V_4) = V_6$.

Clearly, $V_5 = V_1 \oplus V_3$ and $V_6 = V_2 \oplus V_3$ (Fig. 5.19(c)).

The ring sum of two bonds $S_1 \oplus S_2$ consists only of edges that join vertices in $V_5$ to those in $V_6$. Also, there are no edges outside $S_1 \oplus S_2$ that joins vertices in $V_5$ to those in $V_6$. Thus the set of edges $S_1 \oplus S_2$ produces a partitioning of $V$ into $V_5$ and $V_6$ such that $V_5 \cup V_6 = V$ and $V_5 \cap V_6 = \varphi$. Hence $S_1 \oplus S_2$ is a bond if the subgraphs containing $V_5$ and $V_6$ each remain connected after $S_1 \oplus S_2$ is removed from $G$. Otherwise, $S_1 \oplus S_2$ is an edge disjoint union of bonds. ❑



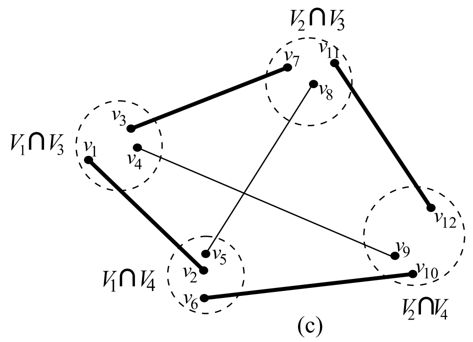



**Fig. 5.19**

**Example** Consider the graph in Figure 5.20. Here $\{d, e, f\} \oplus \{f, g, h\} = \{d, e, g, h\}$ is a bond, $\{a, b\} \oplus \{b, c, e, f\} = \{a, c, e, f\}$ is another bond and $\{d, e, g, h\} \oplus \{f, g, k\} = \{d, e, f, h, k\} = \{d, e, f\} \cup \{h, k\}$ an edge disjoint union of bonds.

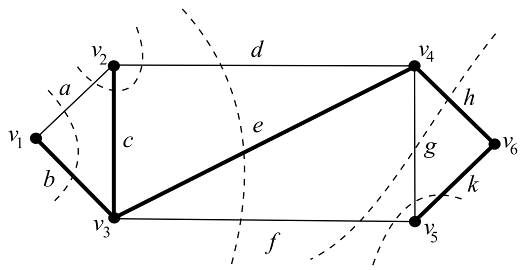



**Fig. 5.20**

**Theorem 5.23** With respect to a given spanning tree $T$, a chord $c_i$ that determines a fundamental cycle $C$ occurs in every fundamental bond associated with the branches in $C$ and in no other.

**Proof** Let $G$ be a connected graph and $T$ be a spanning tree of $G$. Let $c_i$ be a chord with respect to $T$ and let the fundamental cycle made by $c_i$ be called $C$, consisting of $k$ branches $b_1, b_2, \ldots, b_k$ in addition to the chord $c_i$. So $C = \{c_i, b_1, b_2, \ldots, c_k\}$ is a fundamental cycle with respect to $T$.

Now every branch of any spanning tree has a fundamental bond associated with it. So let $S_1$ be the fundamental bond associated with $b_1$, consisting of $q$ chords in addition to the branch $b_1$. Thus, $S_1 = \{b_1, c_1, c_2, \ldots, c_q\}$ is a fundamental bond with respect to $T$.

We know there are even number of edges common to $C$ and $S_1$. Clearly, $b_1$ is in both $C$ and $S_1$. So there is exactly one more edge which is in both $C$ and $S_1$. Obviously, the edge $c_i$ in $C$ can possibly be in $S_1$. Thus $c_i$ is one of the chords $c_1, c_2, \ldots, c_q$.

The same argument holds for fundamental bonds associated with $b_2, b_3, \ldots, b_k$. Thus the chord $c_i$ is contained in every fundamental bond associated with branches in $C$.

Now we show that the chord $c_i$ is not in any other fundamental bond $S'$ with respect to $T$, besides those associated with $b_1, b_2, \ldots, b_k$. Let this be possible. Then since none of the branches in $C$ are in $S'$, there is only one edge $c_i$ common to $S'$ and $C$, which gives a contradiction to the fact that there are even number of edges common to a fundamental bond and a cycle. ❑

**Example** In the graph of Figure 5.20, consider the spanning tree $\{b, c, e, h, k\}$. The fundamental cycle made by the chord $f$ is $C = \{f, e, h, k\}$. The three fundamental bonds determined by the three branches $e$, $h$ and $k$ are as follows: (i) determined by $e$ is $\{d, e, f\}$, (ii) determined by $h$ is $\{f, g, h\}$ and (iii) determined by $k$ is $\{f, g, k\}$. Clearly, chord $f$ occurs in each of these three fundamental bonds and there is no other fundamental bond that contains $f$.

**Theorem 5.24** With respect to a given spanning tree $T$, a branch $b_i$ that determines a fundamental bond $S$ is contained in every fundamental cycle associated with the chords in $S$, and in no others.

**Proof** Let $G$ be a connected graph and $T$ be a spanning tree in $G$. Let the fundamental bond determined by a branch $b_i$ be $S = \{b_i, c_1, c_2, \ldots, c_p\}$.

Let $C_1$ be the fundamental cycle determined by chord $c_1$, so that

$$C_1 = \{c_1, b_1, b_2, \ldots, b_q\}.$$

We know that $S$ and $C_1$ have even number of edges in common. One common edge is obviously $c_1$. Thus the second common edge should be $b_i$, so that $b_i$ is also in $C_1$. Therefore $b_i$ is one of the branches $b_1, b_2, \ldots, b_q$.

The same is true for the fundamental cycles made by the chords $c_2, c_3, \ldots, c_p$.

Now assume that $b_i$ occurs in a fundamental cycle $C_{p+1}$ made by a chord other than $c_1, c_2, \ldots, c_p$. Since none of the chords $c_1, c_2, \ldots, c_p$ is in $C_{p+1}$, there is only one edge $b_i$ common to a cycle $C_{p+1}$ and the bond $S$, which is not possible. Hence the result follows. ❑

**Example** Consider the graph of Figure 5.20. Consider the branch $e$ of spanning tree $\{b, c, e, h, k\}$. The fundamental bond determined by $e$ is $\{e, d, f\}$. The two fundamental cycles determined by chords $d$ and $f$ are respectively $\{d, c, e\}$ and $\{f, e, h, k\}$. Clearly, branch $e$ is contained in both these fundamental cycles and none of the remaining three fundamental cycles contains branch $e$.

**Theorem 5.25** Let $A$, $B$ be two disjoint vertex subsets of a graph $G$ and let any vertex subset of $G$ which meets every $A-B$ path in $G$ have at least $k$ vertices. Then there are $k$ vertex disjoint $A-B$ paths in $G$.

**Proof** Let $G$ be a graph and let $A$ and $B$ be two disjoint vertex subsets of $G$. Let $S$ be any vertex subset of $G$ which meets every $A-B$ path in $G$ and let $|S| \geq k$.

Take two new vertices $s$ and $t$, and join $s$ by an edge to each vertex of $A$, and join $t$ by an edge to each vertex of $B$. Let $G'$ be the resulting graph, and in $G'$ we have $\kappa(s, t) \geq k$.

Hence by Menger's theorem, there are $k$ vertex disjoint paths between $s$ and $t$ in $G'$. Omitting the edges incident with $s$ and $t$ in these paths, we get $k$ vertex-disjoint $A-B$ paths in $G$. ❑

**Definition:** A graph $G$ is *k-connected* if $\kappa(G) = k$, and $G$ is *k-edge connected* if $\lambda(G) = k$. A $k$-connected ($k$-edge connected) graph is $r$-connected ($r$-edge 1-connected) for each $r$, $0 \leq r \leq k-1$. Clearly, a separable graph ($\kappa = 1$) is connected and not 2-connected. A separable graph without cut edge is 2-edge connected. A separable graph with cut edges is only 1-connected.

The following result is due to Whitney [265].

**Theorem 5.26** A graph $G$ with at least three vertices is 2-connected if and only if any two vertices of $G$ are connected by at least two internally disjoint paths.

**Proof** Let $G$ be 2-connected so that $G$ contains no cut vertex. Let $u$ and $v$ be two distinct vertices of $G$. To prove the result, we induct on $d(u, v)$.

If $d(u, v) = 1$, let $e = uv$. Since $G$ is 2-connected and $n(G) \geq 3$, therefore $e$ cannot be a cut edge of $G$. For, if $e$ is a cut edge, then at least one of $u$ and $v$ is a cut vertex. Now, by Theorem 5.2, $e$ belongs to a cycle $C$ in $G$. Then $C-e$ is a $u-v$ path in $G$, internally disjoint from the path $uv$.

Assume any two vertices $x$ and $y$ of $G$, such that $d(x, y) = t-1$, $t \geq 2$, are joined by two internally disjoint $x-y$ paths in $G$. Let $d(u, v) = t$ and let $P$ be a $u-v$ path of length $t$, and $w$ be the vertex before $v$ on $P$. Then $d(u, w) = t-1$. Therefore, by induction hypothesis, there are two internally disjoint $u-w$ paths, say $P_1$ and $P_2$, in $G$. Since $G$ has no cut vertex, $G-w$ is connected and therefore there exists a $u-v$ path $Q$ in $G-w$. Clearly, $Q$ is a $u-v$ path in $G$ not containing $w$. Suppose $x$ is the vertex of $Q$ such that $x-v$ section of $Q$ contains only the vertex $x$ in common with $P_1 \cup P_2$ (Fig. 5.21). Assume $x$ belongs to $P_1$. Then the union of the $u-x$ section of $P_1$ and $x-v$ section of $Q$ together with $P_2 \cup \{wv\}$ are two internally disjoint $u-v$ paths in $G$.

Conversely, assume any two distinct vertices of $G$ are connected by at least two internally disjoint paths. Then $G$ is connected. Also, $G$ has no cut vertex. For, if $v$ is a cut vertex of $G$, then there exist vertices $u$ and $w$ such that every $u-w$ path contains $v$, contradicting the hypothesis. Thus $G$ is 2-connected. ❑

The following property of 3-connected graphs is given in Thomassen [241] and is attributed to Barnette and Grunbaum [14] and Titov [243].

**Theorem 5.27** If $G$ is a 3-connected graph with at least five vertices, then $G$ has an edge $e$ such that $G-e$ is a subdivision of a 3-connected graph.

**Proof** Since $G$ is 3-connected, $\delta \geq 3$, and so by Menger's theorem, $G$ has a subdivision of $K_4$.

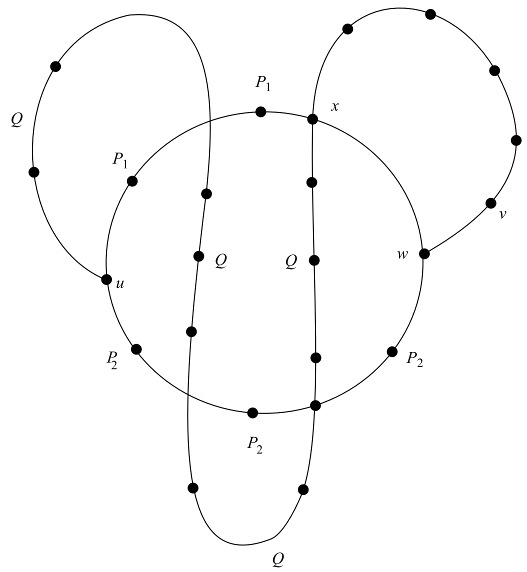



**Fig. 5.21**

Let $H$ be a proper subgraph of $G$ which is a subdivision of a 3-connected graph, and let $H$ have maximum possible number of edges. If $H$ is a 3-connected spanning subgraph of $G$, then by the maximality of $H$, $G$ has an edge $e$ such that $H = G - e$ is a subgraph of $G$ with the desired property.

Now, let $H$ be 3-connected but not spanning (Fig. 5.22). Then there is a vertex $v \in V(G) - V(H)$, so that there are three $v - V(H)$ paths, say $P_1$, $P_2$ and $P_3$ which have only vertex $v$ in common. Let the other end vertices of these paths in $V(H)$ be $v_1$, $v_2$ and $v_3$. If $v_1v_2 = e \in E(H)$, then $H + P_1 + P_2 - e$ is a subdivision of a 3-connected graph. Otherwise, $H + P_1 + P_2$ is a subdivision of a 3-connected graph. In both cases the maximality of $H$ is contradicted.

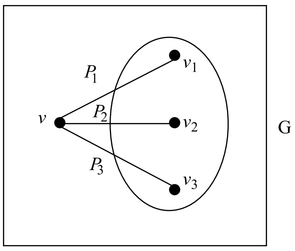



**Fig. 5.22**

Let $H$ be not 3-connected. Then $H$ has a suspended path $P$ of length at least 2. Let $u$ and $v$ be the end vertices of $P$. Since $G$ is 3-connected, $G - \{u, v\}$ has a path $P'$ joining an internal vertex $w$ of $P$ to a vertex in $V(H) - V(P)$. But then $H \cup P'$ is a subdivision of a 3-connected graph. By the choice of $H$, $H \cup P' = G$ and $P'$ consists of a single edge $e'$. ❑

The following property of 3-connected graphs is attributed to Thomassan [241].

**Theorem 5.28** If $G$ is a 3-connected graph with at least five vertices, then $G$ has an edge $e$ such that $G|e$ is 3-connected.

**Proof** Let $G$ be a 3-connected graph with at least five vertices. Let $e = uv$ be an edge of $G$ such that $G|e$ is not 3-connected. Then $G|e$ is 2-connected.

Let $\{x, y\}$ be a vertex cut of $G|e$ and let $z$ be the vertex into which $x$ and $y$ have been coalesced. Assume both $x$ and $y$ are different from $z$. Then $G|e - \{x, y\}$ is a graph obtained by contracting an edge of the connected graph $G - \{x, y\}$. This implies that $G|e - \{x, y\}$ is connected, which is a contradiction. Thus one of $x$ and $y$ coincides with $z$. Renaming the other as $w$, we see that $G$ has a vertex cut $\{u, v, w\}$.

Let $G_1$ be the smallest component of $G - \{u, v, w\}$. Since $G$ is 3-connected, $G_1$ is joined to $w$ by an edge $e_1 = wx_1$. If $G|e_1$ is not 3-connected, by a similar argument, there is a vertex $y_1$ such that $G - \{w, x_1, y_1\} = G_2$ is disconnected. But then the smallest component of $G_2$ is a proper subgraph of $G_1$.

Continuing in this way, we reach a stage when the smallest sub-graph is a single vertex and the edge $f$ joining it to the previous vertex cut is such that $G|f$ is a 3-connected graph. ❑

**Illustration** We illustrate this in Figure 5.23, where graph $G$ is 3-connected having vertex cut $\{u, v, w\}$. $G|e$ is 2-connected with vertex cut $\{z, w\}$ and $G|e_1$ is 3-connected. In $G - \{u, v, w\}$, we observe that the smallest subgraph is a single vertex.

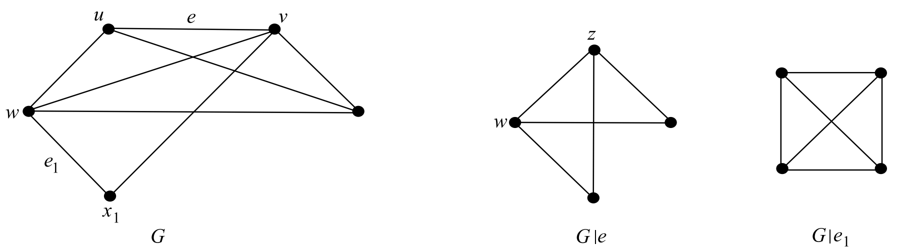



**Fig. 5.23**

## 5.7 Block Graphs and Cut Vertex Graphs

The block graph $B(G)$ of a graph $G$ is a graph whose vertices are the blocks of $G$ and two of these vertices are adjacent whenever the corresponding blocks contain a common cut vertex of $G$. The cut vertex graph $C(G)$ of a graph $G$ has vertices as cut vertices of $G$ and two such vertices are adjacent if the cut vertices of $G$ to which they belong lie on a common block (Fig. 5.24).

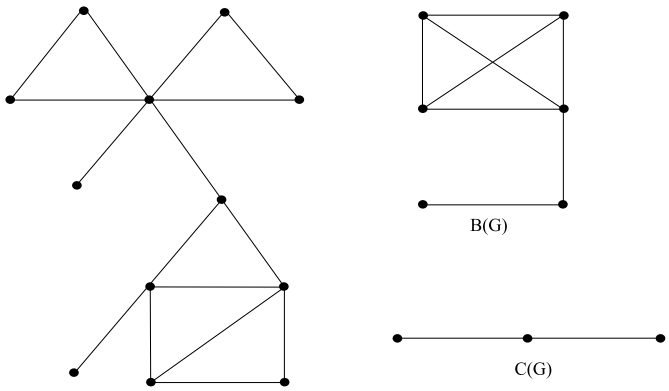



**Fig. 5.24**

The following characterisation of block graphs is due to Harary [103].

**Theorem 5.29** A graph $H$ is the block graph of some graph if and only if every block of $H$ is complete.

**Proof** Let $H = B(G)$ and assume there is a block $H_i$ of $H$ which is not complete. Then there are two vertices in $H_i$ which are non-adjacent and lie on a shortest common cycle $Z$ of length at least 4. But the union of the blocks of $G$ corresponding to the vertices of $H_i$ which lie on $Z$ is then connected and has no cut vertex, so it itself is contained in a block, contradicting the maximality property of a block of a graph.

Conversely, let $H$ be a given graph in which every block is complete. Form $B(H)$, and then form a new graph $G$ by adding to each vertex $H_i$ of $B(H)$ a number of end edges equal to the number of vertices of the block $H_i$ which are not cut vertices of $H$. Then it is easy to see that $B(G)$ is isomorphic to $H$. ❑

**1-isomorphism** A separable graph consists of two or more non-separable subgraphs, and each of the largest non-separable subgraph is a block. The graph in Figure 5.25 has five blocks and three cut vertices $u$, $v$ and $w$. We note that a non-separable connected graph consists of just one block.

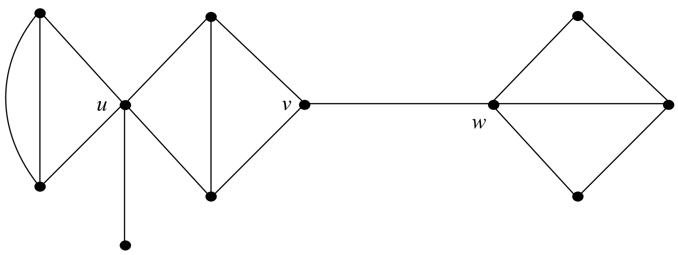



**Fig. 5.25**

Now, compare the disconnected graphs of Figure 5.26 with the graph of Figure 5.25. Clearly, these two graphs are not isomorphic, as they do not have the same number of vertices. Evidently, the blocks of the graph of Figure 5.25 are isomorphic to the components of the graph of Figure 5.26. We call such graphs 1-isomorphic.

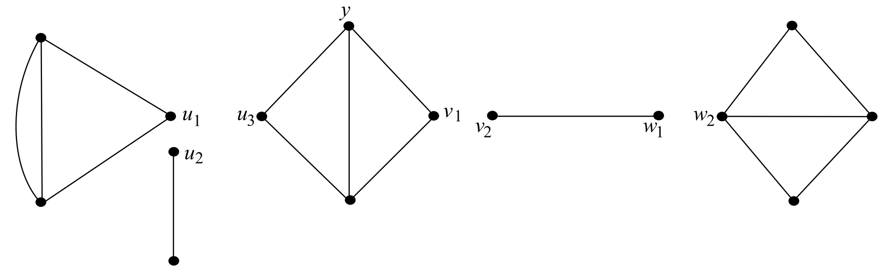



**Fig. 5.26**

These observations lead to the following definition.

**Definition:** Two graphs $G_1$ and $G_2$ are said to be *1-isomorphic* if they become isomorphic to each other under repeated application of the following operation.

**Operation 1** Split a cut vertex into two vertices to produce two disjoint subgraphs.
This definition implies that two non-separable graphs are 1-isomorphic if and only if they are isomorphic.

Two 1-isomorphic graphs have the following property.

**Theorem 5.30** If $G_1$ and $G_2$ are 1-isomorphic graphs, then rank $G_1 =$ rank $G_2$ and nullity $G_1 =$ nullity $G_2$.

**Proof** Under operation 1, whenever a cut vertex in a graph $G$ is split into two vertices, the number of components in $G$ increases by one. Therefore, rank $G =$ number of vertices in $G$-number of components in $G$ remains invariant under operation 1.

Since no edges are destroyed or new edges created by operation 1, two 1-isomorphic graphs have the same number of edges. Two graphs with same rank, and same number of edges have the same nullity, since nullity = number of edges−rank. ❑

Suppose the two vertices $x$ and $y$ belonging to different components of the graph in Figure 5.26 are superimposed, then the graph obtained is shown in Figure 5.27. Clearly, the graph in Figure 5.27 is 1-isomorphic to the graph in Figure 5.26. Also, since the blocks of the graph in Figure 5.27 are isomorphic to the blocks of the graph in Figure 5.25, these two graphs are 1-isomorphic. Hence the three graphs in Figures 5.25, 5.26 and 5.27 are 1-isomorphic.

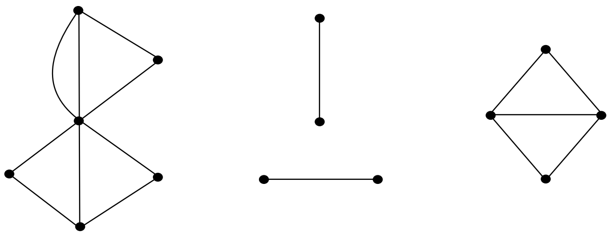


**Fig. 5.27**

We have seen that a graph $G_1$ is 1-isomorphic to a graph $G_2$ if the blocks of $G_1$ are isomorphic to the blocks of $G_2$. Since a non-separable graph is a block, 1-isomorphism for non-separable graphs is same as isomorphism. For separable graphs, obviously 1-isomorphism is different from isomorphism. In fact, graphs that are isomorphic are also 1-isomorphic, but the converse need not be true.

**2-isomorphism** In a 2-connected graph $G$, let $x$ and $y$ be a pair of vertices whose removal from $G$, leaves the remaining graph disconnected. That is, $G$ consists of a subgraph $H$ and its complement $\bar{H}$ such that $H$ and $\bar{H}$ have exactly two vertices $x$ and $y$, in common. Now, we perform the following operation on $G$.

**Operation 2** Split the vertex $x$ into $x_1$ and $x_2$, and the vertex $y$ into $y_1$ and $y_2$ such that $G$ is split into $H$ and $\bar{H}$. Let vertices $x_1$ and $y_1$ go with $H$, and $x_2$ and $y_2$ with $\bar{H}$. Now, rejoin the graphs $H$ and $\bar{H}$ by merging $x_1$ with $y_2$ and $x_2$ with $y_1$. Clearly, edges whose end vertices are $x$ and $y$ in $G$ can go with $H$ or $\bar{H}$, without affecting the final graph.

Two graphs are said to be *2-isomorphic* if they become isomorphic after undergoing operation 1, or operation 2, or both any number of times. For example, Figure 5.28 shows how the two graphs in (a) and (d) are 2-isomorphic.

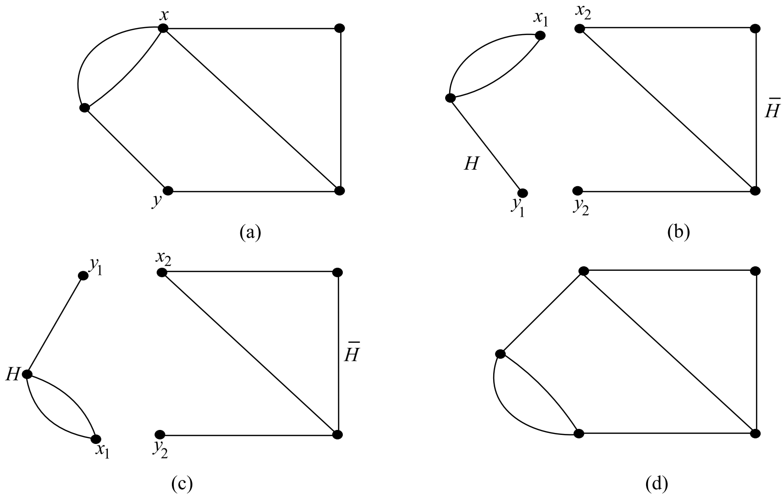



**Fig. 5.28**

It follows from the definition that isomorphic graphs are always 1-isomorphic and 1-isomorphic graphs are always 2-isomorphic. But 2-isomorphic graphs are not necessarily 1-isomorphic and 1-isomorphic graphs are not necessarily isomorphic. However, for graphs with three or more connectivity, isomorphism, 1-isomorphism and 2-isomorphism are same.

Clearly, no edges or vertices are created or destroyed under operation 2. So the rank and nullity of a graph remain unchanged under operation 2. Therefore the 2-isomorphic graphs are equal in rank and equal in nullity.

**Cycle correspondence:** Two graphs $G_1$ and $G_2$ are said to have a cycle correspondence if there is a one-one correspondence between the edges of $G_1$ and $G_2$, and a one-one correspondence between the cycles of $G_1$ and $G_2$, such that a cycle in $G_1$ formed by certain edges of $G_1$ has a corresponding cycle in $G_2$ formed by the corresponding edges of $G_2$, and vice versa. Clearly, isomorphic graphs have cycle correspondence. Since in a separable graph $G$, every cycle is confined to a particular block, every cycle in $G$ retains its edges as $G$ undergoes operation 1. Thus 1-isomorphic graphs have cycle correspondence.

The following result for 2-isomorphic graphs is due to Whitney [266].

**Theorem 5.31** Two graphs are 2-isomorphic if and only if they have cycle correspondence.

## 5.8 Exercises

1. Prove that a vertex $v$ of a tree is a cut vertex if and only if $d(v) > 1$.
2. Prove that a unicentric graph need not be separable.
3. Prove that a graph $H$ is the block-cut vertex graph of some graph $G$ if and only if it is a tree in which the distance between any two end vertices is even.
4. Prove that a unicentric graph need not have $d = 2r$.
5. Prove that a non-separable graph with at least two edges has nullity greater than zero.
6. Prove that a non-separable graph of nullity one is a cycle and its converse.
7. If $v$ is a cut vertex of a simple connected graph $G$, prove that $v$ is not a cut vertex of $\bar{G}$.
8. Prove that a connected $k$-regular bipartite graph is 2-connected.
9. Show that a simple connected graph with at least three vertices is a path if and only if it has exactly two vertices that are not cut vertices.
10. If $b(v)$ denotes the number of blocks of a simple connected graph $G$ containing vertex $v$, prove that the number of blocks $b(G)$ of $G$ is given by

$$b(G) = 1 + \sum_{v \in V(G)} (b(v) - 1).$$

11. Prove that if a graph $G$ is $k$-connected or $k$-edge-connected, then $m \geq \frac{nk}{2}$.
12. Prove that a connected graph with at least two vertices contains at least two vertices that are not cut vertices.
13. Prove that a 3-regular connected graph has a cut vertex if and only if it has a cut edge.
14. Prove that the connectivity and edge connectivity of a cubic graph are equal.
15. Prove that a graph with at least three vertices is 2-connected if and only if any two vertices of $G$ lie on a common cycle.
16. Prove that a graph is 2-connected if and only if for every pair of disjoint connected subgraphs $G_1$ and $G_2$, there exist two internally disjoint paths $P_1$ and $P_2$ of $G$ between $G_1$ and $G_2$.
17. In a 2-connected graph $G$, prove that any two longest cycles have at least two vertices in common.
18. Prove that a connected graph $G$ is 3-connected if and only if every edge of $G$ is the exact intersection of the edge sets of two cycles of $G$.
19. Prove that a connected graph is Eulerian if and only if each of its blocks is Eulerian.
20. Prove that a connected graph is Eulerian if and only if each of its edge cuts has an even number of edges.