14. Score Structure in Digraphs

Landau [145] associated with each tournament an ordered sequence of non-negative integers, its score structure, formed by listing the vertex outdegrees in non-decreasing order. Since then the concept of score structure has been extended to various other classes of digraphs, namely oriented graphs and semicomplete graphs.The score structure property has been used in the study of some structural properties of digraphs.

14.1 Score Sequences in Tournaments

Definition: In a tournament T, the score $s(v_i)$, or simply s_i of a vertex v_i is the number of arcs directed away from v_i and the score sequence $S(T)$ is formed by listing the vertex scores in non-decreasing order. Clearly, $0 \leq s_i \leq \frac{n(n-1)}{2}$. Further, no two scores can be zero and no two scores can be $n-1$. Tournament score sequences have also been called score structures [145], score vectors [165] and score lists [29].

One interpretation of a tournament is as a competition where n participants play each other once in a match that cannot end in a tie and score one point for each win. Player v is represented in the tournament by vertex v and an arc from u to v means that u defeats v. Then player v obtains a total of d_v^+ points in the competition and the vertex scores can be ordered to obtain the score sequence of the tournament. We use $u \to v$ to denote the both, an arc from u to v and the fact that u dominates v. A result of Ryser [227] states that an n-tournament can be obtained from any other having the same score sequence by a sequence of arc reversals of 3-cycles.

Now, we give the characterisation of score sequences of tournaments which is due to Landau [145].This result has attracted quite a bit of attention as nearly a dozen different proofs appear in the literature.Early proofs tested the readers patience with special choices of subscripts,but eventually such gymnastics were replaced by more elegant arguments.Many of the existing proofs are discussed in a survey by Reid [221] and the proof we give here is due to Thomassen [242]. Further, two new proofs can be found in [89].

Theorem 14.1 (Landau) A sequence of non-negative integers $S=[s_i]_1^n$ in non-decreasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n]=\{1, 2, \ldots, n\}$,

$$\sum_{i \in I} s_i \geq \binom{|I|}{2}, \tag{14.1.1}$$

with equality when $|I| = n$.

Because of the monotonicity assumption $s_1 \leq s_2 \leq \ldots \leq s_n$, the inequalities (14.1.1), known as the Landau inequalities, are equivalent to

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2},$$

for $1 \leq k \leq n$, with equality for $k = n$.

Proof

Necessity If a sequence of non-negative integers $[s_i]_1^n$ in the non-decreasing order is the score sequence of an n-tournament T, then the sum of the first k scores in the sequence counts exactly one each arc in the subtournamnent W induced by $\{v_1, v_2, \ldots, v_k\}$ plus each arc from W to $T - W$. Therefore the sum is at least $\frac{k(k-1)}{2}$, the number of arcs in W. Also, since the sum of the scores of the vertices counts each arc of the tournament exactly once, the sum of the scores is the total number of arcs, that is, $\frac{n(n-1)}{2}$.

Sufficiency (Thomassen) Let n be the smallest integer for which there is a non-decreasing sequence S of non-negative integers satisfying Landau's conditions (14.1.2), but for which there is no n-tournament with score sequence S. Among all such S, pick one for which s_1 is as small as possible.

First consider the case where for some $k < n$,

$$\sum_{i=1}^{k} s_i = \binom{k}{2}.$$

By the minimality of n, the sequence $S_1 = [s_1, s_2, \ldots, s_k]$ is the score sequence of some tournament T_1. Further,

$$\sum_{i=1}^{m} (s_{k+i} - k) = \sum_{i=1}^{m+k} s_i - \binom{k}{2} - mk \geq \binom{m+k}{2} - \binom{k}{2} - mk = \binom{m}{2},$$

for each m, $1 \leq m \leq n-k$, with the equality when $m = n-k$. Therefore, by the minimality of n, the sequence $S_2 = [s_{k+1} - k, s_{k+2} - k, \ldots, s_n - k]$ is the score sequence of some tournament T_2. By forming the disjoint union of T_1 and T_2, and adding all arcs from T_2 to T_1, we obtain a tournament with score sequence S.

Now, consider the case where each inequality in 14.1.2 is strict when $k < n$ (in particular $s_1 > 0$). Then the sequence $S_3 = [s_1 - 1, s_2, \ldots, s_{n-1}, s_n + 1]$ satisfies (14.1.2) and by the minimality of s_1, S_3 is the score sequence of some tournament T_3. Let u and v be the vertices with scores s_n+1 and s_1-1 respectively. Since the score of u is larger than that of v, T_3 has a path P from u to v of length ≤ 2. By reversing the arcs of P, we obtain a tournament with score sequence S, a contradiction. ❑

Landau's theorem is the tournament analog of the Erdos-Gallai theorem for graphical sequences. A tournament analog of the Havel-Hakimi theorem for graphical sequences is the following result, the proof of which can be found in Reid and Beineke [218].

Theorem 14.2 A non-decreasing sequence $[s_i]_1^n$ of non-negative integers, $n \geq 2$, is the score sequence of an n-tournament if and only if the new sequence

$$[s_1,\ s_2,\ \ldots,\ s_m,\ s_{m+1}-1,\ \ldots,\ s_{n-1}-1]$$

where $m = s_n$, when arranged in non-decreasing order, is the score sequence of some $(n-1)$-tournament.

Definition: A tournament is *strongly connected* or strong if for every two vertices u and v there is a path from u to v and a path from v to u. A *strong component* of a tournament is a maximal strong subtournament.

The following extension of Theorem 14.1, characterises strong components. The proof is straightforward and consequently omitted.

Theorem 14.3 A non-decreasing sequence $[s_i]_1^n$ of non-negative integers is the score sequence of a strong n-tournament if and only if

$$\sum_{i=1}^{k} s_i > \binom{k}{2},\ 1 \leq k \leq n, \text{ and } \sum_{i=1}^{n} s_i = \binom{n}{2}.$$

We have the following observation from Theorem 14.3. Let $S = [s_i]_1^n$ be a score sequence of an n-tournament T with vertex set $V = \{1,\ 2,\ \ldots,\ n\}$. Let $\sum_{i=1}^{p} s_i = \binom{p}{2}$, $\sum_{i=1}^{q} s_i = \binom{q}{2}$ and $\sum_{i=1}^{k} s_i > \binom{k}{2}$, for $p+1 \leq k \leq q-1$, where $0 \leq p < q \leq n$.

Then the subtournament induced by $\{p+1,\ \ldots,\ q\}$ is a strong component of T with score sequence $[s_{p+1}-p,\ s_{p+2}-p,\ \ldots,\ s_q-p]$.

We say that S is strong if T is strong and the strong components of S are the score sequences of the strong components of T. Theorem 14.3 shows that the strong components of S are determined by the successive values of k for which

$$\sum_{i=1}^{k} s_i = \binom{k}{2},$$

that is, the successive values of k for which equality holds in condition (14.1.2).

For example, consider the score sequence

$$S = [1, 1, 1, 4, 4, 5, 5, 7, 9, 9, 10, 11, 11],$$

$$\sum_{i=1}^{k} s_i = \binom{k}{2} \text{ for } k = 3, 7, 8, \text{ and } 13.$$

Therefore the strong components of S are, in ascending order,

$$[1, 1, 1], [1, 1, 2, 2], [0], \text{ and } [1, 1, 2, 3, 3].$$

The next result due to Brualdi and Shen [40] shows that the score sequence of an n-tournament satisfies inequalities (14.4.1) below, which are individually stronger than the inequalities (14.1.1), although collectively the two sets of inequalities are equivalent.

Theorem 14.4 (Brualdi and Shen) A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for every subset $I \subseteq [n]$,

$$\sum_{i \in I} s_i \geq \frac{1}{2}\sum_{i \in I}(i-1) + \frac{1}{2}\binom{|I|}{2}, \tag{14.4.1}$$

with equality when $|I| = n$.

Proof The sufficiency (14.4.1) imply the inequalities (14.1.1).

We prove that the score sequence S of a tournament satisfies (14.4.1). For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i \in I} s_i - \frac{1}{2}\sum_{i \in I}(i-1) - \frac{1}{2}\binom{|I|}{2}.$$

Choose I firstly to have $f(I)$ minimum and secondly to have $|I|$ minimum.

Claim that $I = \{i : 1 \leq i \leq |I|\}$.

Otherwise, there exists $i \notin I$ and $j \in I$ such that $j = i+1$. Then $s_i \leq s_j$. Since

$$s_j - \frac{1}{2}(j + |I| - 2) = f(I) - f(I - \{j\}) < 0$$

and $s_i - \frac{1}{2}(i + |I| - 1) = f(I \cup \{i\}) - f(I) \geq 0$,

$$\frac{1}{2}(i+|I|-1) \leq s_i \leq s_j < \frac{1}{2}(j+|I|-2) = \frac{1}{2}(i+|I|-1),$$

which is a contradiction. This proves the claim.

Thus, $f(I) = \sum_{i=1}^{|I|} s_i - \frac{1}{2}\sum_{i=1}^{|I|}(i-1) - \frac{1}{2}\binom{|I|}{2} = \sum_{i=1}^{|I|} s_i - \binom{|I|}{2} \geq 0,$

where the inequality follows from 14.1. By the choice of the subset I, Theorem 14.4 follows. ❑

Remark Clearly, the equality can occur often in (14.4.1). For example, equality holds for regular tournaments of odd order n (with score sequence $\left[\frac{n-1}{2}, \ldots, \frac{n-1}{2}\right]$), whenever $|I| = k$, and $I = \{n-k+1, \ldots, n\}$.

Further, Theorem 14.4 is best possible in the sense that, for any real $\in > 0$, the inequality

$$\sum_{i\in I} s_i \geq \left(\frac{1}{2} + \in\right)\sum_{i\in I}(i-1) + \left(\frac{1}{2} - \in\right)\binom{|I|}{2}$$

fails for some I and some tournaments (for example, regular tournaments).

The following set of upper bounds for $\sum_{i\in I} s_i$ is equivalent to the set of lower bounds for $\sum_{i\in I} s_i$ in Theorem 14.4.

Corollary 14.1 A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for any subset $I \subseteq [n]$,

$$\sum_{i\in I} s_i \leq \frac{1}{2}\sum_{i\in I}(i-1) + \frac{1}{4}|I|(2n-|I|-1),$$

with equality when $|I| = n$.

Proof Let $J = [n] - I$. Then,

$$\sum_{i\in I} s_i = \frac{1}{2}\sum_{i\in I}(i-1) + \frac{1}{4}|I|(2n-|I|-1),$$

with equality when $|I| = n$, if and only if

$$\sum_{i\in J} s_i = \binom{n}{2} - \sum_{i\in I} s_i \geq \frac{1}{2}\sum_{i\in J}(i-1) + \frac{1}{2}\binom{|J|}{2},$$

with equality when $|J| = n$. Therefore Corollary 14.1 follows from Theorem 14.4. ❑

Corollary 14.2 If $S=[s_i]_1^n$ is a score sequence of a tournament, then for each i, $\frac{i-1}{2}\le s_i\le \frac{n+i-2}{2}$.

Proof Choose $I=\{i\}$. Then the result follows immediately from Theorem 14.4 and Corollary 14.1. ❑

The next result by Brualdi and Shen [40] shows that when equality occurs in the inequalities (14.4.1), there are implications concerning the strong connectedness and regularity of every tournament with score sequence S. For any integers r and s with $r\le s$, $[r,\ s]$ denotes the set of all integers between r and s.

Theorem 14.5 If $S=[s_i]_1^n$ is a tournament score sequence and if

$$\sum_{i\in I}s_i=\frac{1}{2}\sum_{i\in I}(i-1)+\frac{1}{2}\binom{|I|}{2}$$

for some $I\subseteq[n]$, then one of the following holds.

i. $I=[1,\ |I|]$ and $\sum_{i=1}^{|I|}s_i=\binom{|I|}{2}$.

ii. $I=[t,\ t+|I|-1]$ for some t, $2\le t\le n-|I|+1$,

$$\sum_{i=1}^{t+|I|-1}s_i=\binom{t+|I|-1}{2} \text{ and } s_i=(t+|I|-2)/2, \text{ for all } i\le t+|I|-1.$$

iii. $I=[1,\ r]\cup[r+t,\ t+|I|-1]$, for some r and t such that $1\le r\le |I|-1$ and $2\le t\le n-|I|+1$,

$$\sum_{i=1}^{t+|I|-1}s_i=\binom{t+|I|-1}{2}$$

and $s_i=(r+t+|I|-2)/2$ for all i, $r+1\le i\le t+|I|-1$.

Remark Conditions (i), (ii) and (iii) of Theorem 14.5 are equivalent to the assertion that every tournament with the score sequence S has one of the three structures shown in Figure 14.1. The notation T_r is used to denote a subtournament on r vertices and the double arrows mean that all the arcs between the two parts go in that direction.

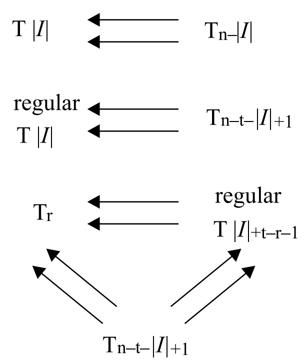


Fig. 14.1

The next result due to Bjelica [26] gives a criterion for score segments and subsequences with arbitrary positions of scores.

Theorem 14.6 If $[t_i]_1^m$ is a sequence of non-negative integers in non-decreasing order and $[s_i]_1^n$ is a score sequence of an n-tournament T with $m < n$, then the following properties are equivalent.

i. $\sum_{i=1}^{j} t_i \geq \binom{j}{2}$, $\quad 1 \leq j \leq m$,
ii. $t_j = s_j$, $\quad 1 \leq j \leq m$, for some T,
iii. $t_j = s_{k+j}$, $\quad 1 \leq j \leq m$, for some T and k,
iv. $t_j = s_{k_j}$, $\quad 1 \leq j \leq m$, for some T and $k_1 < k_2 < \ldots < k_m$.

The following result due to Bjelica and Lakic [27] gives the conditions for a set of integers to be the subset of scores with prescribed positions in some score sequence.

Denote $b(x) = \binom{x}{2}$, $X(k) = \sum_{i=1}^{k} x_i$.

Theorem 14.7 Let $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$ and $0 < k_1 < k_2 < \ldots < k_m$ be two sequences of integers. Then there exists an n-tournament T with score sequence $s_1 \leq s_2 \leq \ldots \leq s_n$ such that $t_j = s_{k_j}$, $1 \leq j \leq m$, if and only if

$$\sum_{i=1}^{j} (k_i - k_{i-1}) t_i \geq \binom{k_j}{2}, \quad 1 \leq j \leq m, \ k_0 = 0. \tag{14.7.1}$$

The size of the tournament can be k_m if and only if in (14.7.1) the equality holds for $j = m$.

Proof

Necessity If, for some tournament, we have $t_j = s_{k_j}$, where $1 \leq j \leq m$, then monotonicity of the score sequence and the Landau theorem give

$$\sum_{i=1}^{j}(k_i-k_{i-1})t_i\sum_{i=1}^{j}(k_i-k_{i-1})S_{k_i}\geq\sum_{i=1}^{k_j}S_i\geq\left(\begin{array}{c}k_j\\2\end{array}\right),\quad 1\leq j\leq m.$$

Sufficiency Let some sequences t and k satisfy (14.7.1). Define the sequence $u_1\leq u_2\leq\ldots\leq u_{k_m}$ which includes the sequence t as subsequence $u_k=t_j$, $k_{j-1}<k<k_j$, $1\leq j\leq m$, and we prove that it satisfies property (i) from Theorem 14.6. In the following minorisation, we apply piecewise linearity of U, inequalities (14.7.1) and convexity of binomial function $bU(k)=U(k_{j-1})+(k-k_{j-1})t_{k_j}=U(k_{j-1})+(k-k_{j-1})\frac{U(k_j)-U(k_{j-1})}{k_j-k_{j-1}}$

$$=\frac{k_j-k}{k_j-k_{j-1}}U(k_{j-1})+\frac{k-k_{j-1}}{k_j-k_{j-1}}U(k_j)$$

$$\geq\frac{k_j-k}{k_j-k_{j-1}}b(k_{j-1})+\frac{k-k_{j-1}}{k_j-k_{j-1}}b(k_j)$$

$$\geq b\left(\frac{k_j-k}{k_j-k_{j-1}}k_{j-1}+\frac{k-k_{j-1}}{k_j-k_{j-1}}k_j\right)=b(k).$$

By property (ii) from Theorem 14.6, there exists an n-tournament T with beginning score segment u. Hence scores from the sequence t appear on the prescribed positions k. ❑

14.2 Frequency Sets in Tournaments

Definition: The number of times that a particular score occurs in a score sequence of a tournament is called the *frequency* of that score. A set of distinct positive integers $F=\{f_1,\ f_2,\ \ldots,\ f_k\}$ is a *frequency set* if there exists a tournament T such that the set of frequencies of the scores in T is exactly F. Note that in such a case T has order at least $f_1+f_2+\ldots+f_k$. For example, the reversal of the orientation of three vertex disjoint arcs in a regular 7-tournament results in a 7-tournament with score sequence [2, 2 , 2, 3, 4, 4, 4] and frequency set F = {1, 3}.

Define $N(F)$ to be the smallest m such that there exists a tournament on m vertices with frequency set F. Clearly, $N(f_1,\ f_2,\ \ldots,\ f_n)\geq\sum_{i=1}^{n}f_i$. An almost regular tournament is an even order tournament in which the scores of the vertices are all as nearly equal as possible.

We have the following observations due to Alspach and Reid [3].

Lemma 14.1 If $f_1<f_2<\ldots<f_n$, $n\geq 2$, are positive integers, f_k is odd, and $N(f_1,\ f_2,\ \ldots,\ f_{k-1},\ f_{k+1},\ \ldots,f_n)=\left(\sum_{i=1}^{n}f_i\right)-f_k$, then

$$N(f_1,\ f_2,\ \ldots,f_n)=\left(\sum_{i=1}^{n}f_i\right).$$

Proof Let R be a tournament on $\left(\sum_{i=1}^{n} f_i\right) - f_k$ vertices with frequency set $\{f_1, f_2, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n\}$. Let Q be a regular tournament on f_k vertices. Let T be the tournament obtained from disjoint copies of Q and R and in which every vertex of Q dominates every vertex of R. It is clear that T has $\left(\sum_{i=1}^{n} f_i\right)$ vertices and frequency set $\{f_1, f_2, \ldots, fn\}$. ❑

Lemma 14.2 Let R be a regular tournament of order r and let $Q_1, Q_2, \ldots, Q_k$ be almost regular tournaments of orders $q_1 < q_2 < \ldots < q_k$. Then there exists a tournament T of order $r + \sum_{i=1}^{k} q_i$ containing disjoint copies of $R, Q_1, Q_2, \ldots, Q_k$ as subtournaments such that $< R \cup Q_i >$ is regular of order $r + q_i$ and each vertex of Q_i dominates each vertex of Q_j when $i > j$.

The following constructive criterion due to Alspach and Reid [3] shows that every set F of positive integers is the frequency set of some tournament and determines the least order $N(F)$ of such a tournament. The proof is omitted as it is lengthy and can be found in [3].

Theorem 14.8 Let $f_1 < f_2 < \ldots < f_n$ be positive integers, at least one of which is odd. Then

$$N(f_1, f_2, \ldots, f_n) = \sum_{i=1}^{n} f_i \tag{14.8.1}$$

unless either

i. $n = 2$, $f_1 \not\equiv f_2$ (mod 2), gcd $\{f_1, f_2\} = 1$, and $f_2 \neq 2$, in which case $N(f_1, f_2) = 2f_1 + f_2$ (14.8.2)

or

ii. $n = 2$, $f_1 = 1$, $f_2 = 2$, in which case
$$N(1, 2) = 5. \tag{14.8.3}$$

Definition: Given the set of even integers $F = \{f_1, f_2, \ldots, f_n\}$, $e(F)$ denotes the largest power of 2 that divides every f_i, i.e., $e(F) = 2^m$ if $2^m / f_i$, $i = 1, 2, \ldots, n$, but $2^{m+1} \times f_j$ for some j.

Lemma 14.3: If T is a tournament with r vertices and frequency set $\{f_1, f_2, \ldots, f_n\}$, then for each $k \geq 1$ there is a tournament with kr vertices and frequency set $\{kf_1, kf_2, \ldots, kf_n\}$.

Proof For each i, $1 \leq i \leq k$, let T_i be a copy of T with vertices $u_{i1}, u_{i2}, \ldots, u_{ir}$. Orient the arcs between T_i and $T_j, i < j$, so that u_{ik} exactly dominates the $r/2$ vertices $u_{j,k+1}$, $u_{j,k+2}, \ldots, u_{j,(k+r)/2}$ where the second subscripts are interpreted modulo r. Every vertex has had its scores increased by $r(k-1)/2$ and since we started with k copies of T, the resulting tournament has kr vertices and frequency set $\{kf_1, kf_2, \ldots, kf_n\}$. ❑

The following result is also due to Alspach and Reid [3].

Theorem 14.9 Let $f_1 < f_2 < \ldots < f_n$ be even positive integers, let $e(f_1, f_2, \ldots, f_n) = 2^m$ and let $f_i = 2^m k_i$, $1 \le i \le n$. Then

i. if an even number of the $k_i's$ is odd, then

$N(f_1, f_2, \ldots, f_n) = \sum_{i=1}^{n} f_i$ and

ii. if an odd number of the $k_i's$ is odd and j is the smallest index for which k_i is odd, then

$N(f_1, f_2, \ldots, f_n) = f_j + \sum_{i=1}^{n} f_i.$

Proof Since 2^m is the highest power of 2 which is a factor of all the $f_i's$, at least one of the $k_i's$ is odd. Hence, in case (i) there are at least two odd $k_i's$ and $n \ge 2$. By Theorem 14.8, there is a tournament T on $\sum_{i=1}^{n} k_i$ vertices with frequency set $\{k_1, k_2, \ldots, k_n\}$. By Lemma 14.3 with $k = 2^m$, there is a tournament on $\sum_{i=1}^{n} f_i$ vertices and frequency set $\{f_1, f_2, \ldots, f_n\}$. This proves (i).

Now, consider Case (ii) with $(k_1, k_2) \ne (1, 2)$. First, we show that $N(f_1, f_2, \ldots, f_n) = \sum_{i=1}^{n} f_i$ is impossible. Let α_i be the score occurring with frequency f_i. Then

$$\sum_{i=1}^{n} \alpha_i f_i = \binom{f_1 + f_2 + \ldots + f_n}{2} = \frac{(f_1 + f_2 + \ldots + f_n)(f_1 + f_2 + \ldots + f_n - 1)}{2}.$$

Clearly, the left hand side is divisible by 2^m, while the right hand side is not. Thus, $N(f_1, f_2, \ldots, f_n) > \sum_{i=1}^{n} f_i$. Also, $N(f_1, f_2, \ldots, f_n) > f_k + \sum_{i=1}^{n} f_i$, for $k < j$, because of the same problem regarding divisibility by 2^m. Thus,

$$N(f_1, f_2, \ldots, f_n) \ge f_j + \sum_{i=1}^{n} f_i.$$

If $n \ge 3$, by Theorem 14.8, there is a tournament on $\sum_{i=1}^{n} k_i$ vertices with frequency set $K = \{k_1, k_2, \ldots, k_n\}$. As k_j is odd, by Lemma 14.3 there is a tournament on $k_j + \sum_{i=1}^{n} k_i$ vertices with frequency set K. If $n = 2$ and k_1 is odd, there is a tournament on $k_1 + k_1 + k_2$ vertices with frequency set K (use Lemma 14.1 if $gcd\{k_1, k_2\} > 1$). If k_1 is even and k_2 is odd, again there is a tournament with $k_1 + k_2 + k_2$ vertices and frequency set K by application of Theorem 14.8 and Lemma 14.1 if $gcd\ \{k_1, k_2\} > 1$, or by Lemma 14.2 if $gcd\ \{k_1, k_2\} = 1$. In case n = 1 and k_1 is odd, then an almost regular tournament of order $2k_1$ has frequency set F. In all of the above cases employ Lemma 14.3 with $k = 2^m$ to obtain the result in (ii).

Finally, let $F = \{2^m,\ 2^{m+1}\}, m \geq 1$. As above, $N(2^m,\ 2^{m+1}) \geq 2^{m+2}$. Any tournament with 2^m scores of $2^{m+1}-2$, 2^m scores of $2^{m+1}+2$ and 2^{m+1} scores of 2^{m+1} shows that $N(2^m,\ 2^{m+1}) = 2^{m+2}$ as required in (ii). ❑

14.3 Score Sets in Tournaments

Definition: The set of distinct scores in a tournament T is called the score set of T.

It is easy to see that every singleton set $\{k\}$, $k \geq 0$, is a score set. Reid [217] showed that a nonempty set S of non-negative integers is a score set whenever $|S|$ = 1, 2, 3 or whenever S is either an arithmetic or geometric progression. Reid conjectured that any nonempty set S of non-negative integers is a score set. Hager [96] proved the conjecture for $|S|$ = 4, 5. This problem proved to be more resistant than that of frequency set discussed above.

If the set $\{x_1,\ x_2,\ \ldots,\ x_k\}$ is the score set of some n-tournament T, then there are multiplicities $m_1, m_2,\ \ldots, m_k$ (positive integers) such that x_i occurs as a score exactly m_i times in T. These m_i are not necessarily distinct, so they are not the frequencies discussed above. Therefore, by Landau's Theorem, $\{x_1,\ x_2,\ \ldots, x_k\}$ is the score set of some n-tournament T if and only if there exist positive integers $m_1,\ m_2,\ \ldots, m_k$ such that

$$\sum_{i=1}^{j} m_i x_i \geq \binom{\sum_{i=1}^{j} m_i}{2}, \text{ for } 1 \leq j \leq k, \text{ with equality for } j = k.$$

Consequently, the connection to tournaments is removed, and Reid's conjecture becomes strictly an arithmetical supposition. Yan [271] proved the conjecture by pure arithmetical analysis.

Pirzada and Naikoo [200] proved by construction that if $s_1,\ s_2,\ \ldots,\ s_p$ are p non-negative integers with $s_1 < s_2 < \ldots < s_p$, then there exists a tournament with score set $S = \left\{s_1,\ \sum_{i=1}^{2} s_i,\ \ldots,\ \sum_{i=1}^{p} s_i\right\}$. More results on score sets in tournaments can be found in [197, 198, 200]. Also, the reconstruction of complete tournament can be seen in [119, 120]. The concept of scores in hyper tournament can be found in Zhou et.at [272]. The literature on kings in tournaments can be found in [130, 131, 132, 133, 155, 182, 184, 219, 220].

14.4 Lexicographic Enumeration and Tournament Construction

Definition: Let $[s_i]_1^n$ be any sequence of integers. The (transitive) deviation sequence of $[s_i]_1^n$ is defined to be the sequence $[d(i)] = [s_i - i + 1]$ and $d(i)$ is called the *deviation* of s_i. It is easy to see that $[s_i]_1^n$ is non-decreasing if and only if $d(i) - d(i+1) \leq 1$ for each $i < n$. Also, for each $k = 1,\ 2,\ \ldots, n$, $\sum_{i=1}^{k} d(i) = \sum_{i=1}^{k} s_i - \binom{k}{2}$. From this, it follows that a sequence

$[s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence if and only if its deviation sequence $[d(i)]_1^n$ satisfies $\sum_{i=1}^{k} d(i) \geq 0$ for $k = 1, 2, \ldots, n$ with equality for $k = n$.

Let $[s_i]_1^n$ and $[s_i']_1^n$ be score sequences of length n. Then, say that $[s_i]_1^n$ precedes $[s_i']_1^n$ if there exists a positive integer $k \leq n$ such that $s_i = s_i'$ for each $1 \leq i \leq k$ and $s_k < s_k'$. They are equal if equality holds for all i. In symbols, $[s_i] \leq [s_i']$ means $[s_i]$ proceeds $[s_i']$. Further, $[s_i']$ is the successor of $[s_i]$ if they are distinct; $[s_i] \leq [s_i']$ and $[s_i'] \leq [s_i'']$ whenever $[s_i] \leq [s_i'']$. An enumeration of all score sequences of a given length with the property that the successor of any score sequence follows it immediately in the list is called a *exicographic enumeration*. Clearly, $[0, 1, 2, \ldots, n-1]$ is not the successor of any score sequence of length n and thus it is always the first in the lexicographic enumeration. Also, $[s_i]$ has no successor if and only if $s_n - s_1 \leq 1$.

If we know the first sequence in a lexicographic enumeration, then we can complete the work provided we know how to get the successor of any given sequence. The following algorithm due to Gervacio [85] gives the successor $[s_i']$ of $[s_i]$, if it exists.

Algorithm

1. Determine the maximum k such that $s_n - s_k \geq 2$.
2. Let $s_i' = s_i$ for all $i < k$.
3. Let $s_k' = s_k + 1$.
4. Let $s_j' = s_k + 1$ until $\sum_{i=1}^{j} s_i' < \binom{j}{2}$.
5. Let t be the minimum j such that $\sum_{i=1}^{j} s_i' < \binom{j}{2}$, set $s_t' = \binom{t}{2} - \sum_{i=1}^{t-1} s_i'$.
6. Let $s_i' = i - 1$ for all i, $t < i \leq n$.

Constructing a tournament: One method of constructing a tournament with a given score sequence can be found in [16]. The method we give here is due to Gervacio [85]. First, we have the following observations.

Lemma 14.4 Let $[s_i]_1^n$ be a score sequence with deviation sequence $[d(i)]$.

a. If max $\{d(i)\} = M > 0$, then for each $1 \leq k \leq M$, there exists a vertex u such that $d(u) = k$.

b. If min $\{d(i)\} = m < 0$, then for each $-1 \geq k \geq m$, there exists a vertex u such that $d(u) = k$.

Proof

a. Clearly, the result is true for $k = M$. Now, we show that it is true for all $1 \leq k \leq M$ by induction on k. Assume the result to be true for $k+1$, i. e., there exists a vertex u such that $d(u) = k+1$. Since $d(n) \leq 0$, there exists $t > u$ such that $d(u) = d(t) > d(t+1)$. Since $d(t) \leq d(t+1)+1, d(t+1) = d(u)-1 = k$. Hence (a) holds.

b. This can be proved by using the argument as in (a).

Lemma 14.5 Let $[s_i]_1^n$ be a score sequence with deviation sequence $[d(i)]$. If c is the number of negative terms, then $c \geq max\{d(i)\}$.

Proof Let p = max $\{d(i)\}$. If p = 0, then c = 0 and the result holds. For $p > 0$, we have the following cases.

Case 1 There exists a vertex k such that $d(k) < 0$ and $|d(k)| \geq p$. By Lemma 14.4, $c \geq |d(k)| \geq p$.

Case 2 For each non-negative deviation $d(k)$, $|d(k)| < p$.

Let $q = \max\{|d(i)| : d(i) < 0\}$ and let $c < p$. Then using Lemma 14.4, $\sum\limits_{d(i)<0} |d(i)| \leq 1+2+\ldots+q$ $+(p-q)q$. But $\sum\limits_{d(i)<0} |d(i)| = \sum\limits_{d(i)>0} |d(i)|$, and by Lemma 14.4, $\sum\limits_{d(i)>0} |d(i)| \geq 1+2+\ldots+p$. Hence, $\dfrac{q(q+1)}{2+(p-q)q} > \dfrac{p(p+1)}{2}$.

This gives the quadratic inequality $p^2-(2q-1)+q(q-1) < 0$ and this implies that $q-1 < p < q$, which is absurd, since p and q are integers. Hence, $c \geq p$. ❑

Now, we describe and validate the above algorithm.

Construction algorithm

Let $[s_i]_1^n$ be a score sequence with deviation sequence $[d(i)]_1^n$. First take n vertices arranged horizontally and labelled $1, 2, \ldots, n$ from left to right.

Step 1 Subdivide $[d(i)]$ into maximal non-increasing segments and denote by p the number of segments in the subdivision. Let n_i be the number of negative deviations in the ith segment, counting from left to right.

Step 2 Let j be the last integer such that $d(j) > 0$. If no such j exists, go to step 6. Else, determine the least integer q such that $\sum\limits_{i \leq q} n_i \geq d(j)$. For each i in the segments to the left of the qth segment such that $d(i) < 0$, let $d'(i) = d(i)+1$ and draw the arc ji.

Step 3 Let $\sigma = d(j) - \sum\limits_{i<q} n_i$ and choose a smallest (negatively largest) deviations $d(i)$ in the qth segment. For each such $d(i)$, let $d'(i) = d(i)+1$ and draw the arc ji. Let $d'(j) = 0$.

Step 4 For all other deviations $d(i)$ not changed in the preceding steps, let $d'(i) = d(i)$.

Step 5 If $[d'(i)] \neq [0]$, go to step 1 using $[d'(i)]$ in place of $[d(i)]$.

Step 6 Whenever $u < v$ and there is no arc between u and v, draw the arc vu.

Step 7 The resulting digraph is a tournament with score sequence $[s_i]_1^n$.

Now, we analyse the algorithm to verify its validity. Clearly, step 1 can always be carried out. Step 2 can be done in view of Lemma 14.5. Step 3 can be implemented because of step 2. Obviously, step 4 can be done, and after this step, $[d'(i)]$ satisfies $d'(i) - d'(i+1) \leq 1$ for all $1 \leq i \leq n$. Let D be the digraph formed when $[d'(i)]$ = [0]. Then for each vertex i in D,

$$d_i^+(D) = \begin{cases} d(i), & if\ d(i) \geq 0 \\ 0, & if\, d(i) < 0 \end{cases}$$

and $$d_i^-(D) = \begin{cases} d(i), & if\ d(i) \geq 0 \\ -d(i), & if\, d(i) < 0 \end{cases}$$

Let T be the tournament formed after step 6 and let i be any vertex of T. If $d(i) \geq 0$, then $s_i = d_i^+ = d_i^+(D) + i - 1 = s_i$. If $d(i) < 0$, then $d_i^- = d_i^-(D) + n - i = -d(i) + n - i$, and thus $s_i = d_i^+ = (n-1) - d_i^- = d(i) + i - 1 = s_i$.

Example Let $[s_i] = [1, 1, 2, 2]$. Then $[d(i)] = [1, 0, 0, -1]$.

The resulting digraph after using above algorithm upto $[d'(i)]$ = [0] is shown in Figure 14.2. To get the tournament, add all arcs ij $(i > j)$.

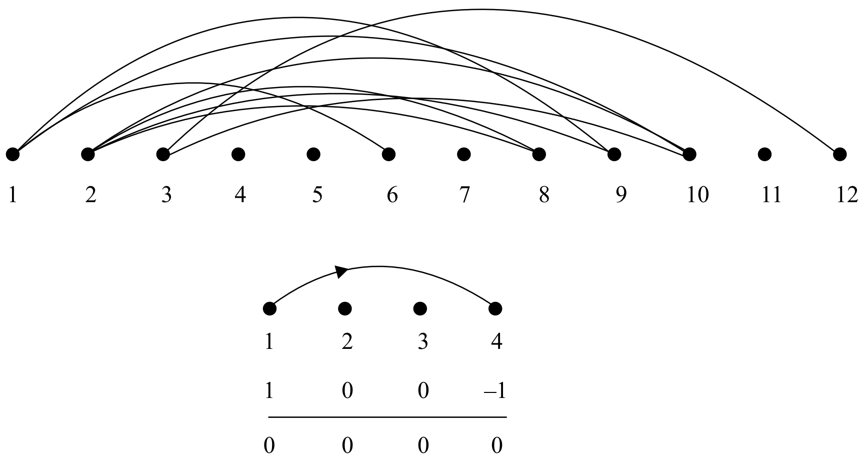


Fig. 14.2

14.5 Simple Score Sequences in Tournaments

Definition: A score sequence is simple (uniquely realisable) if it belongs to exactly one tournament. Every score sequence of tournaments with fewer than five vertices is simple, but the score sequence [1, 2, 3, 3, 3] is not simple, since the tournaments in Figure 14.3 are not isomorphic.

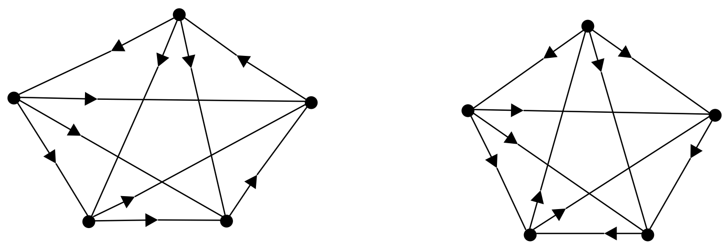


Fig. 14.3

We have the following observations.

Lemma 14.6 A score sequence S is simple if and only if every strong component of S is simple.

The following result due to Avery [8] gives a condition for determining simple score sequence in tournaments.

Theorem 14.10 (Avery) A strong score sequence is simple if it is one of [0], [1, 1, 1], [1, 1, 2, 2] or [2, 2, 2, 2, 2].

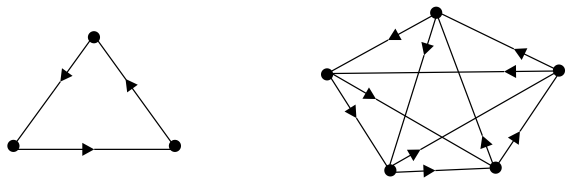


Fig. 14.4

Corollary 14.3 The score sequence S is simple if and only if every strong component of S is one of [0], [1, 1, 1] , [1, 1, 2, 2] or [2, 2, 2, 2, 2].

Hence it is possible to decide whether a given score sequence S is simple by using Theorem 14.10 to determine the strong components of S and then applying Corollary 14.3.

Let $s(n)$ denote the number of simple score sequences of order n. It is easy to show that $s(n)$ satisfies the following recurrence relation, which can be used to evaluate $s(n)$.

Theorem 14.11 $s(n) = s(n-1)+s(n-3)+s(n-4)+s(n-5)$, where $s(k) = 0$ if $k < 0$, and $s(0) = 1$.

14.6 Score Sequences of Self-Converse Tournaments

We know, the converse of an n-tournament T_n is the tournament T_n' obtained by reversing the orientation of all the arcs in T_n. A tournament is called *self-converse* if $T_n \cong T_n'$. The transitive tournaments are examples of self-converse tournaments.

The following characterisation of score sequences of self-converse tournaments is due to Eplett [71].

Theorem 14.12 (Eplett) A score sequence $[s_i]_1^n$ is the score sequence of a self-converse tournament if and only if

$$s_i + s_{n+1-i} = n-1, \tag{14.12.1}$$

for $1 \le i \le n$.

14.7 Score Sequences of Bipartite Tournaments

A *bipartite tournament* T is an orientation of a complete bipartite graph. Clearly, the vertex set of T is the union of two disjoint nonempty sets X and Y, and arc set of T comprises exactly one of the pairs (x, y) or (y, x) for each $x \in X$ and each $y \in Y$. If the orders of X and Y are m and n respectively, T is said to be an $m \times n$ bipartite tournament.

A bipartite tournament may be used to represent competition between two teams and each player competes against everyone on the opposing team. The *score* s_v of vertex v is the number of vertices it dominates and for a bipartite tournament there is a pair of score sequences, one sequence for each set. For example, the bipartite tournament in Figure 14.5 has sequence [4, 3, 2, 0] and [2, 2, 2, 1].

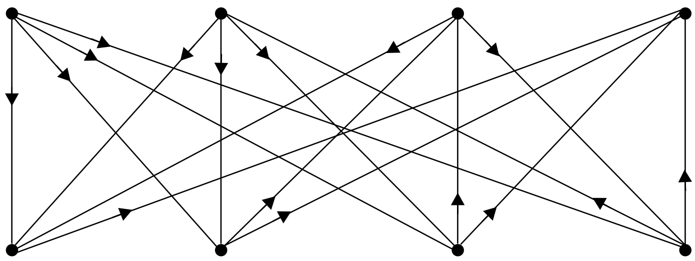


Fig. 14.5

Definition A bipartite tournament is *reducible* if there is a nonempty proper subset of its vertex set to which there are no arcs from the other vertices, otherwise irreducible. A *component* is a maximal irreducible sub-bipartite tournament. A *non-trival component* contains at least two vertices one from each partite set.

A bipartite tournament is consistent if it contains no directed cycles. It can be easily seen that a bipartite tournament is consistent if and only if, for v and w in the same partite set, v dominates every vertex which w dominates if its score is at least that of w.

The *converse* of a bipartite tournament is obtained by reversing the direction of all its arcs and a bipartite tournament is *self-converse* if it is isomorphic to its converse.

Now, assume that the partite sets of bipartite tournaments have a fixed ordering with X first and Y second. Then a given bipartite tournament T has associated with it two bipartite graphs (on the same sets of vertices) in a natural way, one graph containing those edges corresponding to the arcs directed from X to Y, the other from Y to X . For example, two graphs of the bipartite tournament T are shown in Figure 14.6. The two graphs are relative complements as bipartite graphs.

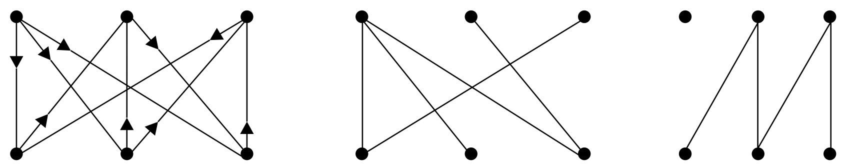


Fig. 14.6

Clearly, the pairs of score sequences of bipartite tournaments and pairs of degree sequences of bipartite graphs are equivalent.

Lemma 14.7 Let $A = [a_1, \ldots, a_m]$ and $B = [b_1, \ldots, b_n]$ be sequences of integers and let $\overline{A} = [n-a_1, \ldots, n-a_m]$ and $\overline{B} = [m-b_1, \ldots, m-b_n]$. Then the following are equivalent.

1. A and B are the score sequences of a bipartite tournament.
2. $\overline{A}$ and $\overline{B}$ are the score sequences of a bipartite tournament.
3. A and $\overline{B}$ are the score sequences of a bipartite graph.
4. $\overline{A}$ and B are the score sequences of a bipartite graph.

The following observation can be easily established.

Lemma 14.8 If v and v' are vertices in the same partite set of a bipartite tournament T, if $s_v \leq s'_v$, and if there is a vertex w which is dominated by v and which dominates v', then there is another vertex w' which is dominated by v' and which dominates v, that is, $v \to w \to v' \to w' \to v$ is a 4-cycle.

The following result is due to Gale [84].

Theorem 14.13 (Gale) If $A = [a_1, \ldots, a_m]$ and $B = [b_1, \ldots, b_n]$ are sequences of non-negative integers in non−decreasing order, then A and B are the score sequences of some bipartite tournament if and only if the sequences $A' = [a_1, \ldots, a_{m-1}]$ and $B' = [b_1, \ldots, b_{a_m}, b_{a_m+1} - 1, \ldots, b_n - 1]$ are.

Proof First assume that A' and B' are the score sequences of a bipartite tournament T'. To the first partite set of T', add a new vertex v with arcs directed from it to vertices (in the second set) with scores $b_1, \ldots, b_{a_m}$, and to it from the others. The result is a bipartite tournament with score sequences A and B.

For the converse, it is sufficient to show that if A and B are the score sequences of a bipartite tournament, then in one realisation, a vertex (in the first set) of score a_m dominates vertices of scores $b_1, \ldots, b_{a_m}$. Among the bipartite tournament realisations of A and B, let T be the one in which a vertex x of score a_m is such that the sum S of the scores of the vertices it dominates is as small as possible. Let $S > \sum_{j=1}^{a_m} b_j$. Then there exist vertices y and y' such that $x \to y'$, $y \to x$ and $s_y < s'_y$. By Lemma 14.8, T has a 4-cycle $x \to y' \to x' \to y \to x$, and if its arcs are reversed, the result is a bipartite tournament with the same sequences, but in which score sum of the vertices dominated by x is less than before. Since the sum was assumed to be minimised, the result follows. ❑

Theorem 14.3 gives a natural construction for a canonical tournament $T^*(A,B)$ from a given pair of score sequences A and B. The only point which needs clarification is getting B' into non-decreasing order, i.e., we must specify dominance when a vertex v_i must dominate some but not all vertices y with a particular score. This is done by forming B' as follows. Let h and k denote the smallest and largest integers j for which $b_j = b_{a_m}$. Let $A' = [a_1, \ldots, a_{m-1}]$, and $B' = [b'_1, \ldots, b'_n]$ with

$$b'_j = \begin{cases} b_j, & for\ 1 \le j < h \ \ and \ \ h + k - a_m \le j \le k, \\ b_j - 1, & otherwise \end{cases}$$

This reduction and the resulting construction is illustrated by the following example, starting with sequences A = [1, 1, 3, 5, 5] and B = [1, 1, 2, 3, 4, 4] .

$A = [1, 1, 3, 5, 5]$ $\qquad$ $B = [1, 1, 2, 3, 4, 4]$

$A_1 = [1, 1, 3, 5]$ $\qquad$ $B_1 = [1, 1, 2, 3, 3, 4]$

(x_5 dominates y_1, y_2, y_3, y_4, y_6)

$A_2 = [1, 1, 3]$ $\qquad$ $B_2 = [1, 1, 2, 3, 3, 3]$

(x_4 dominates y_1, y_2, y_3, y_4, y_5)

$A_3 = [1, 1]$ $\qquad$ $B_3 = [1, 1, 2, 2, 2, 2]$

(x_3 dominates y_1, y_2, y_3)

$A_4 = [1]$ $\qquad B_4 = [0, 1, 1, 1, 1, 1]$

(x_2 dominates y_2)

$A_5 = \varphi$ $\qquad B_5 = [0, 0, 0, 0, 0, 0]$

(x_1 dominates y_1)

Figure 14.7 shows the X to Y arcs resulting from this construction.

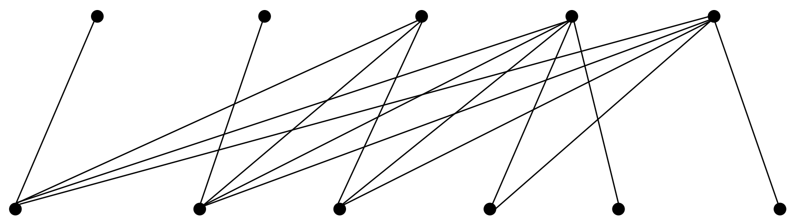


Fig. 14.7

A canonical tournament $T^*(A,B)$ has the following special property. In the subtournament $T^*_{r,\,n}$ induced by $\{x_1, \ldots, x_r\}$ and Y, if $x^*_r \to y^*_i$ and $y^*_j \to x^*_r$, then $s_{y^*_i} \le s_{y^*_j}$, that is, $b_i \le b_j$.

The next result is due to Beineke and Moon [20].

Theorem 14.14 If two bipartite tournaments have the same score sequences, then each can be transformed into the other by successively reversing the arcs of 4-cycles.

It can be noted that Theorem 14.14 does not imply that all bipartite tournaments with given score sequences have the same number of 4-cycles (they need not), although the corresponding statement does hold for 3-cycles in tournaments.

The next result first established by Moon [162] and then in the present form by Beineke and Moon [20] gives a simple criterion for determining whether a pair of sequences are realisable as scores.

Theorem 14.15 (Moon) A pair of sequences A and B of non-negative integers in non-decreasing order are the score sequences of some bipartite tournament if and only if

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \ge kl$$

for $1 \le k \le m$ and $1 \le l \le n$, with equality when $k = m$ and $l = n$.

Further more, the bipartite tournament is irreducible if and only if the inequality is strict except when $k = m$, and $l = n$.

Proof In any bipartite tournament T, the combined scores of any collection of k vertices from the first set and l from the second must be at least kl, so that the inequalities certainly hold. Further, if T irreducible, the inequality is strict unless $k = m$ and $l = n$.

Sufficiency If A and B satisfy the inequalities, we show that A' and B' satisfy the inequalities reordered as in construction of 14.13. It is easily seen that A' and B' are then in non-decreasing order, and further, their combined sum is

$$\sum_{1}^{m-1} a'_i + \sum_{1}^{h} b'_j = mn - a_m - (n - a_m) = (m-1)n.$$

For a fixed value of $k(1 \leq k \leq m-1)$, assume there is a value of l for which the inequality does not hold. And let h denote the least such that

$$\sum_{1}^{k} a'_i + \sum_{1}^{h} b'_j < kh \, ..$$

It follows from the minimality of h that $b'_h < k$, whence $b_h \leq k$. Now, let p and q be the least and greatest values of j for which $b_j = b_{a_m}$ and set r = max (h, q). Since the first $p-1$ values of $b_j$were unchanged, we have $h \geq p$ and thus $b_h = \ldots = b_r$. Finally, let s denote the number of $j \leq h$ such that $b'_j = b_{j-1}$. If $h \leq q$, then $s \leq q - a_m$, and if $h > q$, then $s = (h-q) + (q - a_m) = h - a_m$. In either case, $a_m + s \leq r$. Therefore,

$$\begin{aligned}\sum_{1}^{k+1} a_i + \sum_{1}^{r} b_j = \sum_{1}^{k} a'_i + \sum_{1}^{h} b'_j + \sum_{h+1}^{r} b_j + a_{k+1} + s &< kh + (r-h)b_h + a_m + s \\ \leq kh + (r-h)k + r < (k+1)r,&\end{aligned}$$

which is a contradiction. Therefore A' and B' satisfy the inequalities, as required.

It is easily seen that if the strict inequalities hold for A and B, no realisation can be reducible, completing the proof. ❑

The next criterion derived by Ryser [227] in the context of (0, 1)-matrices with prescribed row and column sums is equivalent to prescribed degrees in bipartite graphs.

Theorem 14.16 If A and B are sequences of non-negative integers with A in non-increasing order, then A and B are the score sequences of some bipartite tournament if and only if

$$\sum_{1}^{k} a_i \leq \sum_{1}^{n} \min(k, m - b_j)$$

for $1 \leq k \leq m$, with equality when $k = m$. Further, the bipartite tournament is irreducible if and only if the inequality, is strict for all $k < m$ and $0 < b_j < m$ for all $j \leq n$.

Remarks In general, one need not check the inequalities for all values of h and k, but only for those for which the next value in the sequence is different. Thus in order to show that [5, 5, 5, 3, 3, 2, 2] and [6, 5, 4, 1, 1, 0] belong to a bipartite tournament, we need to check the inequalities in Theorem 14.16 for k = 3, 5, and 7 only.

Theorem 14.17 If $A = [a_i]_1^m$ and $B = [b_j]_1^n$ are non-decreasing integer sequences with $0 \le a_i \le n$ and $0 \le b_j \le m$, and are such that

a. $A_m + B_n = mn$ and

b. $A_r + B_s \ge rs$, whenever $a_r < a_{r+1}$ and $b_s < b_{s+1}$.

Then (A, B) is realisable.

(Note that for a sequence $L = [x_i]_1^p, L_q = \sum_{i=1}^{q} x_i$ for $1 \le q \le p$ and $L_0 = 0$).

Proof We show that the inequality

$$A_k + B_x \ge kx \tag{4.17.1}$$

holds for all $1 \le k \le m$ and $1 \le x \le n$. If this is not the case for some k and x, let q and s be the smallest, and r and t be the largest indices such that $a_{q+1} = a_k = a_r$ and $b_{s+1} = b_x = b_t (q, s = 0)$. Now, $A_r + B_x < kx$. Claim that at least one of $A_k + B_s < ks$ and $A_k + B_t < kt$ holds. For otherwise, $(x-s)b_x < k(x-s)$ and $(t-x)b_x > k(t-x)$ which is impossible. Thus assume $(i)\ A_k + B_s < ks$.

Now, by hypothesis, (ii) $A_q + B_s \ge qs$ and (iii) $A_r + B_s \ge rs$ (observe that if $r = m$, then $A_m + B_n = mn$ and $0 \le b_j \le m$ together imply (iii). Then (i) and (ii) give $(k-q)a_k < (k-q)s$, while (i) and (iii) give $(r-k)a_k > (r-k)s$. These again lead to a contradiction. The case $A_k + B_t < kt$ can similarly be treated. ❑

The following result can be obtained from Theorem 14.16.

Corollary 14.4 If $C = [c_1, \ldots, c_m]$ and $D = [d_1, \ldots, d_n]$ be two non-increasing sequences having equal sum, then the following are equivalent.

i. $\sum_1^k c_i \le \sum_1^n \min(k, d_j)$, for $k = 1, \ldots, m$.

ii. $\sum_1^x d_j \le \sum_1^m \min(x, c_i)$, for $x = 1, \ldots, n$.

Now, different bipartite tournaments can have the same score sequences and they can differ only within irreducible components. That is, they must have the same numbers of components the same numbers of vertices within components, the same scores within components, and the same dominance between components. The general procedure here is to find a dominating component (a component is called dominating if it has no incoming arcs), delete its vertices and repeat. While an ordinary tournament has precisely one dominating component, the situation in the bipartite case is slightly different. It is described in the following result [19] and is a direct consequence of Theorem 14.15.

Theorem 14.18 Let $A = [a_1, \ldots, a_m]$ and $B = [b_1, \ldots, b_n]$ be score sequences (in non-decreasing order) of a reducible $m \times n$ bipartite tournament.

i. If $a_m = n$ or $b_n = m$, then there is a corresponding trivial dominating component, consisting of one vertex which dominates all the vertices in the other partite set.

ii. Otherwise, if k and x are the largest indices with $k < m$ and $x < n$ such that $\sum_{i=1}^{k} a_i + \sum_{j=1}^{x} b_j < kx$, then the non-trivial dominating component consists of all the vertices in the two partite sets with scores exceeding a_k and b_k respectively.

The following results can be found in [11].

Theorem 14.19 If $A = [a_i]_1^m$ and $B = [b_j]_1^n$ are realisable pair of score sequences with $0 < a_i < n$ and $0 < b_j < m$ and if $|a_i - a_k| \leq 1$ for any i, $k = 1, 2, \ldots, m$, then any bipartite tournament with score sequences A and B is irreducible.

Proof Assume T is a reducible bipartite tournament on partite sets X and Y with score sequences A and B respectively. Since $0 < a_i < n$ and $0 < b_j < m$, T has at least two non trivial components, say C and C', with C being the dominating one. If $x_i \in C \cap X$ and $x_k \in C' \cap X$, then x_i dominates all the vertices in Y dominated by x_k. Also, there exists $y_j \in C \cap Y$ and $y_l \in C' \cap Y$ such that $x_i \to y_j \to x_k$ and $x_i \to y_l \to x_k$. Thus, a_i = score $(x_i) \geq$ score (x_k) + 2 = a_k + 2. This contradicts the hypothesis, and the result follows. ❑

Theorem 14.20 Let $A = [a_i]_1^m$ (in non-decreasing order) and $B = [b^n]$ be sequences such that (A, B) is realisable. Let a_k, a_x be two entries in A with $a_k > 0$ and $a_x < n$. Define a new sequence $A' = [a_i']_1^m$ as follows.

$$a_k' = a_k - 1,\ a_x' = a_x + 1$$

and $a_i' = a_i$, for $i \neq k, x$. Then $(A',\ B)$ is realisable.

Proof: It follows immediately from Theorem 14.17. ❑

Theorem 14.21 If (A, B) is irreducible, i. e., $(A,\ B)$ is realisable and all its realisation are irreducible, and if A' is obtained from A by adding 1 to some entry and B' is obtained from B by subtracting 1 from some entry, then (A', B') is realisable.

14.8 Uniquely Realisable (Simple) Pairs of Score Sequences

If $(A,\ B)$ is realisable, let T denote a realisation on partite sets X and Y. If a pair (A, B) is realisable and all its realisations are isomorphic, then $(A,\ B)$ is called uniquely realisable.

The following observations can be found in [11].

Lemma 14.9 Let $(A,\ B)$ be uniquely realisable. For any entry a_i in A and b_j in B, let X_i and Y_j be the subsets of X and Y consisting of vertices of scores a_i and b_j respectively. Then any cycle in T contains the same number of arcs from X_i to Y_j as from Y_j to X_i.

Proof If this were not the case for some cycle Z, then reversal of the arcs of Z would produce an (A,B) realisation non-isomorphic to T.

Lemma 14.10 If (A,B) is irreducible and uniquely realisable, then A or B is constant.

Proof If neither A nor B is constant, let X_1 be the set of vertices of minimum score in X and let $X_2 = X - X_1$, and similarly define Y_1 and Y_2. Since T is irreducible, every arc is contained in a cycle. Thus by Lemma 14.9, none of the four subtournaments $T(X_i,\ Y_i)$, $i,\ j = 1,\ 2$ is unanimous, i. e., has all its arcs directed from one partite set to the other.

Choose a vertex x_1 in X_1 of minimum score in $T(X_1, Y_1)$. Then x_1 dominates some $y_2 \in Y_2$. Consider two cases depending on whether or not y_2 dominates some vertex in X_2.

Case (i) Every vertex in X_2 dominates y_2. Let uv be an arc from Y_1 to X_2. Since score $(y_2) >$ score(u) in T, there exists an $x \in X_1$ such that $y_2 \to x \to u$ (Fig. 14.8).

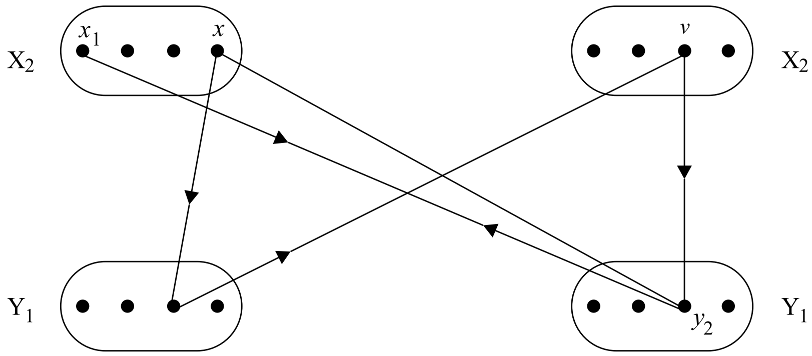


Fig. 14.8

But then $x \to u \to v \to y_2 \to x$ is a cycle which violates Lemma 14.9.

Case (ii) Some vertex $x_2 \in X_2$ is dominated by y_2. By the choice of x_1, there exists $y_1 \in Y_1$ such that $y_1 \to x_1$. If $x_2 \to y_1$, we again get a 4-cycle which has one arc from Y_1 to X_1, but no arcs in the other direction. So assume $y_1 \to x_2$ (Fig. 14.9). Since score $(y_2) >$ score (y_1) in T, there exists an $x \in X_1$ with $y_2 \to x \to y_1$. If $x \in X_2$, we again get a forbidden cycle $x_1 \to y_2 \to x \to y_1 \to x_1$. Thus $x \in X_1$. Likewise, there exists $y \in Y_1$ with $x_2 \to y \to x$. But then $x_1 \to y_2 \to x_2 \to y \to x \to y_1 \to x_1$ is a 6-cycle which violates Lemma 14.9.

Since all the possibilities have been exhausted, the result follows. ❑

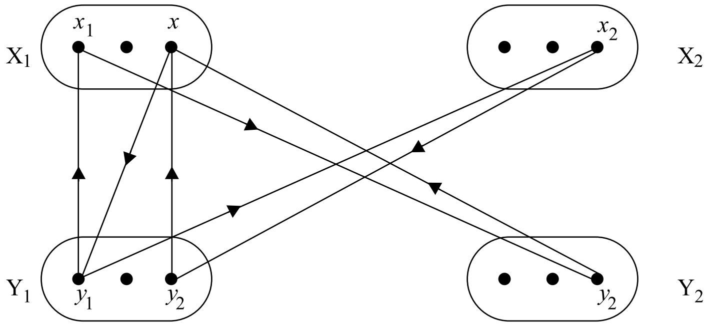


Fig. 14.9

Remarks We note that if in a realisable pair (A, B), one of the sequences has all entries as 1's (the sequence is constantly 1), then (A,B) is uniquely realisable. This is illustrated in Figure 14.10, where $A = [1,1, \ldots, 1]$ and $B = [b_1, b_2, \ldots, b_n]$ and only X to Y arcs are shown.

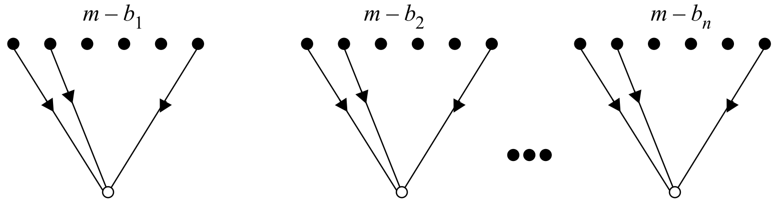


Fig. 14.10

Now onwards assume that none of the sequences A, B, $\overline{A}$ and $\overline{B}$ is constantly 1. Hence the following observations [232] can be easily proved.

Lemma 14.11 If (A, B) is irreducible and uniquely realisable, then A or B is non-constant.

Lemma 14.12 With A and B as above, the sequence B has precisely two distinct values.

Lemma 14.13 If (A,B) is irreducible, uniquely realisable and if $A = [a^m]$, $2 \leq a \leq n-2$, and $B = [b^r,\ c^s]$, $1 \leq b < c \leq m-1$, and $r,s > 0, r+s = n$, then $r = 1$ or $s = 1$.

Remarks It has been shown that if (A,B) is irreducible, uniquely realisable, then one of the sequences (or its dual) consists entirely of 1's, or one of the sequences is constant and the other has exactly two distinct values, one of which appears precisely once.

The following result due to Bagga and Beineke [11] gives necessary and sufficient conditions for unique realisability in the irreducible case.

Theorem 14.22 (Bagga and Beineke) An irreducible pair (A,B) of score sequences is uniquely realisable if and only if one of the following holds.

I. (without loss of generality) $A = [1^m]$ and B is arbitrary,

I′. the dual of (I), that is, $A = [(n-1)^m]$ and B is arbitrary,

II. (without loss of generality) $A = [1^{m-1},\ a]$ and $B = [b^n]$,

II′. the dual of II,

III. (without loss of generality) $A = [1,\ a^{m-1}]$ and $B = [2^n]$,

III′. the dual of III.

Proof The sufficiency of (I) has already been noted in the remarks before Lemma 14.11 where Figure 14.10 shows the unique realisation. Now, let T be a realisation of $A = [1^{m-1}, a]$ and $B = [b^n]$ on partite sets $X = \{x_1,\ x_2,\ \ldots,\ x^m\}$ and $Y = \{y_1,\ y_2,\ \ldots,\ y_n\}$ respectively. If (say) x_m has score a and it dominates (say) $y_1, y_2, \ldots, y_n$, then $T - x_m$ has score sequences $A_1 = [1^{m-1}]$ and $B_1 = [(b-1)^{n-a},\ b^a]$. Thus by (I), $(A_1,\ B_1)$ is uniquely realisable. The unique realisability of (A, B) follows. This proves the sufficiency of (II). The proof of (III) is similar and the dual cases follow by the remarks before Lemma 14.11.

For proving necessity, induct on $m+n$. Since (A, B) is irreducible, so $m, n \geq 2$. If (say) m = 2, then $B = [1^n]$, and the result follows. Now, assume the result holds for all irreducible and uniquely realisable pairs of score sequences with combined length less than $m+n$, and consider such a pair (A, B) with $|A| = m$ and $|B| = n (m, n \geq 3)$.

Assume A and B are not of the type (I) or (I'). Then by the remarks after Lemma 14.13, we have, without loss of generality, $A = [a^m]$ and $B = [b^{n-1}, c]$, with $1 < a < n-1, 1 \leq b, c \leq m-1$ and $b \neq c$.

If y is the vertex of score c in a realisation T of $(A,\ B)$, then $T - y$ has score sequences $A_1 = [(a-1)^{m-c}, a^c]$ and $B_1 = [b^{n-1}]$. Now, the unique realisability of $(A,\ B)$ implies that of (A_1, B_1). Also, by Theorem 14.19, $(A_1,\ B_1)$ is irreducible. Thus, by the induction hypothesis, A_1 and B_1 belong to one of the six given types. Consider these cases one by one.

i. If $B_1 = [1^{n-1}]$, then $B = [1^{n-1}, c]$, so that A and B are of type (II).

ii. If $B_1 = [(m-1)^{n-1}]$, then A and B belong to (II').

iii. If $B_1 = [b^{n-1}]$ and $A_1 = [1^{m-1},\ a]$, then $c = 1$ and $a = 2$, so that $A = [2^m]$ and $B = [1, b^{n-1}]$. This is of type (III).

iv. If $B_1 = [b^{n-1}]$ and $A_1 = [d, (n-2)^{m-1}]$, we get A and B of type (III').

v. If $B_1 = [2^{n-1}]$ and $A_1 = [1,\ a^{m-1}]$, then $b = 2,\ a = 2$ and $c = m-1$. Thus, $A = [2^m]$ and $B = [2^{n-1},\ m-1]$. Using Moon's theorem, we get $2m + 2(n-1) + (m-1) = mn$, so that $m = \frac{2n-3}{n-3}$. It follows that n = 6 and m = 3. But then A and B are both constant, a contradiction. Therefore this case is not possible.

vi. The possibility of $B_1 = [(m-2)^{n-1}]$ and $A_1 = [a^{m-1},\ n-2]$ follows by duality.

This exhausts all the possibilities and hence by induction, the result is completely proved. ❑

Now, assume that $(A,\ B)$ is a realisable pair and $Q_1,\ Q_2,\ \ldots,\ Q_p$ are the irreducible components of a realisation T of $(A,\ B)$. Also, let Q_k has score sequences A_k and $B_k,\ 1 \le k \le p$. Then $(A,\ B)$ is uniquely realisable if and only if $(A_k,\ B_k)$ is uniquely realisable for all k.

14.9 Score Sequences of Oriented Graphs

An oriented graph is a digraph with no symmetric pairs of directed arcs and with no loops. Avery [233] extended the concept of score structure to all oriented graphs.

Definition: Let D be an oriented graph with vertex set $V = \{1,\ 2,\ \ldots,\ n\}$, and let $d^+(v)$ and $d^-(v)$ be the outdegree and indegree respectively of vertex v. Then the *score* of vertex v denoted by a_v is defined as $a_v = n-1+d^+(v)-d^-(v)$ with $0 \le a_v \le 2n-2$. The sequence of scores is called the *score list*, and $A = [a_1, a_2,\ \ldots,\ a_n]$ arranged in non-decreasing order is called the *score sequence* of D.

Any oriented graph can be interpreted as the result of a round robin competition in which ties (draws) are allowed, that is, the participants play each other once, with an arc from u to v if and only if u defeats v. A player receives two points for each win and one point for each tie, as is frequently the case in sports such as soccer, ice hockey and cricket. With this scoring system , player v obtains a total of a_v points. An arc from u to v denoted by $u \to v$ is written as $u(1-0)v$, and $u(0-0)v$ means that neither $u \to v$ nor $v \to u$.

Definition: A *triple* in an oriented graph is an induced subdigraph with three vertices. A *cyclic triple* is an intransitive triple of the form $u \to v \to w \to u$. Any triple can be of the form $u(x_1-x_2)v(y_1-y_2)w(z_1-z_2)u$, where $0 \le x_i,\ y_i,\ z_i \le 1$ with $0 \le \sum x_i,\ \sum y_i,\ \sum z_i \le 1$.

The following result [9] extends a result of Ryser [234] which showed that if two tournaments have the same score structure, then each can be transformed to the other by successively reversing the arcs of appropriate cyclic triples.

Theorem 14.23 Let D and D' be two oriented graphs with the same score sequence. Then D can be transformed to D' by successively transforming appropriate triples in one of the following ways.

Either (a) by changing a cyclic triple $u(1-0)v(1-0)w(1-0)u$ to a transitive triple $u(0-0)v(0-0)w(0-0)u$, which has the same score sequence, or vice versa.

or (b) by changing an intransitive triple $u(1-0)v(1-0)w(0-0)u$ to a transitive triple $u(0-0)v(0-0)w(0-1)u$, which has the same score sequence, or vice versa.

The following result due to Avery [9] gives a constructive condition for a non-negative sequence in non-decreasing order to be a score sequence of some oriented graph. A short proof of this result is due to Pirzada et. al. [199].

Theorem 14.24 (Avery) A sequence of non-negative integers in non-decreasing order is the score sequence of an oriented graph if and only if

$$\sum_{i=1}^{k} a_i \geq k(k-1)$$

for $1 \leq k \leq n$, with equality for $k = n$.

The following result can be found in [9].

Theorem 14.25 Among all oriented graphs with a given score sequence, those with the fewest arcs are transitive.

Proof Let A be a score sequence and let D be a realisation of A that is not transitive. Then D has an intransitive triple. There are two types of intransitive triples, a cyclic triple, which can be transformed by operation (a) of Theorem 14.23 to a triple with the same score sequence and three arcs fewer, and a triple $u(1-0)v(1-0)w(0-0)u$, which can be transformed by operation (b) of Theorem 14.23 to a triple with the same score sequence and one arc fewer. So in either case, we obtain a realisation of A with fewer arcs. ❑

The next result [9] provides a useful recursive test of whether a given sequence of non-negative integers is the score list of an oriented graph. We note that a transmitter is a vertex with indegree zero.

Theorem 14.26 Let A be a sequence of n integers between 0 and $2n-2$ inclusive and let A' be obtained from A by deleting the greatest entry $2n-2-r$ say, and reducing each of the greatest r remaining entries in A by one. Then A is a score list if and only if A' is a score list.

Proof Clearly, in a transitive oriented graph, any vertex of greatest score is a transmitter.

Let A' be a score list of some oriented graph D'. Then an oriented graph D with score list A can be obtained by adding a transmitter that is adjacent to just those vertices of whose scores are not reduced in going from A to A'.

For the converse, we show that there is an oriented graph with score list A in which a transmitter v with score $2n-2-r$ is adjacent to the (other) $n-1-r$ vertices with least scores. By Theorem 14.25, there is a transitive oriented graph D with score list A, in which a vertex v with greatest score $2n-2-r$ is a transmitter. Let U be the set of r vertices, apart from v, with the greatest scores in A, and let W be the set $V - \{v \cup U\}$.

Let v be adjacent in D to vertices $u_1, u_2, \ldots, u_k$ of U. Then there are exactly k vertices, say $w_1, w_2, \ldots, w_k$ of W not adjacent from v. Now, u_i cannot be adjacent to w_i, since D is transitive. Neither can w_i be adjacent to u_i, since taken together with the transitivity of D this implies that the score of w_i is greater than the score of u_i, which is contrary to the assumption. Thus $w_i(0-0)u_i$ for all i.

Now, transforming all triples $v(1-0)u_i(0-0)w_i(0-0)v$ to triples $v(0-0)u_i(0-1)w_i(0-1)v$, the vertex scores remain unchanged. This forms an (not necessarily transitive) oriented graph D_1 with score list A in which the transmitter v is adjacent to all vertices of W and none of U, as required. ❑

Theorem 14.26 provides an algorithm for determining whether a given non-decreasing sequence A of non-negative integers is a score sequence, and for constructing a corresponding oriented graph. At each stage, we form A' according to Theorem 14.26, such that scores of A' are also non-decreasing. If $a_n = 2n-2-r$, this means deleting a_n and reducing the r greatest remaining entries by one each to form $A' = |a'_1, a'_2, \ldots, a'_{n-1}|$ while ensuring that this is also non-decreasing. Arcs of an oriented graph are defined by $n \to v$ if and only if $a'_v = a_v$. If this procedure is applied recursively, then first it tests whether A is a score sequence and if A is a score sequence, an oriented graph $\Delta(A)$ with score sequence A is constructed.

Example Let $n = 5$, $A = [2, 4, 4, 4, 6]$.

Stage	A	B	Arcs of $\Delta(A)$
1	[2, 4, 4, 4, 6]	[2, 3, 3, 4]	$5 \to 4$, $5 \to 1$
2	[2, 3, 3, 4]	[2, 2, 2]	$4 \to 1$
3	[2, 2, 2]	[1, 1]	
4	[1, 1]	[0]	

Thus A is a score sequence.

We have the following observations [9] about $\Delta(A)$.

Theorem 14.27 The oriented graph $\Delta(A)$ is transitive for any score sequence A.

Theorem 14.28 There is no oriented graph with score sequence A which has fewer arcs than $\Delta(A)$.

One more method of constructing oriented graph with a given score sequence can be found in Pirzada [188].

The following is an equivalent statement of Theorem 14.24. A sequence of non-negative integers $A = [a_i]_1^n$ in non-decreasing order is a score sequence of an oriented graph if and only if for each subset $I \subseteq [n] = \{1, 2, \ldots, n\}$

$$\sum_{i \in I} a_i \geq 2\binom{|I|}{2}$$

with equality for $|I| = n$.

The following inequalities for scores in oriented graphs can be found in [235].

Theorem 14.29 A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for every subset $I = [n]$,

$$\sum_{i \in I} a_i \geq \sum_{i \in I} (i-1) + \binom{|I|}{2}$$

with equality when $I = [n]$

Theorem 14.30 A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence if and only if for any subset $I \subseteq [n]$,

$$\sum_{i \in I} a_i \leq \sum_{i \in I} (i-1) + \frac{1}{2}|I|(2n - |I| - 1),$$

with equality for $I = [n]$.

Theorem 14.31 If $A = [a_i]_1^n$ is a score sequence of an oriented graph, then for each i, $i - 1 \leq a_i \leq n + i - 2$.

A necessary condition for a score sequence in oriented graphs to be self-converse, can be found in [192].

14.10 Score Sets in Oriented Graphs

Definition: The set A of distinct scores of vertices in an oriented graph D is called the *score set* of D.

Definition: A digraph D is *diregular* if $d_v^+ = d_v^- = k$ holds for each vertex v in D. In case of an oriented graph D with n vertices, $a_v = n - 1 + d_v^+ - d_v^-$, for each vertex v in D, and when $d_v^+ = d_v^- = k$ (say), then $a_v = n - 1 + k - k = n - 1$ for each v in oriented graph D. Thus an oriented graph D with n vertices is diregular if $a_v = n - 1$, for all v in D.

Now, we have the following result, the proof of which is obvious.

Lemma 14.14 The number of vertices in an oriented graph with at least two distinct scores does not exceed its largest score.

The following result is given by Pirzada and Naikoo [196].

Theorem 14.32 (a) (Pirzada and Naikoo) Let $A = \{a, ad, ad^2, \ldots, ad^n\}$, where a and d are positive integers with $a > 0$ and $d > 1$. Then there exists an oriented graph D with score set A, except for $a = 1$, $d = 2$, $n > 0$ and for $a = 1$, $d = 3$, $n > 0$.

Proof We induct on n. For $n = 0$, there is a positive integer $a > 0$, so that $a + 1 > 0$. Let D be a diregular oriented graph having $a + 1$ vertices. Then $a_v = a + 1 - 1 = a$, for all $v \in D$. Therefore score set of D is $A = \{a\}$. This proves the result for $n = 0$. If $n = 1$, then there are positive integers a and d with $a > 0$ and $d > 1$, and for $a = 1$, $d \neq 2, 3$.

Now, three cases arise: (I) $a > 1,\ d > 2$, (II) $a > 1,\ d = 2$ and (III) $a = 1,\ d > 3$.

(I) Let $a > 1,\ d > 2$. Therefore, $a+1 > 0$. Let D_1 be a diregular oriented graph having $a+1$ vertices. Then $a_v = a+1-1 = a$, for all $v \in D_1$.

Now, $ad-2|V(D_1)|+1 = ad2(a+1)+1 = ad-2a-1 \geq 3a-2a-1 = a-1 > 0$, as $d \geq 3$ and $a > 1$. That is, $ad-2|V(D_1)|+1 > 0$. Let D_2 be a diregular oriented graph having $ad-2|V(D_1)|+1$ vertices. Then $a_u = ad-2|V(D_1)|+1-1 = ad-2|V(D_1)|$, for all $u \in D_2$.

Let there be an arc from every vertex of D_2 to each vertex of D_1, so that we get an oriented graph D (which includes D_1 to D_2 together with all the new arcs from D_2 and D_1) having $|V(D_1)|+|V(D_2)| = a+1+ad-2|V(D_1)|+1 = a+1+ad-2(a+1)+1 = ad-a$ vertices with $a_v = a$, for all $v \in D_1$, and $a_u = ad-2|V(D_1)|+2|V(D_1)| = ad$, for all $u \in D_2$. Therefore score set of D is $A = \{a,\ ad\}$.

(II) Assume $a > 1,\ d = 2$. First take $a = 2,\ d = 2$. Then $ad = 4 > 0$. Let D be an oriented graph having ad = 4 vertices, say, v_1, v_2, v_3, and v_4 in which $v_1 \to v_3$ and $v_2 \to v_4$, so that $a_{v_1} = a_{v_2} = 2+4-2 = 4 = ad$, and $a_{v_3} = a_{4_4}-2 = 2 = a$. Therefore D is an oriented graph having ad vertices with score set $A = \{a,\ ad\}$.

Now, take $a > 2,\ d = 2$. Let D_1 be a diregular oriented graph having 2 vertices, say v_1 and v_2. Then $a_{v_i} = 2-1 = 1$ for all $v_i \in D_1$, where $1 \leq i \leq 2$.

Again, $a > 2$ or $a-2 > 0$. Let D_2 be a diregular oriented graph having $a-2$ vertices, say $v_3,\ v_4,\ \ldots,\ v_a$. Then $a_{v_j} = a-2-1 = a-3$, for all $v_j \in D_2$, where $3 \leq j \leq a$.

Let there be an arc from every vertex of D_2 to each vertex of D_1, so that we get an oriented graph D_3 (which includes D_1 and D_2 together with all the new arcs from D_2 to D_1) having $2+a-2 = a$ vertices with $a_{v_i} = 1$, for all $v_i \in D_1$, where $1 \leq i \leq 2$, and $a_{v_j} = a-3+2(2) = a+1$, for all $v_j \in D_2$, where $3 \leq j \leq a$.

Again, $a > 2 > 0$. Let D_4 be a diregular oriented graph having a vertices, say $w_1,\ w_2,\ \ldots,$ w_a. Then $a_{w_k} = a-1$, for all $w_k \in D_4$, where $1 \leq k \leq a$.

Let there be a arcs from a distinct vertices of D_4 to a distinct vertices of D_3 ($w_q \to v_q$, for all $q = 1, 2,\ \ldots,\ a$), so that we get an oriented graph D (which includes D_3 and D_4 together with all the new arcs from D_4 to D_3) having $a+a = 2a = ad$ vertices with $a_{v_i} = 1+a-1 = a$, for all $v_i \in D_3$, where $1 \leq i \leq 2, = a+1+a-1 = 2a$, for all $v_j \in D_3$, where $3 \leq j \leq a$, and $a_{w_k} = a-1+2(1)+a-1 = 2a$, for all $w_k \in D_4$, where $1 \leq k \leq a$. Therefore score set of D is $A = \{a, 2a\} = \{a, ad\}$.

(III) Finally, let $a = 1,\ d > 3$. Therefore, $a+1 > 0$. Let D_1 be a diregular oriented graph having $a+1$ vertices. Then $a_v = a+1-1 = a$, for all $v \in D_1$.

Now, $ad-2|V(D_1)|+1 = ad-2(a+1)+1 = ad-2a-1 \geq 4a-2a-1 = 2a-1 > 0$, as $d \geq 4$ and a = 1, i.e., $ad-2|V(D_1)|+1 > 0$. Then as in (*I*), we have an oriented graph D having $ad-a$ vertices with score set $A = \{a, ad\}$.

Hence in all these cases, we get an oriented graph D with score set $A = \{a, ad\}$. This shows that the result is also true for n = 1.

Assume the result to be true for all $p \geq 1$. We show that the result is true for $p + 1$.

Let a and d be positive integers with $a > 0$ and $d > 1$, and for $a = 1$, $d \neq 2, 3$. Therefore by induction hypothesis, there exists an oriented graph D_1 having $|V(D_1)|$ vertices with score set $\{a, ad, ad^2, \ldots, ad^p\}$.

Once again, we have either (I) $a > 1$, $d > 2$, or (II) $a > 1$, $d = 2$, or (III) $a = 1$, $d > 3$. Obviously, for $d > 1$, in all the above possibilities, $ad^{p+1} \geq 2ad^p$, and the score set of D_1, namely, $\{a, ad, ad^2, \ldots, ad^p\}$ has at least two distinct scores for $p \leq 1$. Therefore by Lemma 14.14, $|V(D_1)| \leq ad^p$. Hence, $ad^{p+1} \geq 2|V(D_1)|$, or $ad^{p+1} - 2|V(D_1)| + 1 > 0$.

Let D_2 be a diregular oriented graph having $ad^{p+1} - 2|V(D_1)| + 1$ vertices. Then $a_v = ad^{p+1} - 2|V(D_1)| + 1 - 1 = ad^{p+1} - 2|V(D_1)|$, for all $v \in D_2$.

Let there be an arc from every vertex of D_2 to each vertex of D_1, so that we get an oriented graph D (which includes D_1 and D_2 together with all the new arcs from D_2 to D_1) having $|V(D_1)| + |V(D_2)|$ vertices with $a, ad, ad^2, \ldots, ad^p$ as the scores of the vertices of D_1, and $a_v = ad^{p+1} - 2|V(D_1)| + 2|V(D_1)| = ad^{p+1}$, for all $v \in D_2$. Therefore score set of D is $A = \{a, ad, ad^2, \ldots, ad^p, ad^{p+1}\}$, proving the result for $p + 1$. Hence the result follows. ❑

That no oriented graph exists when either $a = 1$, $d = 2$, $n > 0$ or $a = 1$, $d = 3$, $n > 0$, is proved in the following theorem.

Theorem 14.32 (b) There exists no oriented graph with score set $A = \{a, ad, ad^2, \ldots, ad^n\}$, $n > 0$, when either (i) $a = 1$, $d = 2$, or (ii) $a = 1$, $d = 3$.

We now have the following result [201].

Theorem 14.32 (c) If $a_1, a_2, \ldots, a_n$ are n non-negative integers with $a_1 < a_2 < \ldots < a_n$. Then there exists an oriented graph D with score set $A = \{a'_1, a'_2, \ldots, a'_n\}$, where

$$a'_i = \begin{cases} a_{i-1} + a_i + 1, & for\ i > 1, \\ a_i, & for\ i = 1. \end{cases}$$

Remarks

1. From Theorem 14.32 (c), it follows that every singleton set of non-negative integers is a score set of some oriented graph.
2. As we have shown in Theorem 14.32(b), i. e., the sets $\{1, 2, 2^2, \ldots, 2^n\}$ and $\{1, 3, 3^2, \ldots, 3^n\}$ cannot be the score sets of any oriented graph for $n > 0$. It follows, therefore, that the above results cannot be generalised to conclude that any set of non-negative integers forms the score set of some oriented graph. However, there can be other special classes of non-negative integers which can form the score set of an oriented graph, and the problem needs further investigations.

Pirzada and Naikoo [195] have obtained some results on degree frequencies in oriented graphs. More results on scores, score sets and kings in oriented graphs can be seen in [199, 201, 203, 207].

14.11 Uniquely Realisable (Simple) Score Sequences in Oriented Graphs

An oriented graph D is *reducible* if it is possible to partition its vertices into two nonempty sets V_1 and V_2 in such a way that there is an arc from every vertex of V_2 to each vertex of V_1. Let D_1 and D_2 be induced digraphs having vertex sets V_1 and V_2 respectively. Then D consists of D_1 and D_2 and arcs from every vertex of D_2 to each vertex of D_1. We write $D = [D_1, D_2]$. If this is not possible, then the oriented graph D is irreducible. Let $D_1, D_2, \ldots, D_k$ be irreducible oriented graphs with disjoint vertex sets. Then $D = [D_1, D_2, \ldots, D_k]$ denotes the oriented graph having all arcs of D_i, $1 \le i \le k$ and there are arcs from every vertex of D_j to each vertex of D_i, $1 \le i < j \le k$. Here $D_1, D_2, \ldots, D_k$ are called *irreducible components* of D. Such a decomposition is called as *irreducible component decomposition* of D, which is unique.

Definition: A score sequence A is said to be *irreducible* if all the oriented graphs D with score sequence A are irreducible.

In case of ordinary tournaments, the score sequence used to decide whether a tournament T having score sequence S, is strong or not. This is not true in case of oriented graphs. For example, the oriented graphs D_1 and D_2 in Figure 14.11, both have score sequence [2, 2, 2] but D_1 is strong and D_2 is not.

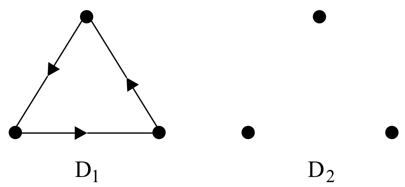


Fig. 14.11

The following result due to Pirzada [186] characterises irreducible oriented graphs.

Theorem 14.33 Let D be an oriented graph having score sequence. Then D is irreducible if and only if, for $k = 1, 2, \ldots, n-1$

$$\sum_{i=1}^{k} a_i \ge k(k-1) \tag{14.33.1}$$

and

$$\sum_{i=1}^{k} a_i = n(n-1) \tag{14.33.2}$$

Proof Suppose D is an irreducible oriented graph having score sequence $[a_i]_1^n$. Condition (14.33.2) holds since Theorem 14.24 has already established it for any oriented graph. To verify inequalities (14.33.1) we observe that for any integer $k < n$, the subdigraph induced by any set of k vertices has a sum of scores $k(k-1)$. Since D is irreducible, there must be an arc from at least one of these vertices to one of the other $n-k$ vertices, or there is no arc from these k vertices to other $n-k$ vertices. Thus, for $1 \leq k \leq n-1$

$$\sum_{i=1}^{k} a_i > k(k-1).$$

For the converse, suppose conditions (14.33.1) and (14.33.2) hold, we know by Theorem 14.24 that there exists an oriented graph D with these scores. Assume that such an oriented graph D is irreducible. Let $D = [D_1, D_2, \ldots, D_k]$ be the irreducible component decomposition of D. If m is the number of vertices in D_1, then $m < n$, and the following equation holds,

$$\sum_{i=1}^{k} a_i = m(m-1),$$

which is a contradiction. This proves the converse part. ❑

The following result due to Pirzada [186] can be proved easily.

Theorem 14.34 Let D be an oriented graph with score sequence $A = [a_i]_1^n$. Suppose that $\sum_{i=1}^{k} a_i = p(p-1)$, $\sum_{i=1}^{q} a_i = q(q-1)$ and for $p+1 \leq k \leq q-1$, where $0 \leq p < q \leq n$. Then the subdigraph induced by the vertices $v_{p+1}, v_{p+2}, \ldots, v_q$ is an irreducible component of D with score sequence $[a_{p+1} - 2_p, \ldots, a_q - 2_p]$.

Now, A is irreducible if D is irreducible and the irreducible components of A are the score sequences of the irreducible components of D. Theorem 14.34 shows that the irreducible components of A are determined by the successive values of k for which

$$\sum_{i=1}^{k} a_i = k(k-1),\ 1 \leq k \leq n. \tag{14.33.3}$$

We illustrate it with the following example.

Let $A = [1, 2, 3, 8, 8, 8, 13, 13]$. Equation (14.33.3) is satisfied for $k = 3, 6, 8$. Thus irreducible components of S are [1, 2, 3], [2, 2, 2] and [1, 1] in ascending order.

Definition: A score sequence is *simple* if it belongs to exactly one oriented graph. We characterise simple score sequences of oriented graphs. First we have the following observation.

Lemma 14.15 The score sequence A of an oriented graph is simple if and only if every irreducible component of A is simple.

The following result due to Pirzada [186] determines which irreducible score sequences are simple.

Theorem 14.35 (S. Pirzada) Let A be an irreducible score sequence. Then A is simple if and only if it is one of [0], or [1, 1].

Proof Suppose A is an irreducible score sequence and let D be an oriented graph having score sequence A. We have three cases to consider. (1) D has $n \geq 3$ vertices, (2) D has two vertices, (3) D has one vertex.

Case (1) D has $n \geq 3$ vertices. Since A is irreducible, there exist vertices u, v and w such that D has a cyclic triple $u(1-0)v(1-0)w(1-0)u$; or an intransitive triple $u(1-0)v(1-0)w(0-0)u$; or a transitive triple $u(0-0)v(0-0)w(0-1)u$; or a transitive triple $u(0-0)v(0-0)w(0-0)$.

Now, if D contains the cyclic triple $u(1-0)v(1-0)w(1-0)u$, it can be changed to the transitive triple $u(0-0)v(0-0)w(0-0)u$ to form an oriented graph with the same score sequence, or vice versa. So the number of arcs in D and D' is different. If D contains the intransitive triple $u(1-0)v(1-0)w(0-0)u$, we can transform it to the transitive triple $u(0-0)v(0-0)w(0-0)u$, to form an oriented graph having the same score sequence, or vice versa. Here also the number of arcs in D and D' is different. Since in every case the number of arcs in D and is not same, D is D' not isomorphic to D' . Thus A is not simple.

Case (2) D has two vertices. Then A = [1, 1] is the only irreducible score sequence and it belongs to exactly one oriented graph, namely $u(0-0)v$.

Case (3) D has just one vertex. Then A = [0] which is obviously simple.

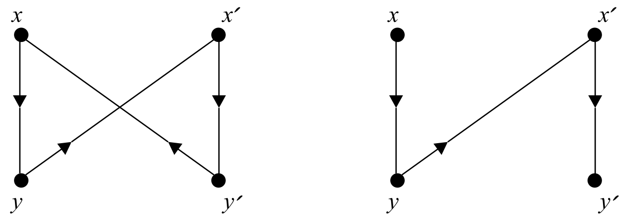
Hence [0] and [1, 1] are the only irreducible score sequences that are simple. ❑

Corollary 14.5 The score sequence A is simple if and only if every irreducible component of A is one of [0], or [1, 1].

14.12 Score Sequences in Oriented Bipartite Graphs

An *oriented bipartite graph* is the result of assigning a direction to each edge of a simple bipartite graph. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the partite sets of an oriented bipartite graph D. For any vertex u in D, let $d^+(u)$ and $d^-(u)$ be the outdegree and indegree respectively. Define $a_x = n + d^+(x) - d^-(x)$ and $b_y = m + d^+(y) - d^-(y)$ as the scores of x in X and y in Y respectively. Clearly, $0 \leq a_x \leq 2n$ and $0 \leq b_y \leq 2m$. Then the sequences $A = [a_i]_1^m$ and $B = [b_j]_1^n$ in non-decreasing order are called a *pair of score sequences* of D. An arc from x to y, that is, $x \to y$ is denoted by $x(1-0)$, and $x(0-0)$ means neither $x \to y$ nor $y \to x$.

Definition: A *tetra* in an oriented bipartite graph is an induced subdigraph with two vertices from each partite set. Define tetras of the form $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ to be of α *-type*, and all other tetras to be of β *-type*. An oriented bipartite graph is said to be of α -type or β -type according as all of its tetras are of α -type or β -type respectively (Fig. 14.12).



α–type

Fig. 14.12

We have the following simple observation.

Theorem 14.36 Among all oriented bipartite graphs with a given pair of score sequences, those with the fewest arcs are of β -type.

Proof Let (A, B) be a given pair of score lists and let D be a realisation of (A, B) that is not β -type. Then D has a tetra of α-type: $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ or $x(1-0)y(1-0)x'(1-0)y'(0-0)x$. Since $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ can be changed to $x(0-0)y(0-0)x'(0-0)y'(0-0)x$ with the same score sequences and four arcs fewer , and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ can be changed to $x(0-0)y(0-0)x'(0-0)y'(0-1)x$ with the same score sequences and two arcs fewer, so in either case we can obtain a realisation of (A, B) with fewer arcs. ❑

A *transmitter* is a vertex with indegree zero. In a β -type oriented bipartite graph with score sequences $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ either the vertex with score a_m, or the vertex with score b_n, or both may act as transmitter.

The next result due to Pirzada, Merajudin and Yin Jianhua [194] provides a useful recursive test to find whether a pair of lists is realisable.

Theorem 14.37 Suppose $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ be two sequences of non-negative integers in nondecreasing order. Let A' be obtained from A by deleting one entry a_m and B' be obtained from B by reducing $2n - a_m$ greatest entries of B by 1 each provided $a_m \geq n$ and $b_n \leq 2m-1$. Then A and B are the score sequences of some oriented bipartite graph if and only if A' and B' are also score sequences of some oriented bipartite graph.

Theorem 14.37 provides an algorithm for determining whether a given pair of sequences (A, B) of non-negative integers in nondecreasing order is a pair of score sequences and for constructing a corresponding oriented bipartite graph. Suppose $A = [a_1, a_2, \ldots, a_m]$ and $B = [b_1, b_2, \ldots, b_n]$ be a pair of score sequences of an oriented bipartite graph with parts $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, where $a_m \geq n$, $b_n \leq 2m-1$. Deleting a_m and reducing $2n - a_m$ greatest entries of B by 1 each to form $B' = [b'_1, b'_2, \ldots, b'_n]$.. Then arcs are defined by $x_m \to y_j$ for which $b_j = b'_j$. Now, if at least one of the conditions $a_m \geq n$ or $b_n \leq 2m-1$ does not hold, then we delete b_n (obviously $b_n \geq m$, $a_m \leq 2n-1$) and reduce $2m - b_n$ greatest entries of A by 1 each to form $A = [a'_1, a'_2, \ldots, a'_m]$. In this case arcs are defined by $y_n \to x_i$ for which $a_i = a'_i$. If this method is applied successively, then it tests whether (A,B) is a pair of score sequences and an oriented bipartite graph $\Delta(A,\ B)$ with score sequences (A,B) is constructed.

We can interpret this algorithm in the following way. Let $A = [a_1,\ a_2,\ \ldots,\ a_m]$ and $B = [b_1,\ b_2,\ \ldots,\ b_n]$ be a pair of score sequences of an oriented bipartite graph, where $a_m \geq n$, $b_n \leq 2m-1$. Let p and q denote the smallest and largest integers j for which $b_j = b_{a_m - n}$. Let $A'[a_1,\ a_2,\ \ldots,\ a_{m-1}]$ as before and let then $B' = [b'_1,\ b'_2,\ \ldots,\ b'_n]$, then

$$b'_j = \begin{cases} b_j, & for\ 1 \leq j \leq p-1\ and\ p+q-(a_m-n) \leq j \leq q, \\ b_j - 1, & otherwise. \end{cases}$$

We illustrate this reduction and the resulting construction with the following example, beginning with lists A_1 and B_1. The oriented bipartite graph constructed is shown in Figure 14.13.

$A_1 = [4,4,5]$	$B_1 = [1,1,4,5]$	$x_3 \to y_2$
$A_2 = [4,4]$	$B_2 = [0,1,3,4]$	$y_4 \to x_1, x_2$
$A_3 = [4,4]$	$B_3 = [0,1,3]$	$x_2 \to y_1$
$A_4 = [4]$	$B_4 = [0,0,2]$	$y_3 \to x_1$
$A_5 = [4]$	$B_5 = [0,0]$	$x_1 \to y_1, y_2$

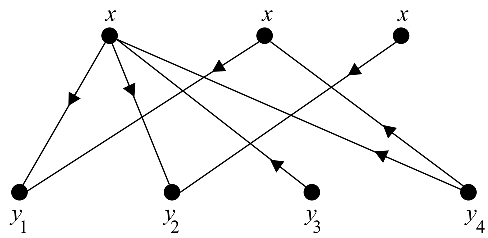


Fig. 14.13

Let D_i be the oriented bipartite graphs with disjoint parts X_i and Y_i for $1 \leq i \leq t$. Let $X = U_{i=1}^t X_i$ and $Y = U_{i=1}^t Y_i$. Clearly, $D = [D_1, D_2,\ \ldots,\ D_t]$ denotes the oriented bipartite graph with parts X and Y, obtained from D_i for $1 \leq i \leq t$ such that the arcs of D are the arcs of D_i and each vertex of Y_j is adjacent to every vertex of X_i for $j < i$ and each vertex of X_i is adjacent to every vertex of Y_j for $i < j$.

The next result [194] gives a criterion for determining whether a pair of sequences are realisable as scores.

Theorem 14.38 Let $A = [a_1,\ a_2,\ \ldots,\ a_m]$ and $B = [b_1,\ b_2,\ \ldots,\ b_n]$ be a pair of non-negative integers in non-decreasing order. Then A and B are scores sequences of some oriented bipartite graph if and only if

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{t} b_j \geq 2kl \tag{14.3.1}$$

for $1 \le k \le m$ and $1 \le l \le n$, with equality when $k = m$ and $l = n$.

The characterisation of scores of oriented tripartite graphs can be found in [190] and scores (marks) in other types of digraphs can be found in [188, 196, 191].

14.13 Score Sequences of Semi Complete Digraphs

Definition: A *semi complete digraph* is a digraph with no directed loops and at least one arc between every pair of distinct vertices. Clearly, a tournament is a semi complete digraph in which there is exactly one arc between every pair of distinct vertices. Therefore every semi complete digraph contains at least one tournament on the same vertex set and is contained in the complete symmetric digraph on the same vertex set. The *score* of a vertex v in a semi complete digraph D is the outdegree of v.

The following result is due to Reid and Zhang [222].

Theorem 14.39 A sequence of non-negative integers $S = [s_i]_1^n$ in non-decreasing order is a score sequence of some semi complete digraph of order n if and only if

$$\sum_{i=1}^{k} \geq \binom{k}{2} \text{ and } s_k \leq n-1, \qquad (14.39.1)$$

for all $k, 1 \le k \le n$.

Proof

Necessity If S is a score sequence of some semi complete digraph D of order n, then any k vertices of D induce a semi complete digraph of order k which, in turn, contains a tournament W of order k. Therefore the sum of the scores in D of these k vertices is at least the sum of their scores in W which is the total number of arcs in W, $\binom{k}{2}$. Also, a vertex of D can dominate at most all of the other vertices, so no score in S can exceed $n-1$. Thus the conditions (14.39.1) are necessary.

We require the following result for proving sufficiency.

Lemma 14.16 If $S = [s_i]_1^n, n \ge 1$, is a sequence of integers in non-decreasing order satisfying (14.39.1), then there exists a tournament T with score sequence $s' = [s'_i]_1^n$, such that $s'_i \le s_i$ for $1 \le i \le n$.

Proof Define an order $\preceq$ on all non-decreasing sequences of integers satisfying (14.39.1) (thus including all sequences satisfying conditions (14.1.2)) as follows. If $B = [b_1, b_2, \ldots, b_n]$ and m is the smallest index for which $b_m = b_n$ (= max $\{b_i : 1 \le i \le n\}$), then B (strictly) covers the sequence $A = [a_1, a_2, \ldots, a_n]$, where A and B are identical such that $a_m = b_m - 1$. Note that if $m > 1$, then $b_{m-1} < b_m = b_{m+1} = \ldots = b_n$ and if $m = 1$, then $b_1 = b_2 = \ldots = b_n \ge (n-1)/2$.

Also, if B covers A, then $\sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n} b_i\right) - 1$.

This implies, by Landau's theorem, that if S satisfies (14.39.1), then S is the score sequence for some tournament if and only if S covers no sequence satisfying (14.39.1). And, if B is not the score sequence for any tournament, then B covers exactly one sequence satisfying (14.39.1). For two non-decreasing sequences of integers X and Y satisfying (14.39.1), define $X \preceq Y$ if either $X = Y$, or there is a sequence $X_0 = X, X_1, X_2, \ldots, X_{j-1}, X_j = Y$ of non-decreasing sequences of integers each satisfying conditions (14.39.1) such that X_i covers $X_{i-1}, 1 \le i \le j$.

Now, let $S = [s_i]_1^n$ be a sequence of integers in non-decreasing order satisfying conditions (14.39.1). Induct on the integer $e(S) \equiv \left(\sum_{i=1}^{n} s_i\right) - \binom{n}{2}$. If $e(S) = 0$, then by Landau's theorem, S itself is a score sequence for some tournament T. If $e(S) > 0$, then by the remarks above, S covers exactly one sequence $Z = [z_1, z_2, \ldots, z_n]$ satisfying (14.39.1), such that $z_i \le s_i$, for $1 \le i \le n$, and $e(Z) = \left(\sum_{i=1}^{n} Z_i\right) - \binom{n}{2} = \left(\sum_{i=1}^{n} s_i\right) - \binom{n}{2} - 1 = e(s) - 1$. By the induction hypothesis applied to Z, there is a score sequence $S' = [s'_i]_1^n$ for some tournament T such that $s'_i \le z_i$, for $1 \le i \le n$. By the transitivity of $\le$, we have $s'_i \le s_i$, for $1 \le i \le n$, so and T suffice for S, as required. ❑

Sufficiency of Theorem 14.39 Let $S = [s_i]_1^n$, $n \ge 1$, be a sequence of integers in non-decreasing order, satisfying conditions (14.39.1). By Lemma 14.16, there is a tournament T of order n with score sequence S', where $S' \preceq S$ In T denote the vertex with score s'_i by v_i, $1 \le i \le n$. Since v_i has indegree $n - 1 - s'_i \ge n - 1 - s_i$, arcs can be added from v_i to any $n - 1 - s_i$ vertices in the inset of v_i in T so as to produce a semi complete digraph D with score sequence S.

14.15 Exercises

1. Prove that any n-tournament can be obtained from any other having the same scores by a sequence of arc reversals of 3-cycles.

2. If an n-tournament has every score s_i satisfying, $\frac{1}{4}(n-1) \le s_i \le \frac{3}{4}(n-1)$ then show that it is irreducible.

3. Construct a proof for Theorem 14.2, and 14.6.

4. If $S = \{a,\ a+d,\ a+d+e\}$, where a, d, e are non-negative integers and $de > 0$, and if $(d,\ e) = g$, $d = a$ and $e \le a + d - d/2g + (1/2)$, then prove S is a score set of some tournament.

5. Prove that every set of three non-negative integers is a score set of some tournament.

6. If a, b, c, d are four non-negative integers with $bcd > 0$, prove that there exists a tournament T with score set $S = \{a, a+b, a+b+c, a+b+c+d\}$.

7. Construct a proof of Theorem 14.11.

8. Prove Lemma 14.7 and Lemma 14.8.
9. Construct a proof of Theorem 14.14.
10. Construct a proof of Theorem 14.16.
11. Construct proofs of Theorem 14.18 and 14.21.
12. If T is a bipartite tournament with score sequences and satisfying $A = [a_i]_1^m$ and $B = [b_j]_1^n$ satisfying $n/4 < a_i < 3n/4$ for $1 \le i \le m$, and $m/4 < b_j < 3m/4$ for $1 \le j \le n$, then prove that T is irreducible.
13. Construct proofs of Theorem 14.37 and Theorem 14.38.