2. Degree Sequences

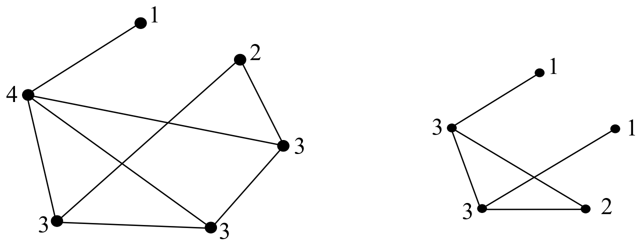
The concept of degrees in graphs has provided a framework for the study of various structural properties of graphs and has therefore attracted the attention of many graph theorists. Here we deliberate on the various criteria for a non-decreasing sequence of non-negative integers to be a degree sequence of some graph.

2.1 Degree Sequences

Let d_i, $1 \le i \le n$, be the degrees of the vertices v_i of a graph in any order. The sequence $[d_i]_1^n$ is called the degree sequence of the graph. The non-negative sequence $[d_i]_1^n$ is called the degree sequence of the graph if it is the degree sequence of some graph, and the graph is said to realise the sequence.

The set of distinct non-negative integers occurring in a degree sequence of a graph is called its *degree set*. A set of non-negative integers is called a *degree set* if it is the degree set of some graph, and the graph is said to realise the degree set.

Two graphs with the same degree sequence are said to be *degree equivalent*. In the graph of Figure 2.1(a), the degree sequence is $D = [1, 2, 3, 3, 3, 4]$ or $D = [1\ 2\ 3^3\ 4]$ and its degree set is $\{1, 2, 3, 4\}$, while the degree sequence of the graph in Figure 2.1(b) is $[1, 1, 2, 3, 3]$ and its degree set is $\{1, 2, 3\}$.



(a) (b)

Fig. 2.1

If the degree sequence is arranged as the non-decreasing positive sequence $d_1^{n_1}, d_2^{n_2}, \ldots d_k^{n_k}$, $(d_1 < d_2 < \ldots < d_k)$, the sequence $n_1, n_2, \ldots, n_k$ is called the *frequency sequence* of the graph.

The two necessary conditions implied by Theorem 1.1 and Theorem 1.12 are not sufficient to ensure that a non-negative sequence is a degree sequence of a graph. To see this, consider the sequence $[1, 2, 3, 4, \ldots, 4, n-1, n-1]$. The sum of the degrees is clearly even and $\Delta = n-1$. However, this is not a degree sequence, since there are two vertices with degree $n-1$, and this requires that each of the two vertices is joined to all the other vertices, and therefore $\delta \geq 2$. But the minimum number in the sequence is 1.

A degree sequence is *perfect* if no two of its elements are equal, that is, if the frequency sequence is $1, 1, \ldots, 1$. A degree sequence is *quasi-perfect* if exactly two of its elements are same.

Definition: Let $D = [d_i]_1^n$ be a non-negative sequence and k be any integer $1 \leq k \leq n$. Let $D' = [d_i']_1^n$ be the sequence obtained from D by setting $d_k = 0$ and $d_i' = d_i - 1$ for the d_k largest elements of D other than d_k. Let H_k be the graph obtained on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ by joining v_k to the d_k vertices corresponding to the d_k elements used to obtain D'. This operation of getting D' and H_k is called *laying off* d_k and D' is called the *residual sequence*, and H_k the subgraph obtained by laying off d_k.

Example Let $D = [2, 2, 3, 3, 4, 4]$. Take $d_3 = 0$. Then $D' = [2, 2, 0, 2, 3, 3]$. The subgraph H_k in this case is shown in Figure 2.2.

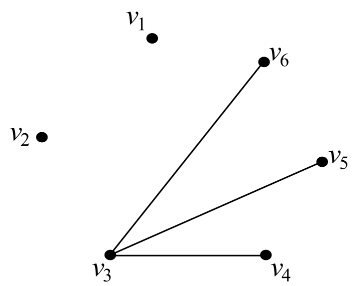


Fig. 2.2

2.2 Criteria for Degree Sequences

Havel [112] and Hakimi [99] independently obtained recursive necessary and sufficient conditions for a degree sequence, in terms of laying off a largest integer in the sequence. Wang and Kleitman [261] proved the necessary and sufficient conditions for arbitrary layoffs.

Theorem 2.1 A non-negative sequence is a degree sequence if and only if the residual sequence obtained by laying off any non-zero element of the sequence is a degree sequence.

Proof

Sufficiency Let the non-negative sequence be $[d_i]_1^n$. Suppose d_k is the non-zero element laid off and the residual sequence $[d_i']_1^n$ is a degree sequence. Then there exists a graph G'

realising $[d_i']_1^n$ in which v_k has degree zero and some d_k vertices, say v_{i_j}, $1 \le j \le d_k$ have degree $d_{i_j} - 1$. Now, by joining v_k to these vertices we get a graph G with degree sequence $[d_i]_1^n$. (Observe that the subgraph obtained by such joining is precisely the subgraph H_k obtained by laying off d_k).

Necessity We are given that there is a graph realising $D = [d_i]_1^n$. Let d_k be the element to be laid off. First, we claim there is a graph realising D in which v_k is adjacent to all the vertices in the set S of d_k largest elements of $D - \{d_k\}$. If not, let G be a graph realising D such that v_k is adjacent to the maximum possible number of vertices in S. Then there is a vertex v_i in S to which v_k is not adjacent and hence a vertex v_j outside S to which v_k is adjacent (since $d(v_k) = |S|$). By definition of S, $d_j \le d_i$. Therefore there is a vertex v_h in $V - \{v_k\}$ adjacent to v_i, but not adjacent to v_j. Note that v_h may be in S (Fig. 2.3).

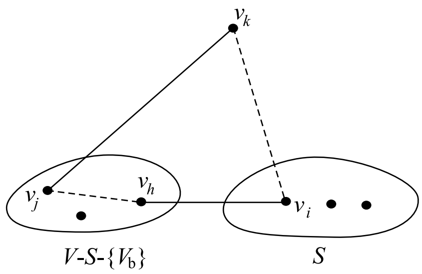


Fig. 2.3

Construct a graph H from G by deleting the edges v_jv_k and v_hv_i and adding the edges v_jv_h and v_iv_k. This operation does not change the degree sequence. Thus H is a graph realising the given sequence, in which one more vertex, namely v_i of S is adjacent to v_k, than in G. This contradicts the choice of G and establishes the claim.

To complete the proof, if G is a graph realising the given sequence and in which v_k is adjacent to all vertices of S, let $G' = G - v_k$. Then G' has the residual degree sequence obtained by laying off d_k. ❑

Definition: Let the subgraph H on the vertices v_i, v_j, v_r, v_s of a multigraph G contain the edges v_iv_j and v_rv_s. The operation of deleting these edges and introducing a pair of new edges v_iv_s and v_jv_r, or v_iv_r and v_jv_s is called an elementary degree preserving transformation (EDT), or simple exchange, or 2-switching, or elementary degree-invariant transformation.

Remarks

1. The result of an EDT is clearly a degree equivalent multigraph.
2. If an EDT is applied to a graph, the result will be a graph only if the latter pair of edges (v_iv_s and v_jv_r), or (v_iv_r and v_jv_s) does not exist in G.

Theorem 2.2 (Havel, Hakimi) The non-negative integer sequence $D = [d_i]_1^n$ is graphic if and only if D' is graphic, where D' is the sequence (having $n-1$ elements) obtained from D by deleting its largest element Δ and subtracting 1 from its Δ next largest elements.

Proof

Sufficiency Let $D = [d_i]_1^n$ be the non-negative sequence with $d_1 \geq d_2 \geq \ldots \geq d_n$. Let G' be the graph realising the sequence D'. We add a new vertex adjacent to vertices in G' having degrees $d_2 - 1, \ldots, d_{\Delta+1} - 1$. Those d_i are the Δ largest elements of D after Δ itself. (But the numbers $d_2 - 1, \ldots, d_{\Delta+1} - 1$ need not be the Δ largest elements in D').

Necessity Let G be a graph realising $D = [d_i]_1^n$, $d_1 \geq d_2 \geq \ldots \geq d_n$. We produce a graph G' realising D', where D' is the sequence obtained from D by deleting the largest entry d_1 and subtracting 1 from d_1 next largest entries.

Let w be a vertex of degree d_1 in G and $N(w)$ be the set of vertices which are adjacent to w. Let S be the set of d_1 number of vertices in G having the desired degrees $d_2, \ldots, d_{d_1+1}$.

If $N(w) = S$, we can delete w to obtain G'. Otherwise, some vertex of S is missing from $N(w)$. In this case, we modify G to increase $|N(w) \cap S|$ without changing the degree of any vertex. Since $|N(w) \cap S|$ can increase at most d_1 times, repeating this procedure converts an arbitrary graph G that realises D, into a graph G^* that realises D, and has $N(w) = S$. From G^*, we then delete w to obtain the desired graph G' realising D'.

If $N(w) \neq S$, let $x \in S$ and $z \notin S$, so that wz is an edge and wx is not an edge, since $d(w) = d_1 = |S|$. By this choice of S, $d(x) \geq d(z)$ (Fig. 2.4).

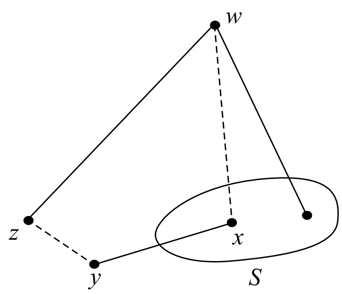


Fig. 2.4

We would like to add wx and delete wz without changing their respective degrees. It suffices to find a vertex y outside $T = \{x, z, w\}$ such that yx is an edge, while yz is not. If such a y exists, then we also delete xy and add zy. Let q be the number of copies of the edge xz (0 or 1). Now x has $d(x) - q$ neighbours outside T, and z has $d(z) - 1 - q$ neighbours outside T. Since $d(x) \geq d(z)$, the desired y outside T exists and we can perform the EDT (elementary degree preserving transformation or 2-switch). ❑

Algorithm: The above recursive conditions give an algorithm to check whether a non-negative sequence is a degree sequence and if so to construct a graph realising it.

The algorithm starts with an empty graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and at the kth iteration generates a subgraph H_k of G by deleting (laying off) a vertex of maximum degree in the residual sequence at that stage. If the given sequence is a degree sequence, we end up with a null degree sequence (i.e., for each i, $d_i = 0$) and the graph realising the original sequence is simply the sum of the subgraphs H_j. If not, at some stage, one of the elements of the residual sequence becomes negative, and the algorithm reports non-realisability of the sequence.

An obvious modification of the algorithm, obtained by choosing an arbitrary vertex of positive degree, gives the *Wang-Kleitman algorithm* for generating a graph with a given degree sequence.

Remarks

1. There can be many non-isomorphic graphs with the same degree sequence. The smallest example is the pair shown in Figure 2.5 on five vertices with the degree sequence [2, 2, 2, 1, 1].

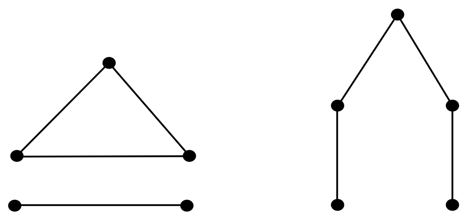


Fig. 2.5

   The problem of generating all non-isomorphic graphs of given order and size involves the problem of graph isomorphism for which a good algorithm is not yet known. So also is the problem of generating all non-isomorphic graphs with given degree sequence. In fact, even the problem of finding the number of non-isomorphic graphs with given order and size, or with given degree sequence (and several other problems of similar nature) has not been satisfactorily solved.

2. The Wang-Kleitman algorithm is certainly more general than the Havel-Hakimi algorithm, as it can generate more number of non-isomorphic graphs with a given degree sequence, because of the arbitrariness of the laid-off vertex. For example, not all the five non-isomorphic graphs with the degree sequence [3, 3, 2, 2, 1, 1] can be generated by the Havel-Hakimi algorithm unlike the Wang-Kleitman algorithm.

3. Even the Wang-Kleithman algorithm cannot always generate all graphs with a given degree sequence. For example, the graph G with degree sequence [3, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1] shown in Figure 2.6, cannot be generated by this algorithm. For

   a. if we lay off a 3, it has to be laid off against the other 3's and will generate a graph in which a vertex with degree 3 is adjacent to three other vertices with degree 3,
   b. if we lay off a 2 it will generate a graph with a vertex of degree 2 adjacent to two vertices of degree 3,

c. if we lay off a one it will generate a graph in which a vertex of degree one is adjacent to a vertex of degree 3. None of these cases is realised in the given graph G.

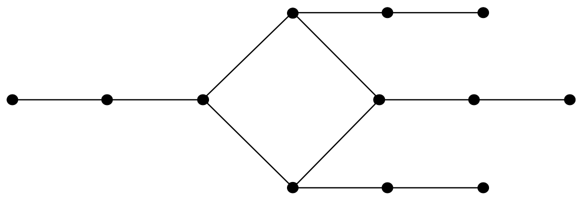


Fig. 2.6

However, there are other methods of generating all graphs realising a degree sequence D from any one graph realising D based on a theorem by Hakimi [98]. But those will also be inefficient unless some efficient isomorphism testing is developed.

4. The graphs in Figure 2.5 show that the same degree sequence may be realised by a connected as well as a disconnected graph. Such degree sequences are called *potentially connected*, where as a degree sequence D such that every graph realising D is connected is called a *forcibly connected* degree sequence.

Definition: If P is a graph property, and $D = [d_i]_1^n$ is a degree sequence, then D is said to be *potentially-P*, if at least one graph realising D is a P-graph, and it is said to be *forcibly-P* if every graph realising it is a P-graph.

Theorem 2.3 (Hakimi) If G_1 and G_2 are degree equivalent graphs, then one can be obtained from the other by a finite sequence of EDTs.

Proof Superimpose G_1 and G_2 such that each vertex of G_2 coincides with a vertex of G_1 with the same degree. Imagine the edges of G_1 are coloured blue and the edges of G_2 are coloured red. Then in the superimposed multigraph H, the number of blue edges incident equals the number of red edges incident at every vertex. We refer to this as blue-red parity. If there is a blue edge v_iv_j and a red edge v_iv_j in H, we call it a blue-red parallel pair.

Let K be the graph obtained from H by deleting all such parallel pairs. Then K is the null graph if and only if G_1 and G_2 are label-isomorphic in H and hence originally isomorphic. If this is not the case, we show that we can create more parallel pairs by a sequence of EDTs and delete them till the final resultant graph is null. This will prove the theorem.

Let B and R denote the sets of blue and red edges in K. If $v_iv_j \in B$, we show that we can produce a parallel pair at v_iv_j, so that the pair can be deleted. This would establish the claim made above.

Now, by construction, there is a blue-red degree parity at every vertex of K. So there are red edges v_iv_k, v_jv_r in K. If $v_k \neq v_r$ (Fig. 2.7(a)) an EDT in G_2 switching the red edges to v_iv_j, v_kv_r produces a blue-red parallel at v_iv_j.

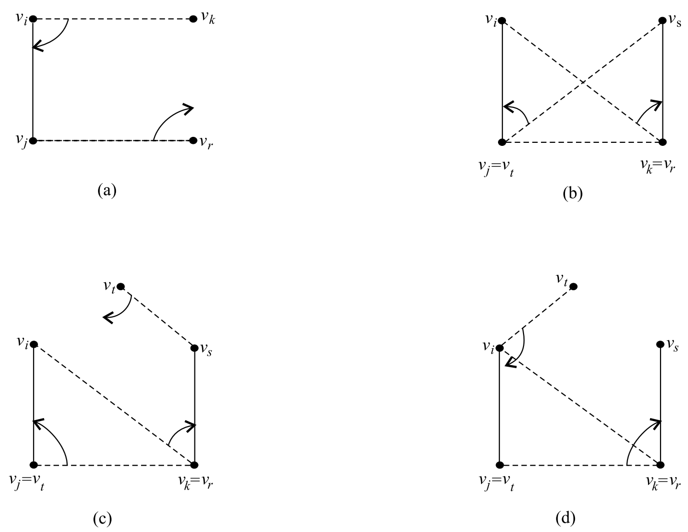


Fig. 2.7

If $v_k = v_r$, again by degree parity, at v_k there are at least two blue edges. Let $v_k v_s$ be one such blue edge. Then v_s is distinct from both v_i and v_j, for otherwise, there is a blue-red parallel pair $v_i v_k$ or $v_j v_r$. Then there is another red edge $v_s v_t$, v_t distinct from v_i or v_j.

Let $v_t \neq v_i$. The two subcases $v_t = v_j$ and $v_t \neq v_j$ are shown in Figure 2.7(b) and (c). In the case of (b), one EDT of G_2 switching $v_i v_k$ and $v_s v_t$ to positions $v_i v_j$ and $v_s v_k$ produces a blue-red pair at $v_i v_j$ and $v_k v_s$. In the case of (c), one EDT of G_2 switching $v_i v_k$ and $v_t v_s$ to positions $v_s v_k$ and $v_t v_i$ produces a blue-red parallel pair at $v_k v_s$ (which can be deleted). Another EDT of G_2 switching the blue-red pair $v_t v_i$ and $v_j v_k$ to positions $v_i v_j$ and $v_s v_k$ produces a blue-red pair $v_i v_j$.

Since in both cases we get a blue-red pair at $v_i v_j$ position, our claim is established and the proof of the theorem is complete. ❑

Remarks In the related context of a (0, 1) matrix A (that is, a matrix A whose elements are 0's or 1's), Ryser [227] defined an interchange as a transformation of the elements of A that changes a minor of type $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ into a minor of the type $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or vice versa and proved an interchange theorem which can be interpreted as EDT theorem for bipartite graphs and digraphs.

The next result is a combinatorial characterisation of degree sequences, due to Erdos and Gallai [73]. Several proofs of the criterion exist; the first proof given here is due to Choudam [58] and the second one is due to Tripathi et al [246].

Theorem 2.4 (Erdos-Gallai) A non-increasing sequence $[d_i]_1^n$ of non-negative integers is a degree sequence if and only if $D = [d_i]_1^n$ is even and the inequality

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i,\, k) \qquad (2.4.1)$$

is satisfied for each integer k, $1 \le k \le n$.

Proof

Necessity Evidently $\sum_{i=1}^{n} d_i$ is even. Let U denote the subset of vertices with the k highest degrees in D. Then the sum $s = \sum_{i=1}^{k} d_i$ can be split as $s_1 + s_2$, where s_1 is the contribution to s from edges joining vertices in U, each edge contributing 2 to the sum, and s_2 is the contribution to s from the edges between vertices in U and $\overline{U}$ (where $\overline{U} = V - U$), each edge contributing 1 to the sum (Fig. 2.8).

s_1 is clearly bounded above by the degree sum of a complete graph on k-vertices, i.e., $k(k-1)$. Also, each vertex v_i of $\overline{U}$ can be joined to at most min $(d_i,\, k)$ vertices of U, so that s_2 is bounded above by $\sum_{i=k+1}^{n} \min(d_i,\, k)$. Together, we get (2.4.1).

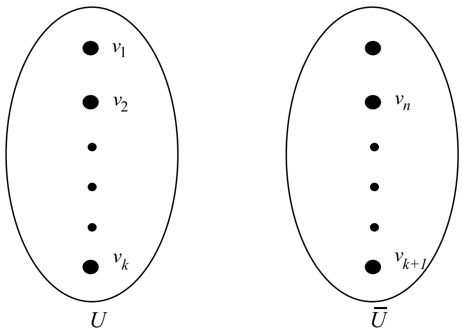


Fig. 2.8

Sufficiency We induct on the sum $s = \sum_{i=1}^{n} d_i$ and use the obvious inequality

$$\min(a,\, b) - 1 \le \min(a-1,\, b), \qquad (2.4.2)$$

for positive integers a and b.

For $s = 2$, clearly $K_2 \cup (n-2)K_1$ realises the only sequence $[1,\, 1,\, 0,\, 0,\, \ldots\, 0]$ or $[1^2 0^{n-2}]$ satisfying the conditions (2.4.1).

As induction hypothesis, let all non-increasing sequences of non-negative integers with even sum at most $s-2$ and satisfying (2.4.1) be degree sequences.

Let $D = [d_i]_1^n$ be a sequence with sum s and satisfying (2.4.1). We produce a new non-increasing sequence D' of non-negative integers by subtracting one each from two positive terms of D and verify that D' satisfies the hypothesis of the theorem. Since the trailing zeros in the non-increasing sequences of non-negative integers do not essentially affect the argument, there is no loss of generality in assuming that $d_n > 0$, and we assume this to simplify the expression.

To define D', let t be the smallest integer (≥ 1) such that $d_t > d_{t+1}$. That is, let D be $d_1 = d_2 = \ldots = d_t > d_{t+1} \geq d_{t+2} \geq \ldots \geq d_n > 0$.

If D is regular (that is, $d_i = d > 0$, for all i) then let t be $n-1$.

Then $d_i' = \begin{cases} d_i, & for \;\; 1 \leq i \leq t-1 \;\; and \;\; t+1 \leq i \leq n-1, \\ d_t - 1, & for \;\; i = t, \\ d_n - 1, & for \;\; i = n. \end{cases}$

Clearly, D' is a non-increasing sequence of non-negative integers and $\sum_{i=1}^{n} d_i' = s-2$ is even.

We verify that D' satisfies (2.4.1) by considering several cases depending on the relative position of k and the magnitudes of d_k and d_n.

Case I Let $k = n$. Therefore, $\sum_{i=1}^{k} d_i' = \sum_{i=1}^{k} d_i - 2 \leq n(n-1) - 2 < n(n-1)$ = RHS of (2.4.1) for D'.

Case II Let $t \leq k \leq n-1$.

Then $\sum_{i=1}^{k} d_i' = \sum_{i=1}^{k} d_i - 1 \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i,\, k) - 1$ (since D satisfies (2.4.1))

$$= k(k-1) + \sum_{i=k+1}^{n-1} \min(d_i',\, k) + \min(d_n,\, k) - 1$$

$$\leq k(k-1) + \sum_{i=k+1}^{n-1} \min(d_i',\, k) + \min(d_n - 1,\, k) \qquad \text{by (2.4.2)}$$

$$= k(k-1) + \sum_{i=k+1}^{n-1} \min(d_i',\, k) + \min(d_n',\, k)$$

Therefore, $\sum_{i=1}^{k} d_i' \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i',\, k)$.

Case III Let $k \leq t-1$.

Subcase III.1 Assume $d_k \leq k-1$.

Then $\sum_{i=1}^{k} d_i' = kd_k \leq k(k-1) \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i',\, k)$,
since the second term is non-negative.

Subcase III.2 Every $d_j = k$, $1 \leq j \leq k$. We first observe that $d_{k+2} + \ldots + d_n \geq 2$.

This is obvious if $k+2 \le n-1$, because $d_n > 0$ gives $d_n \ge 1$ and $d_{n-1} \ge 1$. When $k+2 = n$, we have $k = n-2$. As $k \le t-1$, $t \ge k+1 = n-2+1 = n-1$. Since $t > n-1$ is not possible, $t = n-1$.

The sequence D is $[n-2, n-2, \ldots, n-2, d_n]$, or $[(n-2)^{n-1} d_n]$. Then $s = (n-1)\,(n-2) + d_n$. Since s is even, d_n is even and hence $d_n \ge 2$. Thus, $d_{k+2} + \ldots + d_n \ge 2$.

Therefore, $d_{k+2} + \ldots + d_n - 2 \ge 0$.

Now,

$$\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i = k.k = k^2 = k^2 - k + k$$

$$= k^2 - k + d_{k+1}, \text{ (because } k \le t-1, \text{ and } d_1 = \ldots = d_{t-1} = d_t,$$

so if $d_{t-1} = k$, then $d_t = k$, and if $d_k = k$, $d_{k+1} = k$).

Thus, $\sum_{i=1}^{k} d'_i \le k^2 - k + d_{k+1} + (d_{k+2} + \ldots + d_n - 2) = k(k-1) + \sum_{i=k+1}^{n} \min(d_i,\, k) - 2$,

(because min $(d_{k+1},\, k) = d_{k+1}$, min $(d_{k+2},\, k) = k = d_{k+2},\ \ldots$, min $(d_t,\, k) = k = d_t,\ \ldots$, min $(d_{t+1},\, k) = d_{t+1}$ (as $d_{t+1} < d_t = k$), $\ldots$, min $(d_n,\, k) = d_n$ (as $d_n < d_t = k$)).

Hence, $\sum_{i=1}^{k} d'_i \le k(k-1) + \sum_{\substack{i=k+1\\ i\neq t,n}}^{n} \min(d_i,\, k) + \min(d_t,\, k) + \min(d_n,\, k) - 2$

$$= k(k-1) + \sum_{\substack{i=k+1\\ i\neq t,n}}^{n} \min(d'_i,\, k) + \min(d'_t + 1,\, k) + \min(d'_n + 1,\, k) - 2$$

$$\le k(k-1) + \sum_{\substack{i=k+1\\ i\neq t,n}}^{n} \min(d'_i,\, k) + \min(d'_t,\, k) + 1 + \min(d'_n,\, k) + 1 - 2$$

$$= k(k-1) + \sum_{i=k+1}^{n} \min(d'_i,\, k).$$

Subcase III.3 Let $d_k \ge k+1$.

i. Let $d_n \ge k+1$.

Then $\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i,\, k)$ (since D satisfies (2.4.1))

$$= k(k-1) + \sum_{\substack{k+1\\ i\neq t,n}}^{n} \min(d_i,\, k) + \min(d_t,\, k) + \min(d_n,\, k)$$

$$= k(k-1) + \sum_{\substack{k+1 \\ i \neq t,\, n}}^{n} \min(d_i', k) + \min(d_t - 1, k) + \min(d_n - 1, k),$$

(because $\min(d_t, k) = \min(d_t - 1, k) = k$, $\min(d_n, k) = \min(d_n - 1, k) = k$, as $d_t \geq k + 1$, $d_n \geq k+1$ implies that $d_t - 1 \geq k$, $\quad d_n - 1 \geq k$).

$$\text{So, } \sum_{i=1}^{k} d_i' \leq k(k-1) + \sum_{\substack{k+1 \\ i \neq t,\, n}}^{n} \min(d_i', k) + \min(d_t', k) + \min(d_n', k)$$

$$= k(k-1) + \sum_{i=k+1}^{n} \min(d_i', k).$$

ii. Let $d_n \leq k$ and let r be the smallest integer such that $d_{t+r+1} \leq k$. We verify that in (2.4.1), D can not attain equality for such a choice of k. For, with equality, we have

$$\sum_{i=1}^{k} d_i = kd_k = k(k-1) + \sum_{k+1}^{t+r} \min(d_i, k) + \sum_{t+r+1}^{n} \min(d_i, k)$$

$$= k(k-1) + (t+r-k)k + \sum_{t+r+1}^{n} d_i,$$

$$\text{because } \min(d_i, k) = \begin{cases} k, & for\ i = k+1, \ldots, t+r\ as\ d_i \geq k+1, \\ d_i, & for\ i = t+r+1, \ldots, n\ as\ d_i \leq k. \end{cases}$$

$$\text{So, } kd_k = k(t+r-1) + \sum_{t+r+1}^{k} d_i.$$

$$\text{Then } \sum_{i=1}^{k+1} d_i = (k+1)d_k = (k+1)\left\{(t+r-1) + \frac{1}{k}\sum_{t+r+1}^{n} d_i\right\}, \text{ (using } d_k \text{ from above)}$$

$$= (k+1)(t+r-1) + \frac{k+1}{k}\sum_{t+r+1}^{n} d_i > (k+1)(t+r-1) + \sum_{t+r+1}^{n} d_i$$

$$= (k+1)k - (k+1)k + (k+1)(t+r-1) + \sum_{t+r+1}^{n} d_i$$

$$= (k+1)k + (k+1)(t+r-k-1) + \sum_{t+r+1}^{n} d_i = (k+1)k + \sum_{k+1}^{t+r}(k+1) + \sum_{t+r+1}^{n} d_i$$

$$= (k+1)k + \sum_{t+r+1}^{n} \min(d_i, k+1),$$

because $\min(d_i,\, k+1) = k+1$ for $i = k+1,\, \ldots,\, t+r$, and

$\min(d_i,\, k+1) = d_i$, for $i = t+r+1,\, \ldots,\, n$.

So, $\sum_{i=1}^{k+1} d_i > k(k+1) + \sum_{k+1}^{n} \min(d_i,\, k+1)$.

Therefore, $\sum_{i=1}^{k+1} d_i > k(k+1) + (k+1) + \sum_{k+2}^{n} \min(d_i,\, k+1)$,

which is a contradiction to (2.4.1), for D for $k+1$. Hence D has strict inequality for k.

Therefore, $\sum_{i=1}^{k} d_i' = \sum_{i=1}^{k} d_i < k(k-1) + \sum_{k+1}^{n} \min(d_i,\, k)$.

Thus, $\sum_{i=1}^{k} d_i' = \sum_{i=1}^{k} d_i \le k(k-1) + \sum_{k+1}^{n} \min(d_i,\, k) - 1$

$$= k(k-1) + \sum_{\substack{i=k+1 \\ i \neq t}}^{n-1} \min(d_i,\, k) + \min(d_t,\, k) + \min(d_n,\, k) - 1$$

$$\le k(k-1) + \sum_{\substack{i=k+1 \\ i \neq t}}^{n-1} \min(d_i',\, k) + \min(d_t - 1,\, k) + \min(d_n - 1,\, k),$$

as $\min(d_n,\, k) - 1 \le \min(d_n - 1,\, k)$, $\min(d_t,\, k) = k$ (since $d_t \ge k+1$), $\min(d_t - 1,\, k) = k$ (since $d_t - 1 \ge k$).

Therefore, $\sum_{i=1}^{k} d_i' \le k(k-1) + \sum_{k+1}^{n} \min(d_i',\, k)$.

Hence in all cases D' satisfies (2.4.1).

Therefore by induction hypothesis, there is a graph G' realising D'. If $v_t v_n \notin E(G')$, then $G' + v_t v_n$ gives a realisation G of D. If $v_t v_n \in E(G')$, since $d(v_t|G') = d_t - 1 \le n-2$, there is a vertex v_r such that $v_r v_t \notin E(G')$. Also, since $d(v_r|G') > d(v_n|G')$, there is a vertex v_s such that $v_s v_n \notin E(G')$. Making an EDT exchanging the edge pair $v_t v_n$, $v_r v_s$ for the edge pair $v_t v_r$, $v_s v_n$, we get a realisation G'' of D' with $v_t v_n \notin E(G'')$. Then $G'' + v_t v_n$ realises D.

Second Proof of Sufficiency (Tripathi et al.) Let a subrealisation of a non-increasing sequence $[d_1, d_1, \ldots, d_n]$ be a graph with vertices $v_1, v_1, \ldots, v_n$ such that $d(v_i) = d_i$ for $1 \le i \le n$, where $d(v_i)$ denotes the degree of v_i. Given a sequence $[d_1, d_1, \ldots, d_n]$ with an even sum that satisfies (2.4.1), we construct a realisation through successive subrealisations. The initial subrealisation has n vertices and no edges.

In a subrealisation, the *critical index* r is the largest index such that $d(v_i) = d_i$ for $1 \leq i < r$. Initially, $r = 1$ unless the sequence is all 0, in which case the process is complete. While $r \leq n$, we obtain a new subrealisation with smaller deficiency $d_r - d(v_r)$ at vertex v_r while not changing the degree of any vertex v_i with $i < r$ (the degree sequence increases lexicograpically). The process can only stop when the subrealisation of d.

Let $S = \{v_{r+1}, \ldots, v_n\}$. We maintain the condition that S is an independent set, which certainly holds initially. Write $u_i \leftrightarrow v_j$ when $v_i v_j \in E(G)$; otherwise, $v_i \not\leftrightarrow v_j$

Case 0 $v_r \not\leftrightarrow v_i$ for some vertex v_i such that $d(v_i) < d_i$. Add the edge $u_r v_i$.

Case 1 $v_r \not\leftrightarrow v_i$ for some i with $i < r$. Since $d(v_i) = d_i \geq d_r > d(v_r)$, there exists $u \in N(u_i) - (N(v_r) \cup \{v_r\})$, where $N(z) = \{y : z \leftrightarrow y\}$. If $d_r - d(v_r) \geq 2$, then replace uv_i with $\{uv_r, v_i v_r\}$. If $d_r - d(v_r) = 1$, then since $\sum d_i - \sum d(v_i)$ is even there is an index k with $k > r$ such that $d(v_k) < d_k$. Case 0 applies unless $v_r \leftrightarrow v_k$; replace $\{v_r v_k, uv_i\}$ with $\{uv_r, v_i u_r\}$.

Case 2 $v_1, \ldots, v_{r-1} \in N(v_r)$ and $d(v_k) \neq \min\{r, d_k\}$ for some k with $k > r$. In a subrealisation, $d(v_k) \leq d_k$. Since S is independent, $d(v_k) \leq r$. Hence $d(v_k) < \min\{r, d_k\}$, and case 0 applies unless $u_k \leftrightarrow v_r$. Since $d(v_k) < r$, there exists i with $i < r$ such that $u_k \not\leftrightarrow v_i$. Since $d(v_i) > d(v_r)$, there exists $u \in N(v_i) - (N(v_r) \cup \{u_r\})$. Replace uv_i with $\{uv_r, v_i v_k\}$.

Case 3 $v_1, \ldots, v_{r-1} \notin N(v_r)$ and $v_i \leftrightarrow v_i$ for some i and j with $i < j < r$. Case 1 applies unless $v_i, v_j \in N(v_r)$. Since $d(v_i) \geq d(v_i) > d(v_r)$, there exists $u \in N(v_i) - (N(v_r) \cup \{v_r\})$ and $w \in N(v_j) - (N(v_r) \cup \{v_r\})$ (possibly $u = w$). Since $u, w \notin N(v_r)$, Case 1 applies unless $u, w \in S$. Replace $\{v_i v_j, uv_r\}$ with $\{uv_r, v, v_r\}$.

If none of these case apply, then $v_1, \ldots, v_r$ are pairwise adjacent, and $d(v_k) = \min\{r, d_k\}$ for $k > r$. Since S is independent, $\sum_{i=1}^{r} d(v_i) = r(r-1) + \sum_{k=r+1}^{n} \min\{r, d_k\}$. By (2.4.1), $\sum_{i=1}^{r} d_1$ is bounded by the right side. Hence we have already eliminated the deficiency at vertex r. Increase r by 1 and continue. ❑

Tripathi and Vijay [245] have shown that the Erdos-Gallai condition characterising graphical degree sequences of length n needs to be checked only for as many k as there are distinct terms in the sequence and not for all k, $1 \leq k \leq n$.

2.3 Degree Set of a Graph

The set of distinct non-negative integers occurring in a degree sequence of a graph is called its *degree set*. For example, let the degree sequence be $D = [2, 2, 3, 3, 4, 4]$, then degree set is $\{2, 3, 4\}$. A set of distinct non-negative integers is called a degree set if it is the degree set of some graph and the graph is said to *realise* the degree set.

Let $S = \{d_1, d_2, \ldots, d_k\}$ be the set of distinct non-negative integers. Clearly, S is the degree set as the graph

$$G = K_{d_1+1} \cup K_{d_2+1} \cup \ldots \cup K_{d_k+1},$$

realises S. This graph has $d_1 + d_2 + \ldots + d_k + k$ vertices.

Example Let $S = \{1, 3, 4\}$. Then $G = K_2 \cup K_4 \cup K_5$ (Fig. 2.9).

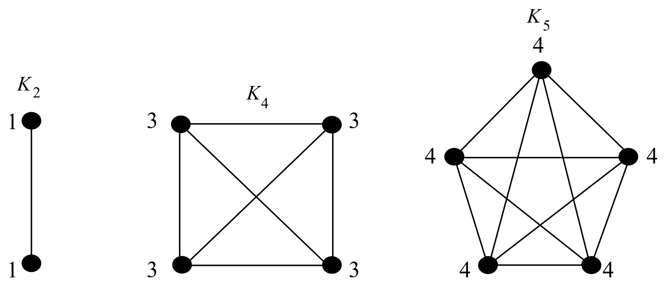


Fig. 2.9

The following result is due to Kapoor, Polimeni and Wall [126].

Theorem 2.5 Any set S of distinct positive integers is the degree set of a connected graph and the minimum order of such a graph is $M + 1$, where M is the maximum integer in the set S.

Proof Let S be a degree set and $n_0(S)$ denote the minimum order of a graph G realising S. As M is the maximum integer in S, therefore in G there is a vertex adjacent to M other vertices, i.e., $n_0(S) \geq M + 1$. Now, if there exists a graph of order $M + 1$ with S as degree set, then $n_0(S) = M + 1$. The existence of such a graph is established by induction on the number of elements p of S.

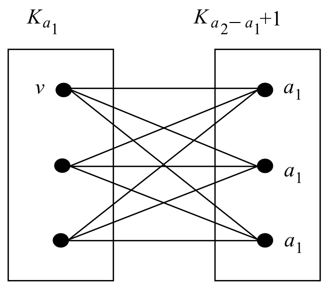
Let $S = \{a_1, a_2, \ldots, a_p\}$ with $a_1 < a_2 < \ldots < a_p$.

For $p = 1$, the complete graph K_{a_1+1} realises $\{a_1\}$ as degree set.

For $p = 2$, we have $S = \{a_1, a_2\}$. Let $G = K_{a_1} V \overline{K}_{a_2-a_1+1}$ (join of two graphs). Here every vertex of K_{a_1} has degree a_2 and every other vertex has degree a_1 and therefore G realises $\{a_1, a_2\}$ (Fig. 2.10(a)).

For $p = 3$, we have $S = \{a_1, a_2, a_3\}$. Then $G = K_{a_1} V(\overline{K}_{a_3-a_2} \cup H)$, where H is the graph realising the degree set $\{a_2 - a_1\}$ with $a_2 - a_1 + 1$ vertices, realises $\{a_1, a_2, a_3\}$ (Fig. 2.10 (b)).

(Note that $d(u) = a_1 - 1 + a_3 - a_2 + a_2 - a_1 + 1 = a_3$, $d(v) = a_1$, $d(w) = a_2 - a_2 + a_1 = a_2$).



(a)

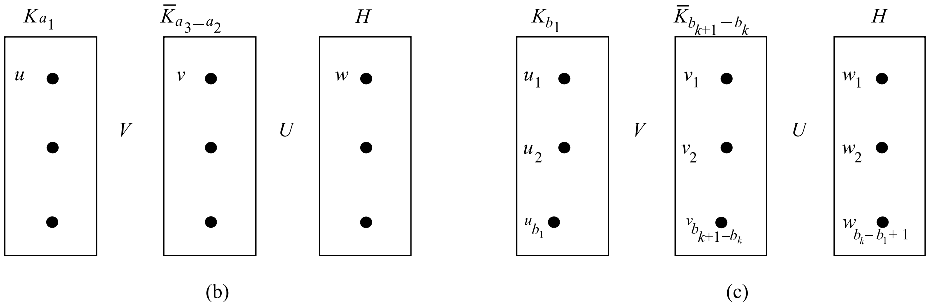


Fig. 2.10

Let every set with h positive integers, $1 \le h \le k$, be the degree set. Let $S_1 = \{b_1, b_2, \ldots, b_{k+1}\}$ be a $(k+1)$ set of positive integers arranged in increasing order. By induction hypothesis, there is a graph H realising the degree set $\{b_2 - b_1, b_3 - b_1, \ldots, b_k - b_1\}$ with order $b_k - b_1 + 1$. The graph $G = K_{b_1} V(\overline{K}_{b_{k+1}-b_k} \cup H)$, with order $b_{k+1} + 1$ realises S_1 (Fig. 2.10 (c)). Clearly by construction, all these graphs are connected.

Hence the result follows by induction. ❑

Note that $d(u_i) = b_1 - 1 + b_{k+1} - b_k + b_k - b_1 + 1 = b_k + 1$, $d(v_i) = b_1$, $d(w_i) = b_{i+1} - b_1 + b_1 = b_{i+1}$, that is $d(w_1) = b_2$, $d(w_2) = b_3$, $\ldots$, $d\left(w_{b_k - b_1 + 1}\right) = b_k - b_1 + b_1 = b_k$. Some results on degree sets in bipartite and tripartite graphs can be seen in [262].

2.4 New Criterion

We have the following notations. Let $D = [d_i]_1^n$ be a non-decreasing sequence of non-negative integers with $0 \le d_i \le n-1$ for all i. Let $n - p_1$ be the greatest integer, $n - p_1 - p_2$, the second greatest integer and $n - \sum_{r=1}^{k} p_r$, the kth greatest integer in D, $1 \le p_r \le n - (r-1)$. Let the number of times the kth greatest integer appears in D be denoted by a_k. Also, we take

$$t_k = n - \left(n - \sum_{r=1}^{k} p_r\right) = \sum_{r=1}^{k} p_r, \quad 1 \le p_r \le n - (r-1) \text{ and } j_k = 1, 2, \ldots, p_{k+1}.$$

The following result due to Pirzada and YinJian [208] is another criterion for a non-negative sequence of integers in non-decreasing order to be the degree sequence of some graph.

Theorem 2.6 A non-decreasing sequence $[d_i]_1^n$ of non-negative integers, where $\sum_{i=1}^{n} d_i$ is even and $0 \le d_i \le n-1$ for all i, is a degree sequence of a graph if and only if »

$$\sum_{i=1}^{t_k + j_k - 1} d_i \ge \sum_{m=1}^{k} \{j_k + (k-m)\} a_m \tag{2.6.1}$$

for all $t_k + j_k - 1 + \sum_{m=1}^{k} a_m \le n$.

Note In the above criterion, the inequalities (2.6.1) are to be checked only for $t_k + j_k - 1 + \sum_{m=1}^{k} a_m \le n$ (but not for greater than n).

We now illustrate the theorem with the help of the following examples.

Example 1 Let $D = [1, 2, 2, 4, 6, 6, 6, 7, 8, 8]$.

Here, $n = 10$, $a_1 = 2$, $a_2 = 1$, $a_3 = 3$, $a_4 = 1$, $p_1 = 2$, $p_2 = 1$, $p_3 = 1$, $p_4 = 2$, so $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 6$.

Also, $j_1 = 1,\ j_2 = 1,\ j_3 = 1,\ 2$.

Now, for $j_1 = 1$, $\sum_{i=1}^{t_1+j_1-1} d_i = \sum_{i=1}^{2+1-1} d_i = \sum_{i=1}^{2} d_i = 1+2 = 3$,

and $\sum_{m=1}^{k} [j_k + (k-m)]\, a_m = \sum_{m=1}^{1} [j_1 + (1-m)]\, a_m = j_1 a_1 = 2$.

So inequalities (2.6.1) hold.

For $j_2 = 1$, $\sum_{i=1}^{t_2+j_2-1} d_i = \sum_{i=1}^{3+1-1} d_i = \sum_{i=1}^{3} d_i = 5$

and $\sum_{m=1}^{k} [j_k + (k-m)]\, a_m = \sum_{m=1}^{2} [j_2 + (2-m)]\, a_m = 2a_1 + a_2 = 4+1 = 5$.

So inequalities (2.6.1) hold.

For $j_3 = 1$, $\sum_{i=1}^{t_3+j_3-1} d_i = \sum_{i=1}^{4+1-1} d_i = \sum_{i=1}^{4} d_i = 9$

and $\sum_{m=1}^{3} [j_3 + (3-m)]\, a_m = \sum_{m=1}^{3} [1 + (3-m)]\, a_m = 3a_1 + 2a_2 + a_3 = 6+2+3 = 11$.

Since the inequalities (2.6.1) do not hold (as 9 > 11 is not true), D is not the degree sequence.

Example 2 Let $D = [1, 2, 3, 4, 5, 6, 6, 7, 8, 8]$.

Here, $n = 10$, $a_1 = 2$, $a_2 = 1$, $a_3 = 2$, $a_4 = 1$, $p_1 = 2$, $p_2 = 1$, $p_3 = 1$, $p_4 = 1$, $p_5 = 1$. So $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 5$.

Also, $j_1 = 1,\ j_2 = 1,\ j_3 = 1,\ j_4 = 1$.

For $j_1 = 1$, $\sum_{i=1}^{t_1+j_1-1} d_i = \sum_{i=1}^{2+1-1} d_i = \sum_{i=1}^{2} d_i = 3$,

and $\sum_{m=1}^{1} [j_1 + (1-m)]\, a_m = a_1 = 2$.

Obviously the inequalities (2.6.1) hold.

For $j_2 = 1$, $\sum_{i=1}^{t_2+j_2-1} d_i = \sum_{i=1}^{3+1-1} d_i = \sum_{i=1}^{3} d_i = 6$

and $\sum_{m=1}^{2} [j_2 + (2-m)]\, a_m = \sum_{m=1}^{2} [1 + (2-m)]\, a_m = 2a_1 + a_2 = 4 + 1 = 5$.

Here again the inequalities (2.6.1) hold.

For $j_3 = 1$, $\sum_{i=1}^{t_3+j_3-1} d_i = \sum_{i=1}^{4+1-1} d_i = \sum_{i=1}^{4} d_i = 10$

and $\sum_{m=1}^{3} [j_3 + (3-m)]\, a_m = \sum_{m=1}^{3} [1 + (3-m)]\, a_m = 3a_1 + 2a_2 + a_3 = 6 + 2 + 2 = 10$.

Therefore the inequalities (2.6.1) hold.

For $j_4 = 1$, $t_4 + j_4 - 1 = 5 + 1 - 1 = 5$ and $a_1 + a_2 + a_3 + a_4 = 2 + 1 + 2 + 1 = 6$, therefore $t_4 + j_4 - 1 + \sum_{m=1}^{4} a_m = 5 + 6 = 11 > 10$ and no further verification of the inequalities is to be done.

Hence D is the degree sequence.

2.5 Equivalence of Seven Criteria

We list the seven criteria for integer sequences to be graphic.

A. The Ryser Criterion (Bondy and Murty [36] and Ryser [227]) A sequence $[a_1, \ldots, a_p;\ b_1, \ldots, b_n]$ is called bipartite-graphic if and only if there is a simple bipartite graph such that one component has degree sequence $[a_1, \ldots, a_p]$ and the other one has $[b_1, \ldots, b_n]$. Define $f = \max\{i : d_i \geq i\}$ and $\tilde{d}_1 = d_i + 1$ if $i \in \langle f \rangle (= \{1, \ldots, f\})$ and $\tilde{d}_1 = d_i$ otherwise. The criterion can be stated as follows.

The integer sequence $[\tilde{d}_1, \ldots, \tilde{d}_n;\ \tilde{d}_1, \ldots, \tilde{d}_n]$is bipartite-graphic. (A)

B. The Berge Criterion (Berge [23]) Define $[\bar{d}_1, \ldots, \bar{d}_n]$ as follows: For $i \in \langle n \rangle$, $\bar{d}_i$ is the ith column sum of the (0, 1) matrix, which has for each k and d_k leading terms in row

k equal to 1 except for the (k, k)th term that is 0 and also the remaining entries are 0. If $d_1 = 3$, $d_2 = 2$, $d_3 = 2$, $d_4 = 2$, $d_5 = 1$, then $\bar{d_1} = 4$, $\bar{d_2} = 3$, $\bar{d_3} = 2$, $\bar{d_4} = 1$, $\bar{d_5} = 0$, and the (0, 1) matrix becomes

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The criterion is

$$\sum_{i=1}^{k} \overline{d_i} \leq \sum_{i=1}^{k} d_i \text{ for each } k \in \langle n \rangle. \tag{B}$$

C. The Erdos-Gallai Criterion. (Bondy and Murty [36])

$$\sum_{i=1}^{k} d_i \leq (k)(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \text{ for each } k \in \langle n \rangle. \tag{C}$$

D. The Fulkerson-Hoffman-McAndrew Criterion (Fulkerson[83] and Grunbaum [92)

$$\sum_{i=1}^{k} d_i \leq (k)(n-m-1) + \sum_{i=n-m+1}^{n} d_i \text{ for each } k \in \langle n \rangle,\ m \geq 0 \text{ and } k+m \leq n. \tag{D}$$

E. The Bollobas Criterion (Bollabas[29]))

$$\sum_{i=1}^{k} d_i \leq \sum_{j=k+1}^{n} d_i + \sum_{i=1}^{k} \min\{d_j, k-1\} \text{ for each } k \in \langle n \rangle. \tag{E}$$

F. The Grunbaum Criterion (Grunbaum [92]).

$$\sum_{i=1}^{k} \max\{k-1, d_i\} \leq (k)(k-1) + \sum_{i=k+1}^{n} d_i \text{ for each } k \in \langle n \rangle. \tag{F}$$

G. The Hasselbarth Criterion (Hasselbarth [111]) Define $[d'_i, \ldots, d'_n]$ as follows. For $i \in \langle n \rangle$, d'_i is the ith column sum of the (0, 1)-matrix in which the d_i leading terms in row i are 1's and the remaining entries are 0's. The criterion is

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} (d_i^* - 1) \text{ for each } k \in \langle f \rangle, \tag{G}$$

with $f = \max\{i : d_i \geq i\}$.

The following result due to Sierksma and Hoogeveen [235] gives the equivalence among the above seven criteria.

Theorem 2.7 (Sierksma and Hoogeveen [235]) Let $[d_1, \ldots, d_n]$ be a positive integer sequence with even sum. Then each of the criteria $(A)-(G)$ is equivalent to the statement that $[d_1, \ldots, d_n]$ is graphic.

Proof Refer to Ryser [227].

2.6 Signed Graphs

A signed graph is a graph in which every edge is labelled with a'+' or a'−'. An edge uv labelled with a'+' is called a *positive edge*, and is denoted by uv^+. An edge uv labelled with a'−' is called a *negative edge*, and is denoted by uv^-. In a signed graph $G(V, E)$, the *positive degree* of a vertex u is $\deg^+(u) = |\{uv : uv^+ \in E\}|$, the *negative degree* of a vertex u is $\deg^-(u) = |\{uv : uv^- \in E\}|$, the *signed degree* of u is $\mathrm{sdeg}(u) = \deg^+(u) - \deg^-(u)$ and the *degree* of u is $\deg(u) = \deg^+(u) + \deg^-(u)$. An edge uv labelled with a'+' is called a *positive edge*, and is denoted by uv^+. An edge uv labelled with a'−' is called a *negative edge*, and is denoted by uv^-.

An integral sequence $[d_i]_1^n$ is the *signed degree sequence* of a signed graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ if $s\deg(v_i) = d_i$, for $1 \le i \le n$.

Chartrand et al. [50] have given the characterisation of signed degree sequences of signed paths, signed stars, signed double stars and complete signed graphs. An integral sequence is *s-graphical* if it is the signed degree sequence of a signed graph. An integral sequence $[d_i]_1^n$ is *standard* if $n-1 \ge d_1 \ge d_2 \ge \ldots \ge d_n$ and $d_1 \ge |d_n|$.

The following lemma shows that a signed degree sequence can be modified and rearranged into an equivalent standard form.

Lemma 2.1 If $[d_i]_1^n$ is the signed degree sequence of a signed graph G, then $[-d_i]_1^n$ is the signed degree sequence of the signed graph G' obtained from G by interchanging positive edges with negative edges.

The following necessary and sufficient condition under which an integral sequence is s-graphical is due to Chartrand et al. [50].

Theorem 2.8 A standard integral sequence $[d_i]_1^n$ is s-graphical if and only if the sequence $[d_2-1,\, d_{d_1+s+1}-1,\, d_{d_1+s+2}, \ldots,\, d_{n-s},\, d_{n-s+1}+1,\, \ldots,\, d_n+1]$ is s-graphical for some $0 \le s \le (n-1-d_1)/2$.

Remark We note that Hakimi's theorem for degree sequences is a case of Theorem 2.8 by taking $s = 0$. This leads to an efficient algorithm for recognising the degree sequences of a graph. But the wide degree of latitude for choosing s in Theorem 2.8 makes it harder to devise an efficient algorithm implementation.

The following result due to Yan et al. [271] provides a good choice for parameter s in Theorem 2.8. It leads to a polynomial time algorithm for recognising signed degree sequences.

Theorem 2.9 A standard sequence $D = [d_i]_1^n$ is s-graphical if and only if $D_m = [d_2 - 1,\, d_{d_1+m+1} - 1,\, \ldots,\, d_{d_1+m+2},\, \ldots,\, d_{n-m},\, d_{n-m+1} + 1\, \ldots,\, d_n + 1]$ is s-graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$.

Proof Let D be the signed degree sequence of a signed graph $G = (V,\, E)$ with $V = \{v_1,\, v_2,\, \ldots, v_n\}$ and $\mathrm{sdeg}(v_i) = d_i$, for $1 \le i \le n$. For each s, $0 \le s \le (n-1-d_1)/2$, consider the sequence

$$D_s = [d_2 - 1,\, \ldots,\, d_{d_1+s+1} - 1, d_{d_1+s+2},\, \ldots,\, d_{n-s},\, d_{n-s+1} + 1,\, \ldots, d_n + 1].$$

By Theorem 2.8, D_s is s-graphical for some s. We may choose s such that $|s-m|$ is minimum. Suppose $G' = (V',\, E')$ is a signed graph with $V' = \{v_2,\, v_3,\, \ldots, v_n\}$ whose signed degree sequence is D_s.

If $s < m$, then $d_a > d_b$ by the choice of m, where $a = d_1 + s + 2$ and $b = n - s$. Since $d_a > d_b$, there exists some vertex v_k of G' different from v_a and v_b and satisfies one of the following conditions.

i. $v_a v_k^+$ is a positive edge and $v_b v_k^-$ is a negative edge.

ii. $v_a v_k^+$ is a positive edge and v_b is not adjacent to v_k

iii. v_a is not adjacent to v_k and $v_b v_k^-$ is a negative edge

For (i), remove $v_a v_k^+$ and $v_b\ v_k^-$ to G', and for (ii), remove $v_a v_k^+$ from G' and add a new positive edge $v_b v_k^+$ to G' and for (iii), remove $v_b v_k^-$ from G' and a new negative edge $v_a v_k^-$ to G'. These modifications result in a signed graph G'' whose signed degree sequence D_{s+1}. This contradicts the minimality of $|s-m|$.

If $s > m$, then $d_{d_1+s+1} = d_{n-s+1}$, and therefore, $d_{d_1+s+1} - 1 < d_{n-s+1} - 1$. An argument similar to the above leads to a contradiction in the choice of s. Therefore, $s = m$ and D_m is s-graphical.

Conversely, suppose D_m is the signed degree sequence of a signed graph $G' = (V',\, E')$ in which $V' = \{v_2,\, v_3,\, \ldots,\, v_n\}$. If G is the signed graph obtained from G' by adding a new vertex v_1 and new positive edges $v_1 v_i^+$ for $2 \le i \le d_1 + m + 1$ and new negative edges $v_1 v_j^-$ for $n - m + 1 \le j \le n$, then D is the signed degree sequence of G. ❑

In a signed graph $G = (V,\, E)$ with $|V| = n$, $|E| = m$, we denote by m^+ and m^- respectively, the numbers of positive edges and negative edges of G. Further, n_+, n_0 and n_- denote respectively, the numbers of vertices with positive, zero and negative signed degrees.

The following result is due to Chartrand et al. [50].

Lemma 2.2 If $G = (V,\, E)$ is a signed graph with $|V| = n$, $|E| = m$, then $k = \sum_{v \in V} \mathrm{s\,deg}(v) \equiv 2m (\mathrm{mod} 4)$, $m^+ = \frac{1}{4}(2m + k)$ and $m^- = \frac{1}{4}(2m - k)$.

The next result is due to Yan et al [271].

Lemma 2.3 For any signed graph $G = (V, E)$ without isolated vertices, $\sum_{v\in V} |s\text{deg}(v)| + 2n_0 \le 2m$.

Proof First, each $|\text{sdeg}(v)| = |\deg^+(v) - \deg^-(v)| \le \deg^+(v) + \deg^-(v)$. Since G has no isolated vertices, $2 \le \deg^+(v) + \deg^-(v)$ when $\text{sdeg}(v) = 0$. Thus,

$$\sum_{v\in V} |\text{s deg }(v)| + 2n_0 \le \sum_{v\in V} (\deg^+(v) + \deg^-(v)| = 2m^+ + 2m^- = 2m. \quad ❑$$

Lemma 2.4 For any connected signed graph $G = (V, E)$, $\sum_{v\in V} |s\deg(v)| + 2 \sum_{s\deg(v)<0} |\text{sdeg}(v)| \le 6m + 4 - 4\alpha - 4n_+ - 4n_0$, where $\alpha = 1$ if $n_+n_- > 0$ and $\alpha = 0$ otherwise.

Proof Consider the subgraph $G' = (V', E')$ of G induced by those edges incident to vertices with non-negative signed degrees. We have,

$$\sum_{\text{sdeg}(v)>0} |\text{sdeg}(v)| \le 2 \text{ (number of positive edges in } G') - \text{(number of negative edges in } G') \le 3m^+ - |E'|.$$

Since G is connected, each component of G' contains at least one vertex of negative signed degree except for the case of $G' = G$.
Therefore, $n_+ + n_0 - 1 + \alpha \le |E'|$. Thus,

$$\sum_{s\deg(v).0} |s\deg(v)| + n_+ + n_0 - 1 + \alpha \le 3m^+ = 3\left(\frac{1}{2}m + \frac{1}{4}\sum_{v\in V} s\deg(v)\right).$$

Hence, $\sum_{v\in V} |s\deg(v)| + 2 \sum_{s\deg(v)<0} |s\deg(v)| \le 6m + 4 - 4\alpha - 4n_+ - 4n_0$. ❑

For any integer k, k copies of v_iv_j means k copies of positive edges $v_iv_j^+$ if $k > 0$, no edges if $k = 0$ and k copies of negative edges $v_iv_j^-$ if $k < 0$. The next result for signed graphs with loops or multiple edges is due to Yan et al. [271].

Theorem 2.10 An integral sequence $[d_i]_1^n$ is the signed degree sequence of a signed if and only if $\sum_{i=1}^{n} d_i$ is even.

Proof The necessity follows from Lemma 2.2.

Sufficiency Let $\sum_{i=1}^{n} d_i$ be even. Then the number of odd terms is even, say $d_i = 2e_i + 1$ for $1 \le i \le 2k$ and $d_i = 2e_i$ for $2k + 1 \le i \le p$. Then $[d_1, d_2, \ldots, d_n]$ is the signed degree sequence of the signed graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{-d_3 = \frac{1}{2}\sum_{i=1}^{n} d_i$ copies of $v_1v_2\} \cup \{d_2 + d_3 - \frac{1}{2}\sum_{i=1}^{n} d_i$ copies of $v_2v_3\} \cup \{d_1 + d_3 - \frac{1}{2}\sum_{i=1}^{n} d_i$ copies of $v_1v_3\} \cup \{d_i$ copies of $v_3v_i : 4 \leq i \leq n\}$. ❑

Various results on signed degrees in signed graphs can be found in [259], [263], [264] and [266].

2.7 Exercises

1. Verify whether or not the following sequences are degree sequences.

   a. $[1, 1, 1, 2, 3, 4, 5, 6, 7]$, b. $[1, 1, 1, 2, 2, 2]$,
   c. $[4, 4, 4, 4, 4, 4]$, d. $[2, 2, 2, 2, 4, 4]$.

2. Show that there is no perfect degree sequence.

3. What conditions on n and k will ensure that k^n is a degree sequence?

4. Give an example of a graph that can not be generated by the Wang-Kleitman algorithm.

5. Draw the five non isomorphic graphs with degree sequence [3, 3, 2, 2, 1, 1].

6. Show that a graph and its complement have the same frequency sequence.

7. Construct a graph with a degree sequence [3, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1] by using Havel-Hakimi algorithm.