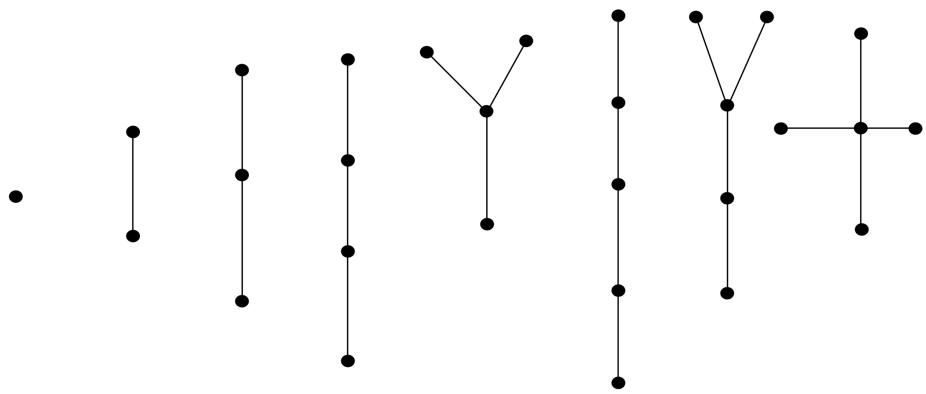
4. Trees

One of the important classes of graphs is the trees. The importance of trees is evident from their applications in various areas, especially theoretical computer science and molecular evolution.

4.1 Basics

Definition: A graph having no cycles is said to be *acyclic*. A *forest* is an acyclic graph.

Definition: A *tree* is a connected graph without any cycles, or a tree is a connected acyclic graph. The edges of a tree are called *branches*. It follows immediately from the definition that a tree has to be a simple graph (because self-loops and parallel edges both form cycles). Figure 4.1(a) displays all trees with fewer than six vertices.



(a)

Fig. 4.1(a)

The following result characterises trees.

Theorem 4.1 A graph is a tree if and only if there is exactly one path between every pair of its vertices.

Proof Let G be a graph and let there be exactly one path between every pair of vertices in G. So G is connected. Now G has no cycles, because if G contains a cycle, say between vertices u and v, then there are two distinct paths between u and v, which is a contradiction. Thus G is connected and is without cycles, therefore it is a tree.

Conversely, let G be a tree. Since G is connected, there is at least one path between every pair of vertices in G. Let there be two distinct paths between two vertices u and v of G. The union of these two paths contains a cycle which contradicts the fact that G is a tree. Hence there is exactly one path between every pair of vertices of a tree. ❑

The next two results give alternative methods for defining trees.

Theorem 4.2 A tree with n vertices has $n-1$ edges.

Proof We prove the result by using induction on n, the number of vertices. The result is obviously true for $n = 1$, 2 and 3. Let the result be true for all trees with fewer than n vertices. Let T be a tree with n vertices and let e be an edge with end vertices u and v. So the only path between u and v is e. Therefore deletion of e from T disconnects T. Now, $T-e$ consists of exactly two components T_1 and T_2 say, and as there were no cycles to begin with, each component is a tree. Let n_1 and n_2 be the number of vertices in T_1 and T_2 respectively, so that $n_1+n_2 = n$. Also, $n_1 < n$ and $n_2 < n$. Thus, by induction hypothesis, number of edges in T_1 and T_2 are respectively n_1-1 and n_2-1. Hence the number of edges in $T = n_1-1+n_2-1+1 = n_1+n_2-1 = n-1$. ❑

Theorem 4.3 Any connected graph with n vertices and $n-1$ edges is a tree.

Proof Let G be a connected graph with n vertices and $n-1$ edges. We show that G contains no cycles. Assume to the contrary that G contains cycles.

Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from one cycle at a time till the resulting graph H is a tree. As H has n vertices, so number of edges in H is $n-1$. Now, the number of edges in G is greater than the number of edges in H. So $n-1 > n-1$, which is not possible. Hence, G has no cycles and therefore is a tree. ❑

Definition: A graph is said to be *minimally connected* if removal of any one edge from it disconnects the graph. Clearly, a minimally connected graph has no cycles.

Here is the next characterisation of trees.

Theorem 4.4 A graph is a tree if and only if it is minimally connected.

Proof Let the graph G be minimally connected. Then G has no cycles and therefore is a tree.

Conversely, let G be a tree. Then G contains no cycles and deletion of any edge from G disconnects the graph. Hence G is minimally connected. ❑

The following results give some more properties of trees.

Theorem 4.5 A graph G with n vertices, $n-1$ edges and no cycles is connected.

Proof Let G be a graph without cycles with n vertices and $n-1$ edges. We have to prove that G is connected. Assume that G is disconnected. So G consists of two or more components and each component is also without cycles. We assume without loss of generality that G has two components, say G_1 and G_2 (Fig. 4.1(b)). Add an edge e between a vertex u in G_1 and a vertex v in G_2. Since there is no path between u and v in G, adding e did not create a cycle. Thus $G \cup e$ is a connected graph (tree) of n vertices, having n edges and no cycles. This contradicts the fact that a tree with n vertices has $n-1$ edges. Hence G is connected.

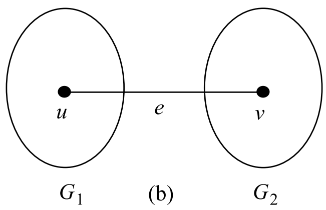


Fig. 4.1(b)

Theorem 4.6 Any tree with at least two vertices has at least two pendant vertices.

Proof Let the number of vertices in a given tree T be $n(n>1)$. So the number of edges in T is $n-1$. Therefore the degree sum of the tree is $2(n-1)$. This degree sum is to be divided among the n vertices. Since a tree is connected it cannot have a vertex of 0 degree. Each vertex contributes at least 1 to the above sum. Thus there must be at least two vertices of degree exactly 1.

Second proof We use induction on n. The result is obviously true for all trees having fewer than n vertices. We know that T has $n-1$ edges, and if every edge of T is incident with a pendant vertex, then T has at least two pendant vertices, and the proof is complete. So let there be some edge of T that is not incident with a pendant vertex and let this edge be $e = uv$ (Fig. 4.2). Removing the edge e, we see that the graph $T-e$ consists of a pair of trees say T_1 and T_2 with each having fewer than n-vertices. Let $u \in V(T_1)$, $v \in V(T_2)$, and $|V(T_1)| = n_1$, $|V(T_2)| = n_2$. Applying induction hypothesis on both T_1 and T_2, we observe that each of T_1 and T_2 has two pendant vertices. This shows that each of T_1 and T_2 has at least one pendant vertex that is not incident with the edge e. Thus the graph $T-e+e=T$ has at least two pendant vertices.

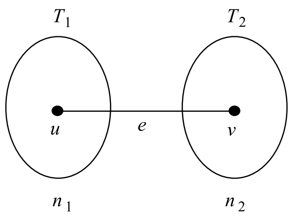


Fig. 4.2

Third proof Let T be a tree with $n(n > 1)$ vertices. The number of edges in T is $n-1$ and the sum of degrees in T is $2(n-1)$, that is, $\sum d_i = 2(n-1)$. Assume T has exactly one vertex v_1 of degree one, while all the other $n-1$ vertices have degree ≥ 2. Then sum of degrees is $d(v_1)+d(v_2)+\ldots+d(v_n) \geq 1+2+2+\ldots+2 = 1+2(n-1)$. So, $2(n-1) \geq 1+2(n-1)$, implying $0 \geq 1$, which is absurd. Hence T has at least two vertices of degree one. ❑

The following result characterises tree degree sequences.

Theorem 4.7 The sequence $[d_i]_1^n$ of positive integers is a degree sequence of a tree if and only if

(i) $d_i \geq 1$ for all i, $1 \leq i \leq n$ and (ii) $\sum\limits_{i=1}^{n} d_i = 2n-2$.

Proof

Necessity Since a tree has no isolated vertex, therefore $d_i \geq 1$ for all i. Also, $\sum\limits_{i=1}^{n} d_i = 2(n-1)$, as a tree with n vertices has $n-1$ edges.

Sufficiency We use induction on n. For $n = 2$, the sequence is $[1, 1]$ and is obviously the degree sequence of K_2. Suppose the claim is true for all positive sequences of length less than n.

Let $[d_i]_1^n$ be the non-decreasing positive sequence of n terms, satisfying conditions (i) and (ii). Then $d_1 = 1$ and $d_n > 1$ (by Theorem 4.5).

Now, consider the sequence $D' = [d_2, d_3, \ldots, d_{n-1}, d_n - 1]$, which is a sequence of length $n-1$. Obviously in D', $d_i \geq 1$ and $\sum d_i = d_2+d_3+\ldots+d_{n-1}+d_n-1 = d_1+d_2+d_3+\ldots+d_{n-1}+d_n-1-1 = 2n-2-2 = 2(n-1)-2$ (because $d_1 = 1$). So D' satisfies conditions (i) and (ii), and by induction hypothesis there is a tree T' realising D'. In T', add a new vertex and join it to the vertex having degree $d_n - 1$ to get a tree T. Therefore the degree sequence of T is $[d_1, d_2, \ldots, d_n]$.

Theorem 4.8 A forest of k trees which have a total of n vertices has $n-k$ edges.

Proof Let G be a forest and $T_1, T_2, \ldots, T_k$ be the k trees of G. Let G have n vertices and $T_1, T_2, \ldots, T_k$ have respectively $n_1, n_2, \ldots, n_k$ vertices. Then $n_1+n_2+\ldots+n_k = n$. Also, the number of edges in $T_1, T_2, \ldots, T_k$ are respectively $n_1-1, n_2-1, \ldots, n_k-1$. Thus number of edges in $G = n_1-1+n_2-1+\ldots+n_k-1 = n_1+n_2+\ldots+n_k-k = n-k$. ❑

The following result characterises trees as subgraphs of a graph.

Theorem 4.9 Let T be a tree with k edges. If G is a graph whose minimum degree satisfies $\delta(G) \geq k$, then G contains T as a subgraph. Alternatively, G contains every tree of order atmost $\delta(G)+1$ as a subgraph.

Proof We use induction on k. If $k = 0$, then $T = K_1$ and it is clear that K_1 is a subgraph of any graph. Further, if $k = 1$, then $T = K_2$ and K_2 is a subgraph of any graph whose minimum degree is one. Assume the result is true for all trees with $k-1$ edges ($k \geq 2$) and consider a tree T with exactly k edges. We know that T contains at least two pendant vertices. Let v be one of them and let w be the vertex that is adjacent to v. Consider the graph $T-v$. Since $T-v$ has $k-1$ edges, the induction hypothesis applies, so $T-v$ is a subgraph of G. We can think of $T-v$ as actually sitting inside G (meaning w is a vertex of G, too). Since G contains at least $k+1$ vertices, and $T-v$ contains k vertices, there exist vertices of G that are not a part of the subgraph $T-v$. Further, since the degree of w in G is at least k, there must be a vertex u not in $T-v$ that is adjacent to w. The subgraph $T-v$ together with u forms the tree T as a subgraph of G (Fig. 4.3). ❑

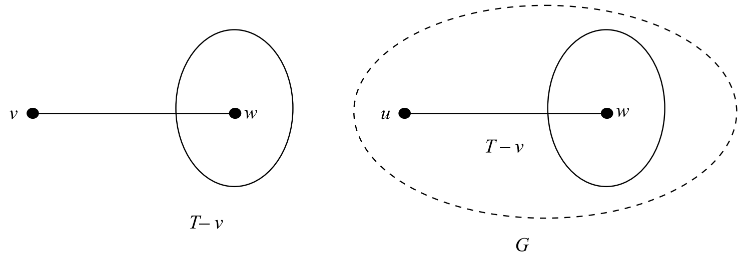


Fig. 4.3

4.2 Rooted and Binary Trees

A tree in which one vertex (called the *root*) is distinguished from all the others is called a *rooted tree*.

A *binary tree* is defined as a tree in which there is exactly one vertex of degree two and each of the remaining vertices is of degree one or three. Obviously, a binary tree has three or more vertices. Since the vertex of degree two is distinct from all other vertices, it serves as a root, and so every binary tree is a rooted tree. ❑

Below are given some properties of binary trees.

Theorem 4.10 Every binary tree has an odd number of vertices.

Proof Apart from the root, every vertex in a binary tree is of odd degree. We know that there are even number of such odd vertices. Therefore when the root (which is of even degree) is added to this number, the total number of vertices is odd.

Corollary 4.1 There are $\frac{1}{2}(n+1)$ pendant vertices in any binary tree with n vertices.

Proof Let T be a binary tree with n vertices. Let q be the number of pendant vertices in T. Therefore there are $n-q$ internal vertices in T and so $n-q-1$ vertices of degree 3. Thus the number of edges in $T = \frac{1}{2}[3(n-q-1)+2+q]$. But the number of edges in T is $n-1$.

Hence, $\frac{1}{2}[3(n-q-1)+2+q]=n-1$, so that $q=\frac{1}{2}(n+1)$. ❑

The following result is due to Jordan [122].

Theorem 4.11 (Jordan) Every tree has either one or two centers.

Proof The maximum distance, $\max d(v,\ v_i)$ from a given vertex v to any other vertex occurs only when v_i is a pendant vertex. With this observation, let T be a tree having more than two vertices. Tree T has two or more pendant vertices. Deleting all the pendant vertices from T, the resulting graph T' is again a tree. The removal of all pendant vertices from T uniformly reduces the eccentricities of the remaining vertices (vertices in T') by one. Therefore the centers of T are also the centers of T'. From T' we remove all pendant vertices and get another tree T''. Continuing this process, we either get a vertex, which is a center of T, or an edge whose end vertices are the two centers of T.

Definition: Trees with center K_1 are called *unicentral* and trees with center K_2 are called *bicentral trees*.

Spanning trees

A tree is said to be a spanning tree of a connected graph G, if T is a subgraph of G and T contains all vertices of G.

Example Consider the graph of Fig. 4.4, where the bold lines represent a spanning tree.

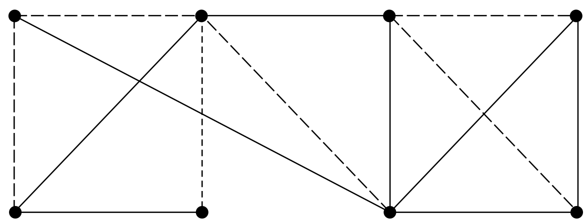


Fig. 4.4

❑

The following result shows the existence of spanning trees in connected graphs.

Theorem 4.12 Every connected graph has at least one spanning tree.

Proof Let G be a connected graph. If G has no cycles, then it is its own spanning tree. If G has cycles, then on deleting one edge from each of the cycles, the graph remains connected and cycle free containing all the vertices of G.

Definition: An edge in a spanning tree T is called a *branch* of T. An edge of G that is not in a given spanning tree T is called a *chord*. It may be noted that branches and chords are defined only with respect to a given spanning tree. An edge that is a branch of one spanning tree T_1 (in a graph G) may be a chord with respect to another spanning tree T_2. In Figure 4.5, $u_1u_2u_3u_4u_5u_6$ is a spanning tree, u_2u_4 and u_4u_6 are chords.

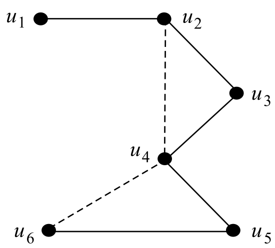


Fig. 4.5

A connected graph G can be considered as a union of two subgraphs T and $\overline{T}$, that is $G = T \cup \overline{T}$, where T is a spanning tree, $\overline{T}$ is the complement of T in G. $\overline{T}$ being the set of chords is called the *co tree*, or chord set.

The following result provides the number of chords in any graph with a spanning tree.

Theorem 4.13 With respect to any of its spanning trees, a connected graph of n vertices and m edges has $n-1$ tree branches and $m-n+1$ chords.

Proof Let G be a connected graph with n vertices and m edges. Let T be the spanning tree. Since T contains all n vertices of G, T has $n-1$ edges and thus the number of chords in G is equal to $m-(n-1) = m-n+1$. ❑

Definition: Let G be a graph with n vertices, m edges and k components. The *rank* r and *nullity* μ of G are defined as $r = n-k$ and $\mu = m-n+k$.

Clearly, the rank of a connected graph is $n-1$ and the nullity is $m-n+1$.

It can be seen that rank of G = number of branches in any spanning tree (or forest) of G. Also, nullity of G = number of chords in G. So, rank + nullity = number of edges in G.

The nullity of a graph is also called its *cyclomatic number*, or first Betti number.

Theorem 4.14 If T is a tree with $2k \geq 0$ vertices of odd degree, then $E(T)$ is the union of k pair-wise edge-disjoint paths.

Proof We prove the result for every forest G, using induction on k. If $k = 0$, then G has no pendant vertex and therefore no edge. Let $k > 0$ and let each forest with $2k-2$ vertices of odd degree has decomposition into $k-1$ paths. Since $k > 0$, some component of G is a tree with at least two vertices. This component has at least two pendant vertices. Let P be the path connecting two pendant vertices. Deleting $E(P)$ changes the parity of the vertex degree only for the end vertices of P and it makes them even. Thus $G-E(P)$ is a forest with $2k-2$ vertices of odd degree. So by the induction hypothesis, $G-E(P)$ is the union of

$k-1$ pair wise edge-disjoint paths. These $k-1$ edge-disjoint paths together with P partition $E(G)$ into k pair wise edge-disjoint paths (Fig. 4.6).

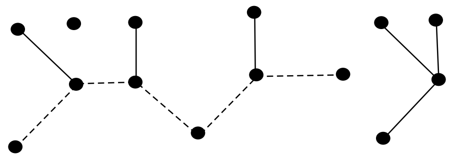


Fig. 4.6

Theorem 4.15 Let T be a non-trivial tree with the vertex set S and $|S| = 2k$, $k \geq 1$. Then there exists a set of k pairwise edge-disjoint paths whose end vertices are all the vertices of S.

Proof Obviously, there exists a set of k paths in T whose end vertices are all the vertices of S. Let $P = \{P_1, P_2, \ldots, P_k\}$ be such a set of k paths and let the sum of their lengths be the minimum.

We show that the paths of P are pairwise edge-disjoint. Assume to the contrary, and let P_i and P_j, $i \neq j$, be paths having an edge in common. Then P_i and P_j have path P_{ij} of length ≥ 1 in common. Therefore, $P_i \Delta P_j$ the symmetric difference of P_i and P_j is a disjoint union of two paths, say Q_i and Q_j, with their end vertices being disjoint pairs of vertices belonging to S (Fig. 4.7).

If P_i and P_j are replaced by Q_i and Q_j in P, then the resulting set of paths has the property that their end vertices are all the vertices of S and that the sum of their lengths is less than the sum of the lengths of the paths in P. This is a contradiction to the choice of P.

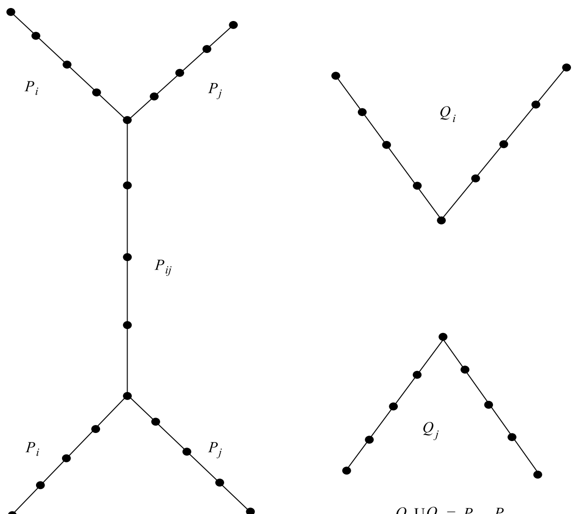


Fig. 4.7

Theorem 4.16 If u is a vertex of an n-vertex tree T, then $\sum_{v \in V(T)} d(u, v) \le \binom{n}{2}$.

Proof Let $T(V, E)$ be a tree with $|V| = n$. Let u be any vertex of T. We use induction on n. If $n = 2$, the result is trivial. Let $n > 2$. The graph $T - u$ is a forest and let the components of $T - u$ be $T_1, T_2, \ldots, T_k$, where $k \ge 1$. Since T is connected, u has a neighbour in each T_i. Also, since T has no cycles, u has exactly one neighbour v_i in each T_i. For any $v \in V(T_i)$, the unique $u - v$ path in T passes through v_i and we have $d_T(u, v) = 1 + d_{T_i}(v_i, v)$. Let $n_i = n(T_i)$ (Fig. 4.8). Then we have

$$\sum_{v \in V(T_i)} d_T(u, v) = n_i + \sum_{v \in V(T_i)} d_{T_i}(v_i, v). \tag{4.16.1}$$

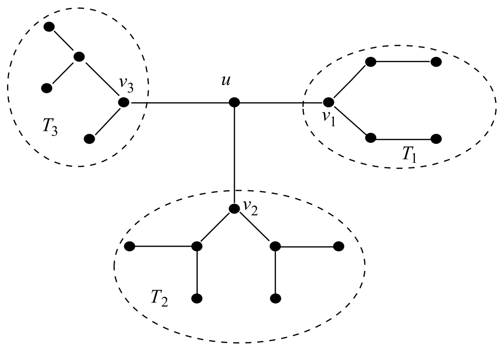


Fig. 4.8

By the induction hypothesis, we have

$$\sum_{v \in V(T_i)} d_{T_i}(v_i, v) \le \binom{n_i}{2}.$$

We now sum the formula (4.16.1) for distances from u over all the components of $T - u$ and we get

$$\sum_{v \in V(T_i)} d_T(u, v) \le (n-1) + \sum_i \binom{n_i}{2}.$$

Now, we have $\sum_i n_i = n - 1$. Clearly, $\sum_i \binom{n_i}{2} \le \binom{\Sigma n_i}{2}$, because the right side counts the edges in $K_{\sum n_i}$ or K_{n-1}, and the left side counts the edges in a subgraph of $K_{\sum n_i}$, the subgraph being union of disjoint cliques $K_{n_1}, K_{n_1}, \ldots, K_{n_k}$.

Thus, $\sum_{v \in V(T)} d_T(u, v) \le (n-1) + \binom{n-1}{2} = \binom{n}{2}$.

Corollary 4.2 The sum of the distances from a pendant vertex of the path P_n to all other vertices is $\sum_{i=0}^{n-1} i = \binom{n}{2}$.

Corollary 4.3 If H is a subgraph of a graph G, then $d_G(u, v) \leq d_H(u, v)$.

Proof Every $u-v$ path in H appears also in G, and G may have additional $u-v$ paths that are shorter than any $u-v$ path in H.

Corollary 4.4 If u is a vertex of a connected graph G, then

$$\sum_{v \in V(G)} d(u, v) \leq \binom{n(G)}{2}.$$

Proof Let T be a spanning tree of G. Then $d_G(u, v) \leq d_T(u, v)$, so that

$$\sum_{v \varepsilon V(G)} d_G(u, v) \leq \sum_{v \varepsilon V(G)} d_T(u, v) \leq \binom{n(G)}{2}.$$ ❑

The sum of the distances over all pairs of distinct vertices in a graph G is the Wiener index $W(G) = \sum_{u,\, v \varepsilon V(G)} d(u, v)$. On assigning vertices for the atoms and edges for the atomic bonds, we can use graphs to study molecules. Wiener [268] originally used this to study the boiling point of paraffin.

Theorem 4.17 Let v be any vertex of a connected graph G. Then G has a spanning tree preserving the distances from v.

Proof Let G be a connected graph. We find a spanning tree T of G such that for each $u \in V = V(G) = V(T)$, $d_G(v, u) = d_T(v, u)$.

Consider the neighbourhoods of v,

$N_i(v) = \{u \in V : d_G(v, u) = i\}$, $1 \leq i \leq e$, where $e = e(v)$.

Let H be the graph obtained from G by removing all edges in each $< N_i(v) >$. Clearly, H is connected. Let $< B_i(v) >_H$ denote the induced subgraph of H, induced by the ball $B_i(v)$. Clearly, $< B_1(v) >_H$ does not contain any cycle. If $< B_2(v) >_H$ contains cycles, remove edges from $[N_1(v), N_2(v)]$ sequentially, one edge from each cycle, till it becomes acyclic. Proceeding successively by removing edges from $[N_i(v), N_{i+1}(v)]$ to make $< B_{i+1}(v) >_H$ acyclic for $1 \leq i \leq e-1$, we get a spanning tree of H and hence of G.

Since in this procedure one distance path from v to each of the other vertices remains intact, we have $d_G(v, u) = d_T(v, u)$ for each $u \in V$. ❑

Remarks The above result implies that for any vertex v of a connected graph G, there exists an image $\Phi_v(G)$ which is a spanning tree of G preserving distances from v. This is called an *isometric tree* of G at v. If there is only one such tree (upto isomorphism) at v, we say that G has a *unique isometric tree* at v. If G has the same unique isometric tree at each vertex v, then G is said to have a *unique isometric tree* (or unique distance tree). $K_{2,2}$ and the Peterson graph are examples of graphs having unique isometric trees, while $K_{3,3}$ does not have a unique isometric tree at any vertex. Every tree has a unique isometric tree.

The next result due to Chartrand and Stewart [52] gives the necessary condition for a graph to have a unique isometric tree.

Theorem 4.18 Let G be a connected graph with $d = 2r$, which has a unique isometric tree. Then the end vertices of every diametral path of G has degree 1.

Proof Let G be a connected graph with $d = 2r$ and let P be a diametral path with end vertices u and v. If possible let $d(u|G) > 1$. Let T_u be the isometric tree at u. It is easy to see that T_u can be chosen to contain P.

Since u has degree at least 2 in G, there is a vertex u_i adjacent to u and not lying in P. Clearly, $d_{T_u}(u_i, v) = 1 + d$.

Let c be a central vertex of G. Then for any two vertices w_1 and w_2 of G, we have

$$d_G(w_1, c) \le r = \tfrac{1}{2}d \text{ and } d_G(w_2, c) \le r = \tfrac{1}{2}d.$$

Therefore, $d_G(w_1, w_2) \le d(w_1, c) + d(c, w_2) \le d$.

Since T_c is isometric with G at c, we also have $d_{Tc}(w_1, w_2) \le d$.

Thus no path T_c has length greater than d, whereas there is a path in T_u of length $1 + d$. Therefore $T_c \neq T_u$ and G does not have a unique isometric tree. This contradicts the hypothesis. Hence the result follows. ❑

Remark The above condition is necessary but not sufficient. To see this, consider the graph given in Figure 4.9.

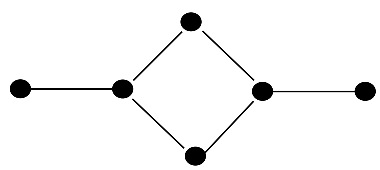


Fig. 4.9 *Graph without a unique isometric tree*

Chartrand and Schuster [54], and Kundu [142] have given some more results on the graphs with unique isometric trees.

Definition: The *complexity* $\tau(G)$ of a graph G is the number of different spanning trees of G.

The following result gives a recursive formula for $\tau(G)$.

Theorem 4.19 For any cyclic edge e of a graph G, $\tau(G) = \tau(G-e) + \tau(G|pe)$.

Proof Let S be the set of spanning trees of G and let S be partitioned as $S_1 \cup S_2$, where S_1 is the set of spanning trees of G not containing e and S_2 is the set of the spanning trees of G containing e.

Since e is a cyclic edge, $G-e$ is connected and there is a one-one correspondence between the elements of S_1 and the spanning trees of $G-e$. Also, there is a one-one correspondence between the spanning trees of $G|$þe and the elements of S_2.

Thus, $\tau(G) = |S_1| + |S_2| = \tau(G-e) + \tau(G|$þ$e)$. ❑

Remarks

1. The above recurrence relation is valid even if e is a cut edge. This is because $\tau(G-e) = 0$ and every spanning tree of G contains every cut edge.

2. The recurrence relation is valid even if G is a general graph and e is a multiple edge, but not when e is a loop.

3. The complexity of any graph G is computed by repeatedly applying the above recurrence. We observe that on applying the elementary contraction to a multiple edge, the resulting graph can have a loop and by remark (2) the procedure can be still continued. At each stage of the algorithm, only an edge belonging to the proper cycle is chosen. The algorithm starts with a given graph and produces two graphs (possibly general) at the end of the first stage. At each subsequent stage one proper cyclic edge from each graph is chosen (if it exists) for applying the recurrence. On termination of the algorithm, we get a set of graphs (or general graphs) none of which have a proper cycle. Then $\tau(G)$ is the sum of the number of these graphs. If H is any of these graphs, then $\tau(H)$ is the product of its edges, ignoring the loops.

Example Consider the graph G given in Figure 4.10.

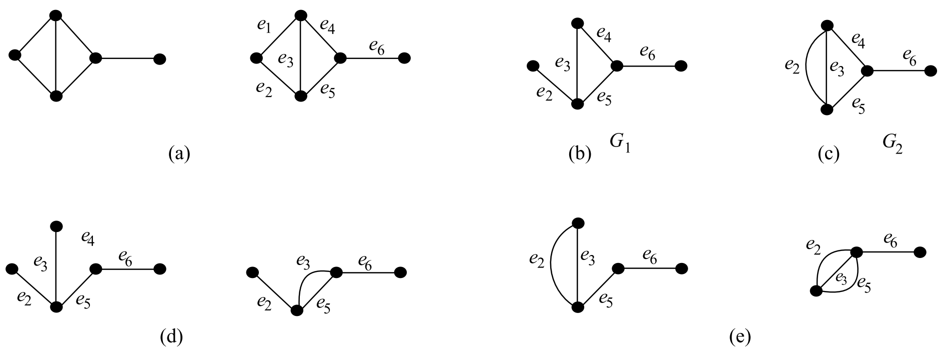


Fig. 4.10

Label the edges of G arbitrarily. Choose e_1 as the first cyclic edge. Then $\tau(G)$ is the sum of the complexities of the graphs given in Figure 4.10(b) and (c). Now, choose e_4 in both G_1 and G_2 as the next cyclic edge. Then $\tau(G)$ is the sum of the complexities of the graphs in Figure 4.10(d) and (e). Since there are no more cyclic edges, the algorithm terminates, and we have $\tau(G) = 1+2+2+3 = 8$.

4.3 Number of Labelled Trees

Let us consider the problem of constructing all simple graphs with n vertices and m edges. There are $n(n-1)/2$ unordered pairs of vertices. If the vertices are distinguishable from each other (i.e., labelled graphs), then the number of ways of selecting m edges to form the graph is $\binom{\frac{n(n-1)}{2}}{m}$.

Thus the number of simple labelled graphs with n vertices and m edges is

$$\binom{\frac{n(n-1)}{2}}{m}. \qquad \text{(A)}$$

Clearly, many of these graphs can be isomorphic (that is they are same except for the labels of their vertices). Thus the number of simple, unlabelled graphs of n vertices and m edges is much smaller than that given by (A) above.

Theorem 4.20 The number of simple, labelled graphs of n vertices is $2^{\frac{n(n-1)}{2}}$.

Proof The number of simple graphs of n vertices and 0, 1, 2, ..., $n(n-1)/2$ edges are obtained by substituting 0, 1, 2, ..., $n(n-1)/2$ for m in (A). The sum of all such numbers is the number of all simple graphs with n vertices.

Therefore the total number of simple, labelled graphs of n vertices is

$$\binom{\frac{n(n-1)}{2}}{0} + \binom{\frac{n(n-1)}{2}}{1} + \binom{\frac{n(n-1)}{2}}{2} + \ldots + \binom{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2}} = 2^{\frac{n(n-1)}{2}},$$

by using the identity $\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k} = 2^k$.

The following result was proved independently by Tutte [252] and Nash-Williams [167]. We prove the necessity and for sufficiency the reader is referred to the original papers of Tutte and Nash-Williams.

Theorem 4.21 A simple connected graph G contains k pairwise edge-disjoint spanning trees if and only if, for each partition π of $V(G)$ into p parts, the number $m\,(\pi)$ of edges of G joining distinct parts is at least $k(p-1)$.

Proof
Necessity Let G has k pairwise edge-disjoint spanning trees. If T is one of them, and if $\pi = \{V_1, V_2, \ldots, V_p\}$ is a partition of $V(G)$ into p parts, then identification of each part V_i into a single vertex v_i, $1 \leq i \leq p$, results in a connected graph G_0 (possibly with multiple edges) on $\{V_1, V_2, \ldots, V_p\}$. Clearly, G_0 contains a spanning tree with $p-1$ edges, and each such edge belongs to T, and joins distinct partite sets of π. Since this is true for each of the k edge −disjoint spanning trees of G, the number of edges joining distinct parts of π is at least $k(p-1)$. ❑

Cayley [46] in 1889 discovered the formula $\tau(K_n) = n^{n-2}$. Clearly, the number of spanning trees of K_n is same as the number of non-label-isomorphic trees on n vertices. Several proofs of this result have appeared since Cayley's discovery. Moon [164] has outlined ten such proofs, and a complete presentation of some of these can also be found in Lovasz [152]. Here we give two proofs, and the first is due to Prufer [212].

Theorem 4.22 (Cayley) There are n^{n-2} labelled trees with n vertices, $n \geq 2$.

Proof Let T be a tree with n vertices and let the vertices be labelled $1, 2, \ldots, n$. Remove the pendant vertex (and the edge incident to it) having the smallest label, say u_1. Let v_1 be the vertex adjacent to u_1. From the remaining $n-1$ vertices, let u_2 be the pendant vertex with the smallest label and let v_2 be the vertex adjacent to u_2. We remove u_2 and the edge incident on it. We repeat this operation on the remaining $n-2$ vertices, then on $n-3$ vertices, and so on. This process completes after $n-2$ steps, when only two vertices are left.

Let the vertices after each removal have labels $v_1, v_2, \ldots, v_{n-2}$. Clearly, the tree T uniquely defines the sequence

$$(v_1, v_2, \ldots, v_{n-2}). \tag{4.22.1}$$

Conversely, given a sequence of $n-2$ labels, an n-vertex tree is constructed uniquely as follows. Determine the first number in the sequence

$$1, 2, 3, \ldots, n, \tag{4.22.2}$$

that does not appear in (4.22.1). Let this number be u_1. Thus the edge (u_1, v_1) is defined. Remove v_1 from sequence (4.22.1) and u_1 from (4.22.2). In the remaining sequence of (4.22.2), find the first number which does not appear in the remaining sequence of (4.22.1). Let this be u_2 and thus the edge (u_2, v_2) is defined. The construction is continued till the sequence (4.22.1) has no element left. Finally, the last two vertices remaining in (4.22.2) are joined.

For each of the $n-2$ elements in sequence (4.22.1), we choose any one of the n numbers, thus forming n^{n-2} $(n-2)$-tuples, each defining a distinct labelled tree of n vertices. Since each tree defines one of these sequences uniquely, there is a one−one correspondence between the trees and the n^{n-2} sequences. ❑

Example Consider the tree shown in Figure 4.11. Pendant vertex with smallest label is u_1. Remove u_1. Let v_1 be adjacent to u_1 (label of v_1 is 1). Pendant vertex with smallest label is 4. Remove 4. Here 4 is adjacent to 1. Pendant vertex with smallest label is 1. Remove 1. Here 1 is adjacent to 3. Remove 3. Then 3 is adjacent to 5. Remove 6. So 6 is adjacent to 5. Remove 5. Remove 7. 7 is adjacent to 5. So 5 is adjacent to 9. Sequence $(v_1, v_2, \ldots, v_{n-2})$ is $(1, 1, 3, 5, 5, 5, 9)$.

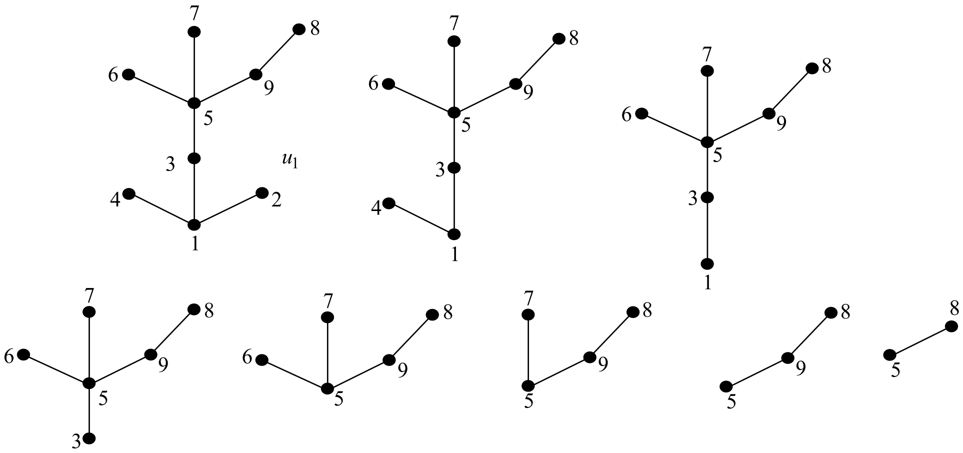


Fig. 4.11

Theorem 4.23 If $D = [d_i]_1^n$ is the degree sequence of a tree, then the number of labelled trees with this degree sequence is

$$\frac{(n-2)!}{(d_1-1)!(d_2-1)!\ldots(d_n-1)!}.$$

Proof We first observe that, when asking for all possible trees with the vertex label set $V = \{v_1, v_2, \ldots, v_n\}$ with degree sequence $D = [d_i]_1^n$, it is not necessary that $d_i = d\,(v_i)$ and it is not necessary that the sequence be monotonic non-decreasing.

Therefore we assume that $D = [d_i]_1^n$ is an integer sequence satisfying the conditions $\sum d_i = 2(n-1)$ and $d_i \geq 1$. We use induction on n. The result is obvious for $n = 1, 2$. For $n = 2$, the sequence is $[d_1, d_2]$ and the only degree sequence in this case is $[1, 1]$. Clearly, there is only one labelled tree with this degree sequence.

Also, $$\frac{(n-2)!}{(d_1-1)!\ldots(d_n-1)} = \frac{(2-2)!}{(1-1)!(1-1)!} = 1.$$

Now, assume that the result is true for all sequences of length $n-1$. Let $D = [d_i]_1^n$ be an n length sequence. By assumption there is a $d_i = 1$ and let it be $d_n = 1$. Let T_n be a tree realising $D = [d_i]_1^n$. Now, removing v_n, we get a tree T_{n-1} on the vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$ with degrees $d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1}$, where v_j is the vertex to which v_n is adjacent in

T_n. Clearly, the converse is also true. Therefore, by induction hypothesis, the number of trees T_{n-1} is

$$\frac{(n-3)!}{(d_1-1)!\ldots(d_{j-1}-1)!(d_j-1-1)!(d_{j+1}-1)!\ldots(d_{n-1}-1)!}$$

$$= \frac{(n-3)!(d_j-1)}{(d_1-1)!\ldots(d_{j-1}-1)!\,[(d_j-1)\,(d_j-2)!]\,(d_{j+1}-1)!\ldots(d_{n-1}-1)!}$$

$$= \frac{(n-3)!(d_j-1)}{(d_1-1)!\ldots(d_j-1)!\ldots(d_{n-1}-1)!(d_n-1)!}$$

$$= \frac{(n-3)!(d_j-1)}{\prod\limits_{j=1}^{n}(d_j-1)!}.$$

Since v_j is any one of the vertices $v_1, \ldots, v_{n-1}$, the number of trees T_n is

$$\sum_{j=1}^{n-1} \frac{(n-3)!(d_j-1)}{\prod\limits_{j=1}^{n}(d_j-1)!} = \frac{(n-3)!}{\prod\limits_{j=1}^{n}(d_j-1)!} \sum_{j=1}^{n}(d_j-1), \text{ as } d_n = 1 \text{ and } d_n - 1 = 1 - 1 = 0$$

$$= \frac{(n-3)!}{\prod\limits_{j=1}^{n}(d_j-1)!}(n-2), \text{ since } \sum_{j=1}^{n}(d_j-1) = 2(n-1) - n = n-2$$

$$= \frac{(n-2)!}{\prod\limits_{j=1}^{n}(d_j-1)!}.$$ ❑

Now, we use Theorem 4.22 to obtain $\tau(K_n) = n^{n-2}$, which forms the second proof of Cayley's Theorem.

Second Proof of Theorem 4.22 We know the number of labelled trees with a given degree sequence $[d_i]_1^n$ is

$$\frac{(n-2)!}{\prod\limits_{j=1}^{n}(d_j-1)!}.$$

The total number of labelled trees with n vertices is obtained by adding the number of labelled trees with all possible degree sequences.

Therefore, $\tau(K_n) = \sum_{\substack{d_i \geq 1 \\ \sum_{i=1}^{n} d_i = 2n-2}} \left[\frac{(n-2)!}{\prod_{j=1}^{n} (d_j - 1)!} \right].$

Let $d_i - 1 = k_i$. So $d_i \geq 1$ gives $d_i - 1 \geq 0$, or $k_i \geq 0$.

Also, $\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} (d_i - 1) = \sum_{i=1}^{n} d_i - n = 2n - 2 - n = n - 2$.

$$\text{Thus, } \tau(K_n) = \sum_{\substack{k_i \geq 0 \\ \sum_1^n k_i = n-2}} \frac{(n-2)!}{k_1!k_2!\ldots k_n!} = \sum_{\substack{k_i \geq 0 \\ \sum_1^n k_i = n-2}} \frac{(n-2)!}{k_1!k_2!k_n!} 1^{k^1} 1^{k2} \ldots 1^{kn}$$

$$= (1 + 1 + \ldots + 1)^{n-2}, \text{ by multinomial theorem.}$$

Hence, $\tau(K_n) == n^{n-2}$. ❑

Note The multinomial distribution is given by

$$\frac{n!}{x_1!x_2!\ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k} = (p_1 + p_2 + \ldots + p_k)^n, \text{ where } \sum_{i=1}^{n} x_i = n.$$

4.4 The Fundamental Cycles

Definition: Let T be a spanning tree of a connected graph G. Let $\overline{T}$ be the spanning subgraph of G containing only the edges of G which are not in T (i.e., $\overline{T}$ is the relative complement of T in G). Then $\overline{T}$ is called the co tree of T in G. The edges of T are called branches and the edges of $\overline{T}$ are called chords of G relative to the spanning tree T.

Theorem 4.24 If T is a spanning tree of a connected graph G and f is a chord of G relative to T, then $T + f$ contains a unique cycle of G.

Proof Let $f = uv$. Then there is a unique $u - v$ path P in T. Clearly, $P + f$ is a cycle of G, since T is acyclic, any cycle C of $T + e$ should contain e, and $C - e$ is a $u - v$ path in T. Since there is a unique path in T, $T + e$ contains a unique cycle of G. ❑

Remarks

1. If f_1 and f_2 are two distinct chords of the connected graph G relative to a spanning tree T, then there are two unique distinct cycles C_1 and C_2 of G containing respectively f_1 and f_2.
2. If $e \in E(\overline{G})$ and T is a spanning tree of G, then $T + e$ contains a unique cycle of K_n.

Definition: Let G be a connected graph with n vertices and m edges. The number of chords of G relative to a spanning tree T of G is $m-n+1=\mu$. The μ distinct cycles of a connected graph G corresponding to the distinct chords of G relative to a spanning tree T of G are said to form a set of fundamental cycles of G.

If G is a disconnected graph with k components $G_1, G_2, \ldots, G_k$ and T_i, $1 \le i \le k$, are a set of k spanning trees of G_i, then the union of the set of fundamental cycles of G_i with respect to T_i is a set of fundamental cycles for G. It is to be noted that different spanning trees give different sets of fundamental cycles.

The following result characterises cycles in terms of the set of all spanning trees.

Theorem 4.25 Any cycle of a connected graph G contains at least one chord of every spanning tree of G.

Proof Let C be a cycle and assume the result is not true. So there exists a spanning tree T of G such that C is contained in the edge set $E(G)-E(\overline{T})$, where $\overline{T}$ is the cotree of G corresponding to T. This means that the tree T contains the cycle C, which is a contradiction. ❑

Theorem 4.26 A set of edges C of a connected graph G is a cycle of G if and only if it is a minimal set of edges containing at least one chord of every spanning tree of G.

Proof Let C be a cycle of G. Then it contains at least one chord of every spanning tree of G. If C' is any proper subset of C, then C' does not contain a cycle and is a forest. A spanning tree T of G can therefore be constructed containing C'. Clearly, C' does not contain any chord of T. Thus no proper subset of C has the stated property, proving that C is minimal with respect to the property.

To prove sufficiency, let C be minimal set with the stated property. Then C is not acyclic. Therefore C contains at least a cycle C'. But by the necessary part, C' is minimal with respect to the property and hence $C'=C$, that is, C is a cycle. ❑

4.5 Generation of Trees

Definition: Let T_1 and T_2 be two spanning trees of a connected graph G and let there be edges $e_1 \in T_1$ and $e_2 \in T_2$ such that $T_1-e_1+e_2=T_2$ (and hence $T_2-e_2+e_1=T_1$). The transformation $T_1 \leftrightarrow T_2$ is called an *elementary tree transformation* (ETT), or a fundamental exchange. If e_1 and e_2 are adjacent in G, then the ETT is called a *neighbour transformation* (NT). If e is a pendant edge of T_1 (and hence e_2 is a pendant edge of T_2) the ETT is called a *pendant-edge transformation* (PET) or an end-line transformation.

Definition: Let I be the collection of all spanning trees of a connected graph G. Let $Tr(G)$ be the graph whose vertices t_i correspond to the elements T_i of I, and in which t_i and t_j are adjacent if and only if there is an ETT between T_i and T_j, that is, if and only if $E(T_i)\Delta E(T_j)=\{e_i, e_j\}$. Then $Tr(G)$ is called the *tree graph* of G. The distance $d(T_i,\, T_j)$

between the spanning trees T_i and T_j of G is defined to be the distance between t_i and t_j in $Tr(G)$.

Theorem 4.27 The tree graph $Tr(G)$ of a connected graph is connected.

Proof Let G be a connected graph with n vertices and let $Tr(G)$ be its tree graph. To prove that $Tr(G)$ is connected, it is enough to show that any two spanning trees of G can be obtained from each other by a finite sequence of ETT's.

Let T and T' be two distinct spanning trees of G. Then there is a set $S = \{e_1, e_2, \ldots, e_k\}$ of some k edges of T which are not in T'. Since a spanning tree has $n-1$ edges, there is a corresponding set $S' = \{e'_1, e'_2, \ldots, e'_k\}$ of edges of T' which are not in T. Thus $T + e'_1$ contains a unique fundamental cycle $T\,e'_1$. As T' is a tree, at least one edge of $T\,e'_1$ (which is a branch of T) will not be in T' and thus is a member of S. Without loss of generality, let this edge be e_1. Define $T_1 = T - e_1 + e'_1$. Then T_1 can be obtained from T by an ETT and therefore T_1 and T' have one more edge in common.

Repeating this process $k-1$ more times, we get a sequence of spanning trees $T_0 = T, T_1, T_2, \ldots, T_{k-1}, T_k = T'$ such that there is an ETT $T_i \longleftrightarrow T_{i+1}$, $0 \le i \le k-1$. ❑

Theorem 4.28 An elementary tree transformation can be obtained by a sequence of neighbour transformations.

Proof Let T and $T' = T - x + y$ be spanning trees of the graph G, where x and y are non-adjacent edges of G. Then we can choose a set of edges $e_1, e_2, \ldots, e_k$ such that $x, e_1, e_2, \ldots, e_k, y$ is a path in $T + y$. Define $T_1 = T - x + e_1$ and $T_i = T_{i-1} + e_{i-1} + e_i$, $2 \le i \le k$ and $T_{k+1} = T_k - e_k + y$. Then $T_{k+1} = T'$, and is obtained from T by a sequence of $k+1$ neighbour transformations through the intermediate trees T_i, $1 \le i \le k$. ❑

Definition: A spanning tree of a graph G corresponding to a central vertex of the tree $Tr(G)$ is called a *central tree*.

The set of diameters of the spanning trees of a connected graph G is the *tree diameter set* of G. A set of positive integers is a *feasible tree diameter set* if it is the tree diameter set of some graph. For example, the graph in Figure 4.12 has one spanning tree of diameter seven and all others of diameter five.

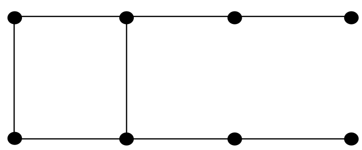


Fig. 4.12

The *girth* $g(G)$ of a graph G is the length of a smallest cycle of G. A cycle of smallest length is called a *girdle* of G. The *circumference* $c(G)$ of a graph G is the length of the longest cycle of G. A cycle of maximum length is called a *hem* of G.

Let $\underline{n}\,(\delta,\, g)$ denote the minimum order (minimum vertices) of a graph with minimum degree at least δ (≥ 3) and girth at least g (≥ 2). Let $\bar{n}\,(\Delta,\, g)$ denote the maximum order of a graph with degree at most Δ and girth at most g.

The following upper bound for $\underline{n}\,(\delta,\, g)$ can be found in Bollobas [29].

Theorem 4.29 (Bollobas) $\underline{n}\,(\delta,\, g) \leq (2\delta)^g$.

Proof Clearly, $\underline{n}\,(\delta,\, g)$ denotes the minimum order of a graph with minimum degree at least δ (≥ 3) and girth at least g (≥ 2). Therefore we construct a graph with atmost $(2\delta)^g$ vertices with these properties. Let $n = (2\delta)^g$.

Consider all graphs with vertex set $V = \{1, 2, \ldots, n\}$ and having exactly δn edges.

Since there are $\binom{n}{2}$ possible positions to accommodate these δn edges, the number of such graphs

$$= \binom{\binom{n}{2}}{\delta n}.$$

Among the n available vertices, the number of ways an h-cycle can be formed is

$$= \frac{1}{2}\binom{n}{h}(h-1)!$$

Obviously, $\frac{1}{2}\binom{n}{h}(h-1)! < \frac{1}{2h}n^h$.

The number of graphs in the set which contain a given h-cycle is

$$= \binom{\binom{n}{2} - h}{\delta n - h}.$$

Hence the average number of cycles of length at most $g-1$ in these graphs

$$< \sum_{h=3}^{g-1} \frac{1}{2h} n^h \binom{\binom{n}{2} - h}{\delta n - h} \Big/ \binom{\binom{n}{2}}{\delta n}$$

$$< \sum_{h=3}^{g-1} (2\delta)^h < (2\delta)^g = n.$$

Since the average is less than n, there is an element in the set with value less than or equal to $n-1$. Thus there is a graph G on n vertices with δn edges and at most $n-1$ cycles of length at most $g-1$. Removing one edge from each of these cycles, we get a graph G_0 with girth at least g. The number of edges removed is atmost $n-1$, so that $m(G_o) \geq n\delta - (n-1) \geq n\,(\delta - 1) + 1$ and $n\,(G_0) = n$. Thus $G_0 \in G_{\delta-1}$, and hence G_0 contains a subgraph H with $\delta(H) \geq \delta$. By construction, $g(H) \geq g$ and $n(H) \leq n = (2\delta)^g$. Thus we have constructed a graph H with the desired properties. ❑

Note If G is a graph with at least n_0 vertices and at least $n_0 n(G) - \binom{n_0+1}{2} + 1$ edges, then G contains a subgraph H with $\delta(H) \geq n_0 + 1$.

We denote by $G_{n_o} = \left\{G : n(G) > n_0,\ m(G) \geq n_0.n(G) - \binom{n_0+1}{2} + 1\right\}$.

The following lower bound for $\underline{n}\,(\delta,\, g)$ is due to Tutte [248].

Theorem 4.30 (Tutte)

$$\underline{n}(\delta,\, g) \geq \begin{cases} \dfrac{\delta(\delta-1)^{\frac{g-1}{2}} - 2}{\delta - 2}, & \text{if } g \text{ is odd,} \\ \dfrac{2(\delta-1)^{\frac{g}{2}} - 1}{\delta - 2}, & \text{if } g \text{ is even.} \end{cases}$$

Proof

i. Let g be odd, say $g = 2d+1$. Then clearly the diameter of G is at least d. Let v be a vertex with eccentricity at least d. Consider the neighbourhoods

$$N_i = N_i(V),\ 1 \leq i \leq d = (g-1)/2.$$

Obviously, no vertex of N_i is adjacent to more than one vertex of N_{i-1}, because otherwise, there will be a cycle of length $1 \leq 2i < g$. Similarly, there is no edge in $< N_i >$.

Therefore, for every $u \in N_i$, we have

$$|N(u) \textstyle\bigcap N_{i-1}| = 1,\ |N(u) \bigcap N_{i+1}| = d(u) - 1 \text{ and}$$

$$|N_{i+1}| = \sum_{u \in N_i} \{d(u) - 1\} \geq (\delta - 1)\,|N_i|\,. \qquad (4.30.1)$$

As $V \supseteq \{v\} \bigcup \bigcup_{i=1}^{d} N_i(v)$, therefore

$$n \geq 1 + \sum_{i=1}^{d} |N_i| \geq 1 + \delta + \delta(\delta-1) + \ldots + \delta(\delta-1)^{d-1}$$

$$= 1 + \frac{\delta}{\delta-2}\left\{(\delta-1)^d - 1\right\} = \frac{\left\{\delta(\delta-1)^{\frac{g-1}{2}} - 2\right\}}{\delta - 2}.$$

ii. Let g be even, say $g = 2d$. Then again the diameter is at least d. Let xy be an edge of G and let

$$S_i = \{v \in V : d(x,\ v) = i, \text{ or } d(y,\ v) = i\}, \text{ for } 1 \le i \le d-1 \text{ and } S_0 = \{x, y\}.$$

The girth requirement forces that there are no edges in $< S_i >$, for $1 \le i \le d-2$, and that each vertex of S_i be adjacent to at most one vertex of S_{i-1}, for $1 \le i \le d-1$.

Thus, for each $u \in S_i$, we have $|N(u) \cap S_{i-1}| = 1$, $|N(u) \cap S_{i+1}| = d(u) - 1$ and

$$|S_{i+1}| = \sum_{u \epsilon S_i} (d(u) - 1) \ge (\delta - 1)\,|S_i|. \qquad (4.30.2)$$

Since $V \supseteq \{x, y\} \bigcup \bigcup_{i=1}^{d-1} S_i$,

$$n = \sum_{i=0}^{d-1} |S_i| \ge 2 \sum_{i=0}^{d-1} (\delta - 1)^i = \frac{2}{\delta - 2} \left[(\delta - 1)^{\frac{g}{2}} - 1 \right]. \qquad \square$$

By using arguments as in Theorem 4.30 and by replacing δ by Δ, we obtain the following result.

Theorem 4.31

$$\bar{n}(\Delta, g) \le \begin{cases} \dfrac{\Delta(\Delta-1)^{\frac{g-1}{2}} - 2}{\Delta - 2}, & \text{if g is odd,} \\ \dfrac{2\left[(\Delta-1)^{\frac{g}{2}} - 1\right]}{\Delta - 2}, & \text{if g is even.} \end{cases}$$

Definition: A k-regular graph with girth g and with minimum order $\underline{n}(k,\ g)$ is called a *(k, g)-cage*.

$$\text{The integer } n_0 = \begin{cases} \dfrac{k(k-1)^{\frac{g-1}{2}} - 2}{k-2}, & if\ g\ is\ odd, \\ \dfrac{2\left[(k-1)^{\frac{g}{2}} - 1\right]}{k-2}, & if\ g\ is\ even, \end{cases}$$

is called the *Moore bound* for a k-regular graph with g.

4.6 Helly Property

Definition: A family $\{A_i : i \in I\}$ of subsets of a set A is said to satisfy the Helly property if $J \subseteq I$, and $A_i \bigcap A_j \ne \phi$, for every $i,\ j \in J$, then $\bigcap_{j \in J} A_j \ne \phi$.

The following result is reported by Balakrishnan and Ranganathan [13].

Theorem 4.32 A family of subtrees of a tree satisfies the Helly property.

Proof Let $\tau = \{T_i : i \in I\}$ be a family of subtrees of a tree T. Suppose for all $i, j \in J \subseteq I$, $T_i \cap T_j \neq \phi$. We have to prove $\bigcap_{j\in J} T_j \neq \phi$. If some tree $T_i \in \tau$, $i \in J$, is a single vertex tree $\{v\}$ (that is, K_1), then clearly, $\bigcap_{j\in J} T_j = \{v\}$. So assume that each tree $T_i \in T$ with $i \in J$ has at least two vertices.

We induct on the number of vertices of T. Suppose the result is true for all trees with at most n vertices and let T be a tree with $(n+1)$ vertices. Let v_0 be an end vertex of T and u_0 its unique neighbour in T. Let $T_i' = T_i - v_0$, $i \in J$ and $T' = T - v_0$. By induction hypothesis, the result is true for the tree T'. Also, $T_i' \cap T_j' \neq \phi$, for any $i, j \in J$. In fact, if T_i and T_j have a vertex u $(\neq v_0)$ in common then T_i' and T_j' also have u in common, whereas if T_i and T_j have v_0 in common, then T_i and T_j have u_0 also in common, and so do T_i' and T_j'. Hence by induction hypothesis, $\bigcap_{j\in J} T_j' \neq \phi$ and therefore $\bigcap_{j\in J} T_j \neq \phi$. ❑

4.7 Signed Trees

The following result by Yan et al. [271] characterises signed degree sequences in signed trees.

Theorem 4.33 Let $D = [d_i]_1^n$ be an integral sequence of $n \geq 2$ terms and let D has n_+ positive terms, n_0 zero and n_- negative terms. Let $\alpha = 1$ if $n_+ n_- > 0$, and $\alpha = 0$, otherwise. Then D is the signed degree sequence of a signed tree if and only if (i) to (iv) hold.

i. $\sum_{i=1}^{n} d_i \equiv 2n-2 (\mathrm{mod} 4)$.

ii. $\sum_{i=1}^{n} |d_i| \leq 2n-2-2n_0$.

iii. $\sum_{i=1}^{n} |d_i| + 2\sum_{d_i>0} |d_i| \leq 2n-2-4\alpha+4p_-$.

iv. $\sum_{i=1}^{n} |d_i| + 2\sum_{d_i>0} |d_i| \leq 2n-2-4\alpha+4p_+$.

Proof Note that condition (iv) for D is same as condition (iii) for $-D$. The necessity of the theorem follows from the fact that $m = n-1$ and Lemmas 2.2, 2.3 and 2.4.

We prove the sufficiency by induction on n. For $n = 2$, by (i) and (iii), $d_1 = d_2 = 1$ or -1. Therefore D is the signed degree sequence of K_2 with positive edge or a negative edge. Assume that the theorem is true for $n-1$. Let $n \geq 3$.

By (ii), D has at least two terms in which $|d_i|=1$. After rearranging the terms in D or taking $-D$, we may assume without loss of generality that $d_n = 1$ and one of the following holds.

1. $|d_i|=1$, for $1 \le i \le n$, $d_1 \ge 0$ and $d_1 = 0$, if $n_0 > 0$.
2. $d_1 \ge 2$.
3. $d_i \le 1$ but $d_i \ne -1$ for $1 \le i \le n$ and $d_1 = 0$ and $\alpha = 1$.
4. $d_i = 1$ or $d_i \le -2$, for $1 \le i \le n$ and $d_1 = \alpha = 1$.

For any of the above, consider the sequence $D' = [d_i']_i^{n'}$, where $n' = n-1$ and $d_i' = d_1 - 1$ and $d_i' = d_i$, for $2 \le i \le n-1$.

Note that $\sum_{i=1}^{n'} d_i' = \left(\sum_{i=1}^{n} d_i\right) - 2 \equiv (2n-2) - 2 \equiv 2n' - 2 (\mathrm{mod} 4)$, that is, (i) holds for D'. We check conditions (ii) to (iv) for D' according to the four cases above.

Case 1 In this case, $|d_i'| \le 1$, for $1 \le i \le n-1$, we have

$$\sum_{i=1}^{n'} |d_i'| = n_+' + n_-', \ \sum_{d_i'>0} |d'| = n_+', \ \sum_{d_i'<0} |d_i'| = n_-'.$$

Thus (ii) to (iv) holds for D' as $n_+' + n_-' \ge 2$.

Case 2 In this case, since $d_1 \ge 2$ and $d_n = 1$, we have

$$n' = n-1, \ n_+' = n_+ - 1, \ n_0' = n_0, \ n_-' = n_-, \ \alpha' = \alpha,$$

$$\sum_{i=1}^{n'} |d_i'| = \sum_{i=1}^{n} |d_i'| - 2, \ \sum_{d_i'>0} |d_i'| = \sum_{d_i'>0} |d_i'| - 2, \ \sum_{d_i'<0} |d_i'| = \sum_{d_i'<0} |d_i'|.$$

Therefore (ii) to (iv) holding for D imply that (ii) to (iv) hold for D'.

Case 3 In this case, since $d_1 = 0$ and $d_n = 1$, we have

$$n' = n-1, \ n_+' = n_+ - 1, \ n_0' = n_0 - 1, \ n_-' = n_- + 1, \ \alpha' \le \alpha,$$
$$\sum_{i=1}^{n'} |d_i'| = \sum_{i=1}^{n} |d_i|, \ \sum_{d_i>0} |d_i'| = \sum_{d_i>0} |d_i'| - 1, \sum_{d_i'<0} |d_i'| = \sum_{d_i'<0} |d_i'| + 1.$$

So (ii) and (iii) holding for D imply that (ii) and (iii) hold for D'. Since $d_i' \le 1$ for $1 \le i \le n-1$, $\sum_{d_i'}^{n'} |d_i'| = n_+'$. By (iii) for D and the fact that $d_i \le -2$ when $d_i < 0$,

$$n_+ + 6n_- \le \sum_{i=1}^{n'} |d_i| + 2 \sum_{d_i<0} |d_i| \le 2n - 2 - 4\alpha + 4n_- = 2n_+ + 2n_0 + 6n_- 6,$$

and so $6 \leq n_+ + 2n_0$. Therefore, $3 \leq n'_+ + 2n'_0$ and then $4 \leq 2n'_+ + 2n'_0$.

This together with (ii) for D' and $\sum_{d'_i > 0} |d'_i| = n'_+$ implies (iv) for D'.

Case 4 In this case, since $d_1 = d_n = 1$, therefore

$$n' + n - 1,\ n'_+ = n_+ - 2,\ n'_0 = n_0 + 1 = 1,\ n'_- = n_-,\ \alpha' \leq \alpha,$$

$$\sum_{i=1}^{n'} |d'_i| = \sum_{i=1}^{n} |d_i| - 2,\ \sum_{d'_i>0} |d'_i| = \sum_{d'_i>0} |d'_i| - 2,\ \sum_{d'_i<0} |d'_i| = \sum_{d_i<0} |d'_i|.$$

(iii) for D implies that (iii) holds for D'. As in the argument for Case 3, we have $\sum_{d'_i>0} |d_i| = n'_+$ and $6 \leq n_+ + 2n_0$. Therefore, $4 \leq n'_+$. Adding $2 \sum_{d'_i>0} |d'_i| = 2n'_+$ to the equality in (iii) for D' and dividing the resulting equality by 3, we get (ii) for D' as $2n'_0 \leq 2n'_+$. Adding $2 \sum_{d'_i>0} |d'_i| = 2n'_+$ to the equality in (ii) for D', we get (iv) for D' as $4\alpha' \leq 2n'_0 + 2n'_+$.

From the above discussion, D' satisfies (i) to (iv). By the induction hypothesis, there exists a signed tree T' with the vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$ and signed degree $T'(v_i) = d'_i$, for $1 \leq i \leq n-1$. Suppose T is the signed tree obtained from T' by adding a new vertex v_n and a new positive edge $v_1 v_n^+$, then T has a signed degree sequence D. ❑

Corollary Let $D = [d_i]_1^n$ be an integral sequence of $n \geq 3$ terms. Let D has at least two terms in which $|d_i| = 1$, $|d_n| = 1$ and one of the following condition holds.

1. $|d_i| \leq 1$, for $1 \leq i \leq n$, $d_i \geq 0$, and $d_1 = 0$ if $n_o > 0$.
2. $d_1 \geq 2$.
3. $d_i \leq 1$ but $d_i \neq -1$ for $1 \leq i \leq n$, and $d_1 = 0$ and $\delta = 1$
4. $d_i = 1$ or $d_i \leq -2$ for $1 \leq i \leq n$, and $d_1 = \delta = 1$.

Then D is the signed degree sequence of a signed tree if and only if $D' = [d_1 - 1, d_2, \ldots, d_{n-1}]$ is the signed degree sequence of a signed tree.

4.8 Exercises

1. Draw all unlabelled trees with seven and eight vertices.
2. Draw a tree which has radius five and diameter ten.
3. If a tree has an even number of edges, then show that it contains at least one vertex of even degree.

4. If the maximum degree of a vertex in a tree is Δ, then show that it has Δ pendant vertices.
5. If T is a tree such that every vertex adjacent to a pendant vertex has degree at least three, then prove that some pair of pendant vertices in T has a common neighbour.
6. Show that a path is its own spanning tree.
7. Prove that every tree is a bipartite graph.
8. If for a simple graph G, $m(G) \geq n(G)$, prove that G contains a cycle.
9. Show that for a unicentral tree, $d = 2r$, and for a bicentral tree, $d = 2r - 1$.
10. Prove that if $K_{r,s}$ is a tree, then it must be a star.
11. How many spanning trees does K_4 have?
12. Prove that each spanning tree of a connected graph G contains all the pendant edges of G.
13. Prove that each edge of a connected graph G belongs to at least one spanning tree of G.