1. Introduction

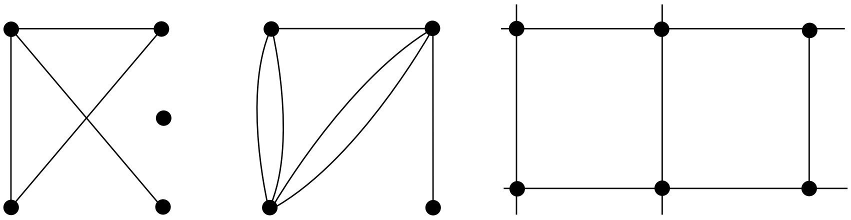
Graph theory owes its evolution to the study of some physical problems involving sets of objects and binary relations among them. It is difficult to pinpoint its formulation to a single source; in fact, graph theory can be said to have been discovered many times, each discovery being independent of the other. The earliest known studies appear in the works of Euler, Kirchoff, Cayley and Hamilton. The twentieth century witnessed considerable activity in this area, with new discoveries and proofs being proposed as solutions to classical problems including the celebrated four colour problem.

1.1 Basic Concepts

Let V be a nonempty set. The cartesian product of V with itself, denoted by $V \times V$, is the set of all unordered pairs of elements of V. That is, $V \times V = \{(u,\ v) : u,\ v \in V\}$. We denote by $V_{(2)}$ the set of unordered pairs of distinct elements of V, by $V_{[2]}$ the set of unordered pairs of elements of V, not necessarily distinct, and by $V^{(2)}$ the set of ordered pairs of distinct elements of V.

Definition: A simple graph (or briefly, a graph) G is a finite nonempty set V together with a symmetric, irreflexive relation R on V. The elements of the set V are called the *vertices* of the graph and the relation R is called the *adjacency relation*. If u is related to v by R, then u is said to be *adjacent* to v and we write uRv. An example of a simple graph is given in Figure 1.1(a).

Since R is a symmetric relation, it defines a subset E of $V_{(2)}$. The elements of the set E are called the *edges* of the graph.



(a) *Simple graph* (b) *Multigraph* (c) *Infinite graph*

Fig. 1.1

From the above definition, we observe that a graph G is a pair $(V,\ E)$, where V is a nonempty set whose elements are called the vertices of G and E is a subset of $V_{(2)}$ whose elements are called the edges of G.

We observe that there is an incidence relation I between the vertex set V and the edge set E of a graph. If the element $e \in E$, there is a pair of distinct vertices u and v such that $e = \{u,\ v\}$. The vertices u and v are called *end vertices* of e, and u and v are said to be *incident* with e (uIe and vIe). Also, e is said to be incident with u and v, and in that case we write $eI'u$ and $eI'v$, where I' is the relation converse to I.

A graph with a finite number of vertices and finite number of edges is called a *finite graph*, otherwise it is an *infinite graph* (Figure 1(c)).

We represent a graph G with vertex set V and edge set E by $(V(G),\ E(G))$. Since we are only going to deal with finite graphs, we write $V(G) = \{v_1, v_2, \ldots, v_n\}$, $E(G) = \{e_1, e_2, \ldots, e_m\}$.

We define $|V| = n$ to be the *order* of G and $|E| = m$ to be the *size* of G. Such a graph is called an $(n,\ m)$ graph. If there is an edge e between the vertices u and v, we briefly write $e = uv$ and say edge e joins the vertices u and v. A vertex is said to be *isolated* if it is not adjacent to any other vertex.

Multigraph: A multigraph is a pair (V, E), where V is a nonempty set of vertices and E is a multiset of edges, being a multi-subset of $V_{[2]}$. The number of times an edge $e = uv$ occurs in E is called the *multiplicity* of e and edges with multiplicity greater than one are called *multiple edges*. An example of a multigraph is given in Figure 1(b).

General graph: A general graph is a pair $(V,\ E)$, where V is a nonempty set of Moment and E is a multiset of edges, being a multi-subset of $V_{(3)}$. An edge of the form $e = uu,\ u \in V$ is called a *loop*. An edge which is not a loop is called a *proper edge* or *link*. The number of times edge e occurs is called its multiplicity, and proper edges with multiplicity greater than one are called multiple edges. Loops with multiplicity greater than one are called *multiple loops* (Fig. 1.2).

If $u,\ v \in V$ in a general graph or multigraph G, then the multiplicity of the edge uv is denoted by $q_G[u,\ v]$. If uv is not an edge, then $q_G[u,\ v] = 0$. Similarly, if u is a vertex of a general graph G, then the number of loops at u in G is denoted by $q_G[u]$.

The graph obtained by replacing all multiple edges by single edges in a multigraph G is called the *underlying graph* of G. Similarly, if G is a general graph, the graph H obtained by removing all its loops and replacing all multiple edges by single edges is called the underlying graph of G.

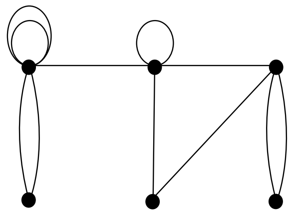


Fig. 1.2 *General graph*

1.2 Degrees

In a graph G, the degree of a vertex v is the number of edges of G which are incident to v and is denoted by $d(v)$ or $d(v|G)$. We have $d(v) = |\{e \in E\colon e = uv, \text{ for } u \in V\}|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. In the graph of Figure 1.3(a), $d(v_1) = d(v_3) = d(v_6) = 3$, $d(v_2) = 1$, $d(v_4) = 0$ and $d(v_5) = 2$.

A graph is said to be *regular* if all its vertices are of same degree and *k-regular* if all its vertices are of degree k. A 3-regular graph is also called a *cubic graph*. A vertex with degree zero is an *isolated vertex*, a vertex with degree one is a *pendant vertex* and the unique edge incident to a pendant vertex is a *pendant edge*. A vertex of odd degree is an *odd vertex* and a vertex of even degree is an *even vertex.* In Figure 1.3(a), v_4 is an isolated vertex and v_2 is a pendant vertex. The graph shown in Figure 1.3(b) is 2-regular and that given in Figure 1.3(c) is 3-regular, i.e., cubic.

In a general graph G, a loop incident to a vertex v is counted as two edges incident to v. Therefore $d(v)$ is the number of non-loop edges incident to v plus twice the number of loops at v.

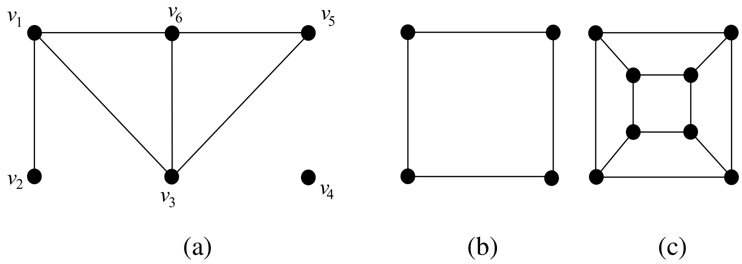


Fig. 1.3 *Graphs (b) and (c) are regular*

The following result is due to Euler [76].

Theorem 1.1 The sum of the degrees of a graph is even, being twice the number of edges.

Proof Let m be the number of edges in a graph $G = (V, E)$. Since each edge contributes two to the degrees, one at the beginning vertex and one at the end vertex of the edge, the sum of the degrees is even and equal to twice the number of edges. Hence,

$$\sum_{v \epsilon V} d(v) = 2m. \qquad \square$$

Theorem 1.2 In any graph there is an even number of vertices of odd degree.

Proof Let $G = (V, E)$ be a graph and $d(v)$ be the degree of the vertex $v \in V$. Let $|E| = m$.

Then $\sum_{v \epsilon V} d(v) = 2m$ and therefore,

$$\underbrace{\sum_{\substack{\text{odd degree}\\ \text{vertices}}} d(v_j)}_{\text{I}} + \underbrace{\sum_{\substack{\text{even degree}\\ \text{vertices}}} d(v_k)}_{\text{II}} = 2m. \tag{1.2.1}$$

Since the right hand side of (1.2.1) is even, and (II) in (1.2.1) is also even, therefore (I) in (1.2.1) is even. Hence,

$$\sum_{\substack{\text{odd degree}\\ \text{vertices}}} d(v_j) = \text{even}.$$

This is only possible when the number of vertices with odd degree is even. ❑

1.3 Isomorphism

Let G and H be general graphs. Let f be a one–one mapping of $V(G)$ onto $V(H)$, and g be a one–one mapping of $E(G)$ onto $E(H)$. Let θ denote the ordered pair (f, g). Then θ is an isomorphism of G onto H, when the vertex x is incident with the edge e in G if and only if the vertex fx is incident with the edge ge in H (Fig. 1.4). If such an isomorphism θ exists, the graphs G and H are said to be *isomorphic* and is denoted by $G \cong H$. We have, $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$.

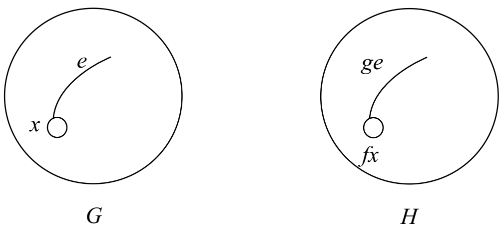
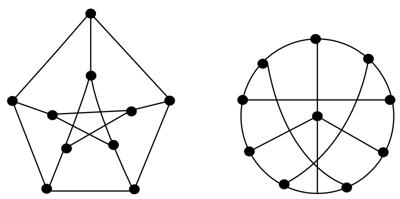


Fig. 1.4

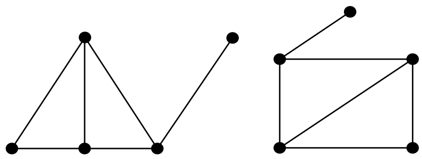
We can think θ to be an operation transforming G into H and write $\theta G = H$. Also, we write $\theta v = fv$ and $\theta e = ge$ for each vertex v and each edge e of G. Clearly, G and H can be represented by the same diagram. The representative of an edge or vertex x of G can be reinterpreted as the representative of θx in H.

An isomorphism of a graph G onto itself is called an *automorphism* of G. Any graph G has the *identical* or *trivial automorphism* I such that $Ix = x$ for each edge or each vertex x of G.

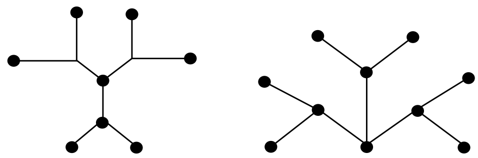
Clearly, two graphs G and G' are *isomorphic* to each other if there is a one–one correspondence between their vertices, and between their edges such that the incidence relationship is preserved. For example, the graphs shown in Figure 1.5(a), Figure 1.5(b) and Figure 1.5(c) are isomorphic.



(a) *Isomorphic graphs*



(b) *Isomorphic graphs*



(c) *Isomorphic graphs*

Fig. 1.5

It follows from the definition of isomorphism that two isomorphic graphs have

i. the same number of vertices,

ii. the same number of edges, and

iii. an equal number of vertices with a given degree.

However, these conditions are not sufficient. To see this, consider the two graphs given in Figure 1.6.

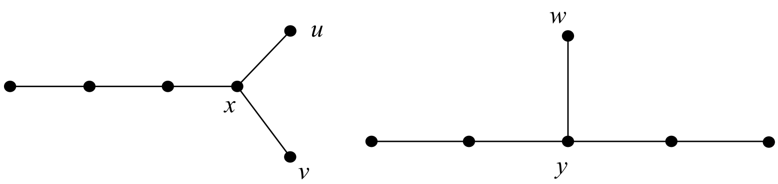


Fig. 1.6 *Non-isomorphic graphs*

These graphs satisfy all the three conditions, but they are not isomorphic because the vertex x in (a) corresponds to vertex y in (b) as there are no other vertices of degree three, and in (b) there is only one pendant vertex w adjacent to y, while in (a) there are two pendant vertices u and v adjacent to x.

The graphs in Figure. 1.7 also satisfy conditions (i), (ii) and (iii) but are not isomorphic.

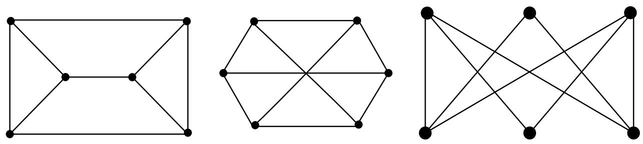


Fig. 1.7 *Non-isomorphic graphs*

We have the following observation on isomorphism of graphs.

Theorem 1.3 The relation isomorphism in graphs is an equivalence relation.

Proof The relation of isomorphism between graphs is reflexive because of the trivial automorphisms.

Let $\theta = (f, g)$ be an isomorphism of a graph G onto a graph H, so $G \cong H$. Then there is an inverse isomorphism $\theta^{-1} = (f^{-1}, g^{-1})$ of H onto G. So $H \cong G$. Therefore $\cong$ is symmetric.

Now, let $\theta = (f, g)$ be an isomorphism of G onto H, and $\phi\theta = (f_1 f, g_1 g)$ of G onto K. Here $f_1 f$ is a mapping obtained by applying first f and then f_1. Similarly, $\phi\theta$ is the isomorphism obtained applying first θ and then ϕ. Thus $\cong$ is transitive.

Hence the relation isomorphism is an equivalence relation. ❑

Remark The multiplication of the isomorphism defined above is associative.

Since the relation isomorphism is an equivalence relation, it partitions the class of all graphs into disjoint nonempty subclasses called *isomorphism classes*, such that two graphs belong to the same isomorphism class if and only if they are isomorphic.

Theorem 1.4 Let G and H be graphs and let f be a one–one mapping $V(G)$ onto $V(H)$ such that two distinct vertices x and y of G are adjacent if and only if the corresponding vertices fx and fy of H are adjacent in H. Then there is a uniquely determined one–one mapping g of $E(G)$ onto $E(H)$ such that (f, g) is an isomorphism of G onto H.

Proof Let e be any edge of G, having distinct ends x and y. By hypothesis, there is a uniquely determined edge e' of H whose ends are fx and fy. We define a one–one mapping g by the rule $ge = e'$, for each edge e of G. It is then clear that (f, g) is an isomorphism of G onto H.

Conversely, let g be a mapping such that (f, g) is an isomorphism of G onto H. Then for each edge e of G, there is an edge e' of H such that $ge = e'$. ❑

An isomorphism of a graph G onto a graph H is defined as a one–one mapping of $V(G)$ onto $V(H)$ that preserves adjacency. This specialisation can be regarded as an application of Theorem 1.4.

Definition: Two graphs $G(V, E)$ and $H(U, F)$ are label-isomorphic if and only if $V = U$, and for any pair u, v in V, $uv \in E$ if and only if $uv \in F$. The graphs of Figure 1.8 are isomorphic, but not label-isomorphic.

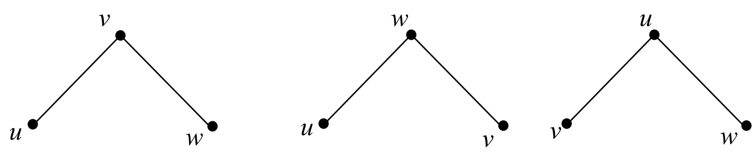


Fig. 1.8 *Isomorphic but not label-isomorphic graphs*

Definition: A graph invariant is a function f from the set of all graphs to any range of values (numerical, vectorial or any other) such that f takes the same value on isomorphic graphs. When the range of values is numerical (real, rational or integral) the invariant is called a *parameter*. The order and size of a graph are *graph parameters*.

1.4 Types of graphs

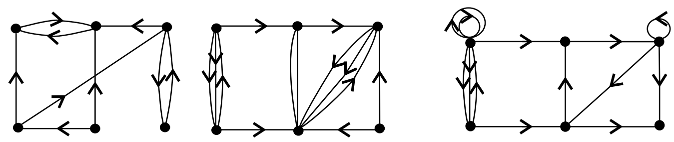
Simple directed graph or simple digraph: A simple digraph (or simply digraph) D is a pair (V, A), where V is a nonempty set of vertices and A is a subset of $V^{(2)}$ whose elements are called *arcs* of D.

Multidigraph: A multidigraph D is a pair (V, A), where V is a nonempty set of vertices, and A is a multiset of arcs of $V^{(2)}$. The number of times an arc occurs in D is called its multiplicity and arcs with multiplicity greater than one are called multiple arcs of D.

General digraph: A general digraph D is a pair (V, A), where V is a nonempty set of vertices, and A is a multiset of arcs, being a multisubset $V \times V$. An arc of the form uu is called a *loop* of D and arcs which are not loops are called *proper arcs* of D. The number of times an arc occurs is called its multiplicity. A loop with multiplicity greater than one is called a *multiple loop*.

An arc $(u, v) \in A$ of a digraph is denoted by uv, implying that it is directed from u to v, u being the *initial vertex* and v the *terminal vertex*. Clearly, a digraph is an irreflexive binary relation on V.

Various types of digraphs are shown in Figure 1.9.



(a) *Digraph* (b) *Multidigraph* (c) *General digraph*

Fig. 1.9

If $D(V, A)$ is a digraph, the graph $G(V, E)$, where $uv \in E$ whenever uv or vu or both are in A, is called the underlying graph of D (also called the *covering graph* $C(D)$ of D).

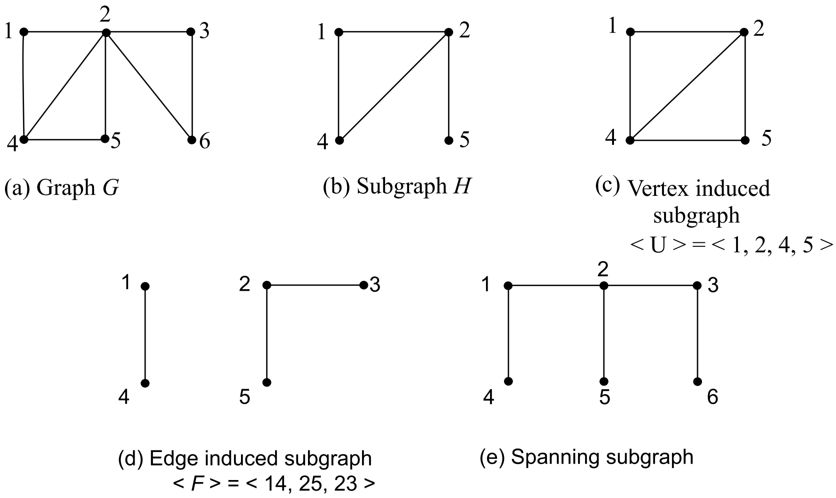
If $D(V, A)$ is a general digraph, the digraph $D_1(V, A_1)$ obtained from D by removing all loops, and by replacing all multiple arcs by single arcs is the *digraph underlying D*. The underlying graph of D_1 is the underlying graph of D.

Mixed graph: A mixed graph $G\{V, A \cup E\}$ consists of a nonempty set V of vertices, a set A of arcs ($A \subseteq V^{(2)}$), and a set E of edges ($E \subseteq V_{(2)}$), such that if $uv \in E$ then neither uv, nor vu is in A.

We represent a general, multi and simple graph by g-graph, m-graph and s-graph respectively.

Subgraphs: A subgraph of a graph $G(V, E)$ is a graph $H(U, F)$ with $U \subseteq V$ and $F \subseteq E$. We denote it by $H < G$ (G is also called the *super graph* of H.) If $U = V$ then H is called the *spanning subgraph* of G, and is denoted by $H \leq G$. Here G is called the *spanning super graph* of H and is denoted by $G \geq H$.

If F consists of all those edges of G joining pairs of vertices of U, then H is called the *vertex induced subgraph* of G and is denoted by $H = < U >$. If $F \subseteq E$, and U is the set of end vertices of the edges of F, then $H(U, F)$ is called an *edge induced subgraph* of G and is denoted by $H = < F >$. These definitions are illustrated in Figure 1.10.



(a) Graph G

(b) Subgraph H

(c) Vertex induced subgraph $< U > = < 1, 2, 4, 5 >$

(d) Edge induced subgraph $< F > = < 14, 25, 23 >$

(e) Spanning subgraph

Fig. 1.10

If S and T are two disjoint subsets of the vertex set V of a graph $G(V, E)$, we define $[S, T] = \{uv \in E : u \in S, v \in T\}$ and $q[S, T] = |[S, T]|$.

If $D = (V, A)$ is a digraph and S and T are disjoint subsets of V, we denote $(S, T) = \{uv \in A : u \in S$ and $v \in T\}$ and $q(S, T) = |(S, T)|$.

Complete graph: A graph of order n with all possible edges $\left(m = \dfrac{n(n-1)}{2}\right)$ is called a complete graph of order n and is denoted by K_n. A graph of order n with no edges is called an *empty graph* and is denoted by $\overline{K_n}$. Each graph of order n is clearly a spanning subgraph of K_n. Some examples of complete graphs are shown in Figure 1.11.

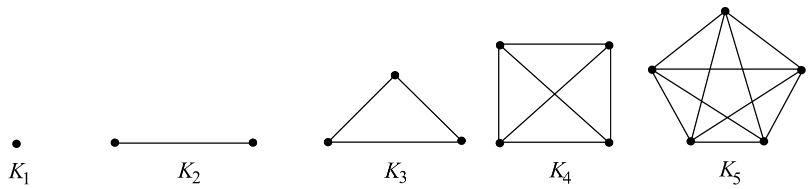


Fig. 1.11 *Complete graphs*

r-partite graph: A graph $G(V, E)$ is said to be r-partite (where r is a positive integer) if its vertex set can be partitioned into disjoint sets $V_1, V_2, \ldots, V_r$ with $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that uv is an edge of G if u is in some V_i and v in some V_j, $i \neq j$. That is, every one of the induced subgraphs $< V_i >$ is an empty graph. We denote r-partite graph by $G(V_1, V_2, \ldots, V_r, E)$.

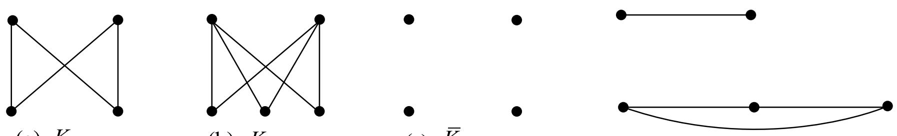
If an r-partite graph has all possible edges, that is, $uv \in E$, for every $u \in V_i$ and every $v \in V_j$, for all $i, j, i \neq j$, then it is called a *complete r-partite graph*. If $|V_i| = n_i$, we denote it by $K_{n_1, n_2, \ldots, n_r}$.

Bipartite graph: A graph $G(V, E)$ is said to be bipartite, or 2-partite, if its vertex set can be partitioned into two different sets V_1 and V_2 with $V = V_1 \cup V_2$, such that $uv \in E$ if $u \in V_1$ and $v \in V_2$. The bipartite graph is said to be *complete* if $uv \in E$, for every $u \in V_1$ and every $v \in V_2$. When $|V_1| = n_1$, $|V_2| = n_2$, we denote the complete bipartite graph by K_{n_1, n_2}. For example, $K_{2,2}$ and place $K_{2,3}$ are shown in Figure 1.12(a) and (b).

The complete bipartite graph $K_{1,n}$ is called an *n-star* or *n-claw*. For example, 3-star can be seen in Figure 1.12(e).

Complement of a graph: The complement $\overline{G}(V, \overline{E})$ of a graph $G(V, E)$ is the graph having the same vertex set as G, and its edge set $\overline{E}$ is the complement of E in $V_{(2)}$, that is, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. In Figure 1.12(c) and (d) the complements of K_4 and $K_{2,3}$ respectively are shown.

A graph G is said to be *self-complementary* if $\overline{G} \cong G$. The complement K_n of the complete graph of order n is clearly the empty graph of order n.



(a) $K_{2,2}$ (b) $K_{2,3}$ (c) $\overline{K}_4$ (d) $\overline{K}_{2,3}$

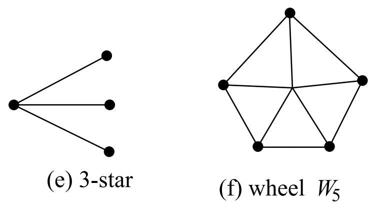


Fig. 1.12 *Complete bipartite graphs*

Removal of edges: Let $G(V,\ E)$ be a graph and let $F \subseteq E$. The graph $H(V, E - F)$ with vertex set V and edge set $E - F$ is said to be obtained from G by removing the edges in F. It is denoted by $G - F$. If F consists of a single edge e of G, the graph obtained by removing e is denoted by $G - e$.

Now, $G - F$ may contain isolated vertices which are not isolated vertices of G. The graph obtained by removing these newly created isolated vertices from $G - F$ is denoted by $G \setminus F$. Similarly, the graph obtained by removing isolated vertices from $G - e$ is denoted by $G \setminus e$. Figure 1.13 illustrates this operation.

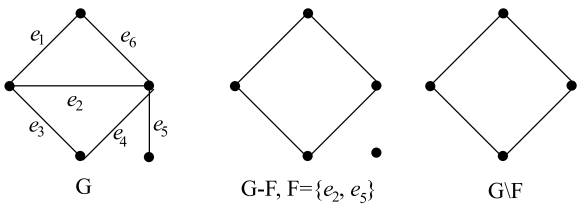


Fig. 1.13

Removal of vertices: Let $G(V,\ E)$ be a graph and let $v \in V$. Let E_v be the set of all edges of G incident with v. The graph $H(V - \{v\},\ E - E_v)$ is said to be obtained from G by the removal of the vertex v and is denoted by $G - v$.

If U is a subset of V, the graph obtained by removing the vertices of G which are in U is denoted by $G - U$. If H is a subgraph of G, we denote $G - V(H)$ by $G - H$, and $G - E(H)$ by $\overline{H}(G)$. Here $\overline{H}(G)$ is called the *relative complement* of H in G (Fig. 1.14).

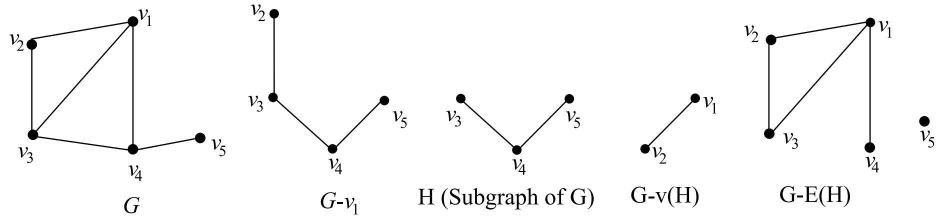


Fig. 1.14

Addition of edges: Let $G(V, E)$ be a graph and let f be an edge of $\overline{\overline{G}}$. The graph $H(V, E \cup \{f\})$ is said to be obtained from G by the addition of the edge f and is denoted by $G+f$. If F is a subset of edges of $\overline{\overline{G}}$, the graph obtained from G by adding the edges of F is denoted by $G+F$ (Fig. 1.15(a) and (b)).

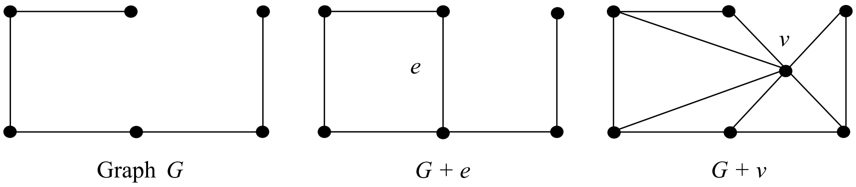


Fig. 1.15

Addition of vertices: Let $G(V, E)$ be a graph and let $v \notin V$. The graph H with vertex set $V \cup \{v\}$ and edge set $E \cup \{uv$, for all $u \in V\}$ obtained from G by adding a vertex v, is denoted by $G+v$. Thus $G+v$ is obtained from G by adding a new vertex and joining it to all vertices of G. For illustration, refer to Figure 1.15(c).

Join of graphs: Let $G(V, E)$ and $H(U, F)$ be two graphs with disjoint vertex sets ($V \cap U = \Phi$). The join of G and H denoted by GVH is the graph with vertex set $V \cup U$ and edge set E⊔ F ⊔[V, ⊔]. So the join is obtained from G and H by joining every vertex of G to each vertex of H by an edge. Clearly, $G+v = GVK_1 = GV\overline{K_1}$. This operation is illustrated in Figure 1.16.

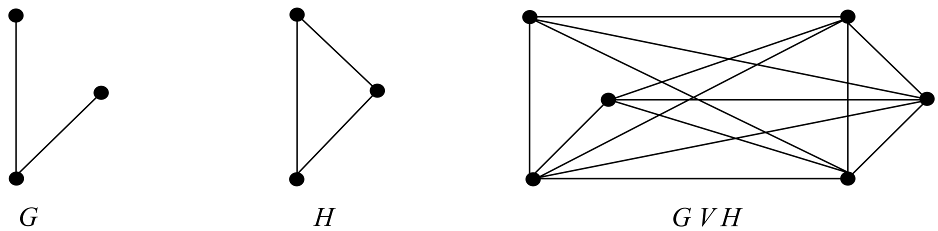


Fig. 1.16

1.5 Graph properties

Parametric property: A property P is called a parametric property of a graph if for a graph G having property P, every graph isomorphic to G also has property P. A graph with property P is denoted as P-graph. A subgraph H of a graph G is said to be *maximal* with respect to a property P (or P-maximal) if H is a P-graph, and there is no subgraph K of G having property P which properly contains H. So H is P-maximal if H is a P-graph, and for every $e \in E(G) - E(H)$, $H + e$ is not a P-graph, that is, the addition of any edge to H destroys the property P of H. A subgraph H of a graph G is said be *minimal* with respect to a property P (or P-minimal) if H is a P-graph and H has no proper subgraph K which is also a P-graph. So H is minimal if and only if H is a P-graph, and for every $e \in E(H)$, $H - e$ is not a P-graph, that is, if and only if the removal of any edge destroys the property P.

P-critical: A graph G is said to be P-critical if G is a P-graph and for every $v \in V$, $G - v$ is not a P-graph.

Hereditary property: A property P is said to be a hereditary property of a graph G if a graph G has the property P, then every subgraph of G also has the property P. It is called an *induced-hereditary property* if every induced subgraph of G also has the property.

Monotone property: A property P is said to be a monotone property of a graph G, if G has the property P, then for every $e \in E(\overline{G})$, $G + e$ also has the property P.

A vertex subset U of a graph G is said to be an *independent set* of G if the induced subgraph $< U >$ is an empty graph. An independent set of G with maximum number of vertices is called a *maximum independent set* (MIS) of G. The number of vertices in a *MIS* of G is called the *independent number* of G and is denoted by $\alpha_0(G)$.

A maximal complete subgraph of G is a *clique* of G. The order of a maximum clique of G is the *clique number* of G and is denoted by $w(G)$. Consider the graph given in Figure 1.17. Here $\{1, 4\}$ is independent set, $\{1, 3, 5\}$ is an *MIS*, $\alpha_0(G) = 3$, $\{1, 2, 6\}$ is a clique, $\{2,3,4,6\}$ is a maximum clique and $w(G) = 4$.

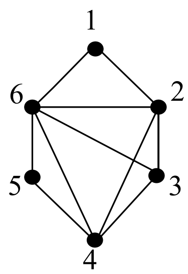


Fig. 1.17

The property of being a complete subgraph is an induced hereditary property. The property of being a planar graph is hereditary and the property of being a nonplanar graph is a monotone property.

1.6 Paths, Cycles and Components

The incidence relation I between the elements of V and the elements of E induced by the adjacency relation R further induces an adjacency relation among the edges, namely, two edges e and f are adjacent if and only if they have an end vertex in common. This relation is denoted by L. Two edges e and f of a graph $G(V, E)$ are adjacent if and only if $e = uv$ and $f = vw$, for some three vertices u, v, w. If e and f are adjacent, it is denoted by eLf.

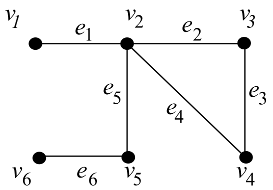
Walks: An alternating sequence of vertices and edges, beginning and ending with vertices such that no edge is traversed or covered more than once is called a walk. A vertex may appear more than once in a walk. $v_1e_1v_2e_2v_3\dots v_ke_kv_{k+1}$ is called a $v_1 - v_{k+1}$ walk W, v_1 and v_{k+1} are called the *initial* and *terminal vertices* of the walk W, and k, the number of edges in W is called *length* of W. The walk is said to be *open* if v_1 and v_{k+1} are distinct, and *closed* if $v_1 = v_{k+1}$. If v_r and v_s are two vertices in W, $s > r$, then the walk $v_re_rv_{r+1}e_{r+1}\dots e_{s-1}v_s$ is called a *subwalk* of W, or $v_r - v_s$ *section* of W, and is denoted by $Wv_r - v_s$. The walk $v_{k+1}e_kv_k\dots e_1v_1$ is called the *reverse walk* of W and is denoted by W^{-1}.

Paths: A path is an open walk in which no vertex (and therefore no edge) is repeated. A closed walk in which no vertex (and edge) is repeated is called a *cycle*. A path of length n is called an *n-path* and is denoted by P_n. A cycle of length n is called an *n-cycle* and is denoted by C_n. A loop is 1-cycle and a pair of edges joining two vertices form a 2-cycle. An n-cycle is proper only if $n \geq 3$.

As the edges and vertices in a path or cycle are not repeated, these are denoted by the sequence of vertices only. For example $u_1u_2\dots u_k$, where $u_i \in V$ is a $k-1$ path.

Two distinct vertices u and v of a graph G are said to be *connected* or *joined* if there is a $u - v$ walk in G. By convention, a vertex is connected to itself. A graph is said to be connected if every two of its vertices are connected, otherwise it is *disconnected*. The relation of connectedness is an equivalence relation on the vertex set V of a graph. The graphs induced on the equivalence classes of this relation are called the *components* of the graph.

Component of a graph: A maximal connected subgraph of a graph G is called a component of G. A component which is K_1 is called a *trivial component*. The number of components of a graph G is denoted by $k(G)$. A component of G with an odd (even) number of vertices is called an *odd (even) component* of G. The number of odd components of G is denoted by $k_0(G)$. Consider the graph shown in Figure 1.18. Here $v_1e_1v_2e_2v_3e_3v_4e_4v_2e_5v_5e_6v_6$ is a walk, $v_1e_1v_2e_2v_3e_3v_4$ is a path and $v_2e_2v_3e_3v_4e_4v_2$ is a 3-cycle.



Graph G

Fig. 1.18

Union of graphs: Let $G(V, E)$ and $H(U, F)$ be two graphs with $V \cap U = \Phi$. The union of G and H, denoted by $G \cup H$, is the graph with vertex set $V \cup U$ and edge set $E \cup F$. Clearly, when G and H are connected graphs, $G \cup H$ is a disconnected graph whose components are G and H (Fig. 1.19).

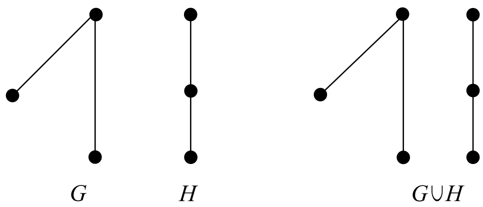


Fig. 1.19

A graph G which has k components, all isomorphic to H, is $H \cup H \cup \ldots \cup H$, or $G = kH$. We write $G = k_1 G_1 \cup k_2 G_2 \cup \ldots \cup k_r G_r$, if G has k_i components isomorphic to G_i, $1 \leq i \leq r$.

Theorem 1.5 A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2, such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2.

Proof Let G be a graph whose vertex set can be partitioned into two nonempty disjoint subsets V_1 and V_2, so that no edge of G has one end in V_1 and the other in V_2. Let v_1 and v_2 be any two vertices of G such that $v_1 \in V_1$ and $v_2 \in V_2$. Then there is no path between vertices v_1 and v_2, since there is no edge joining them. This shows that G is disconnected.

Conversely, let G be a disconnected graph. Consider a vertex v in G. Let V_1 be the set of all vertices that are joined by paths to v. Since G is disconnected, V_1 does not contain all vertices of G. Let V_2 be the set of the remaining vertices. Clearly, no vertex in V_1 is joined to any vertex in V_2 by an edge, proving the converse. ❑

Theorem 1.6 If a graph has exactly two vertices of odd degree, they must be connected by a path.

Proof Let G be a graph with all its vertices of even degree, except for v_1 and v_2 which are of odd degree. Consider the component C to which v_1 belongs. Then C has an even number of vertices of odd degree. Therefore C must contain v_2, the only other vertex of odd degree. Thus v_1 and v_2 are in the same component, and since a component is connected, there is a path between v_1 and v_2. ❑

Lemma 1.1 For any set of positive integers $n_1, n_2, \ldots, n_k$

$$\sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i \right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k \right).$$

Proof We have, $\sum\limits_{i=1}^{k} (n_i - 1) = \sum\limits_{i=1}^{k} n_i - k$.

Squaring both sides, we get

$$\left[\sum_{i=1}^{k}(n_i-1)\right]^2 = \left[\sum_{i=1}^{k} n_i - k\right]^2,$$

or $[(n_1-1)+(n_2-1)+\ldots+(n_k-1)]^2 = \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2,$

or $(n_1-1)^2+(n_2-1)^2+\ldots+(n_k-1)^2 + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ i\neq j}}^{k} (n_i-1)(n_j-1) = \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2.$

Therefore, $\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ i\neq j}}^{k} (n_i-1)(n_j-1) = \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2,$

or $\quad \sum_{i=1}^{k} n_i^2 \leq 2\sum_{i=1}^{k} n_i - k + \left(\sum_{i=1}^{k} n_i\right)^2 - 2k\sum_{i=1}^{k} n_i + k^2,$

or $\quad \sum_{i=1}^{k} n_i^2 \leq \left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right).$ ❑

Theorem 1.7 A graph with n vertices and k components cannot have more than $\frac{1}{2}(n-k)(n-k+1)$ edges.

Proof The number of components is k and let the number of vertices in ith component be n_i, $1 \leq i \leq k$.

So, $\quad n_1+n_2+\ldots+n_k = \sum_{i=1}^{k} n_i = n.$

Now, a component with n_i vertices will have the maximum possible number of edges when it is complete, and in that case it has $\frac{1}{2}n_i(n_i-1)$ edges.

Thus the maximum number of edges in $G = \frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \leq \frac{1}{2}\left\{\left(\sum_{i=1}^{k} n_i\right)^2 - (k-1)\left(2\sum_{i=1}^{k} n_i - k\right)\right\} - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$= \frac{1}{2}\left\{n^2-(k-1)(2n-k)\right\} - \frac{1}{2}n = \frac{1}{2}\left[n^2-2nk+k^2+n-k\right] = \frac{1}{2}(n-k)(n-k+1).$$ ❑

The following result is due to Turan [247].

Theorem 1.8 The maximum number of edges among all n vertex graphs with no triangles is $\left[\frac{n^2}{4}\right]$, where $\left[\frac{n^2}{4}\right]$ is the greatest integer not exceeding the number $\frac{n^2}{4}$.

Proof The result is proved by induction, taking the cases of n odd and n even separately.

Let n be even. The result is obvious for small values of n. Assume the result to be true for all even $n \le 2p$. We then prove it for $n = 2p+2$. So assume G is a graph with $n = 2p+2$ vertices and no triangles. Since G is not totally disconnected, there are adjacent vertices u and v. The subgraph $G' = G - \{u, v\}$ has $2p$ vertices and no triangles, so that by the induction hypothesis, G' has at most $\left[\frac{4p^2}{4}\right] = p^2$ edges.

There can be no vertex w such that u and v are both adjacent to w, for then u, v and w form a triangle of G. Therefore, if u is adjacent to k vertices of G', v is adjacent to at most $2n-k$ vertices. Then G has at most

$$p^2 + k + (2p-k) + 1 = p^2 + 2p + 1 = (p+1)^2 = \frac{n^2}{4} = \left[\frac{n^2}{4}\right] \text{ edges.}$$

It can be shown that for all even p, there exists a $(p, \frac{p^2}{4})$ graph with no triangles. Such a graph is formed as follows. Take two sets V_1 and V_2 of $\frac{p}{2}$ vertices each, and join each vertex of V_1 with each vertex of V_2. ❑

Example When $n = 6$, the graph formed is shown in Figure 1.20.

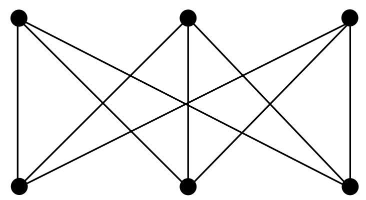


Fig. 1.20

The following result due to Konig [136] characterises bipartite graphs.

Theorem 1.9 A graph is bipartite if and only if it contains no odd cycles.

Proof Let G be a bipartite graph. Then its vertex set V can be partitioned into two sets V_1 and V_2, so that every edge of G joins a vertex of V_1 with a vertex of V_2. Thus every cycle $v_1 v_2 \ldots v_n v_1$ in G has its oddly subscripted vertices in V_1, say, and the others in V_2, so that its length is even.

Conversely, we assume, without loss of generality, that G is connected, for otherwise we can consider the components of G separately. Take any vertex $v_1 \in V$, and let V_1 consist of v_1 and all vertices of G whose shortest paths from v_1 (in terms of the number of edges) contain an even number of edges, while $V_2 = V\_V_1$. Since G has no odd cycles, every edge of G joins a vertex of V_1 with a vertex of V_2 (Fig. 1.21).

For suppose there is an edge uv joining two vertices of V_1. The length of v_1 to u is even, the length of v_1 to v is even and the union of edges from v_1 to u and from v_1 to v together with the edge uv contains an odd cycle, which is a contradiction. Hence G is bipartite. ❑

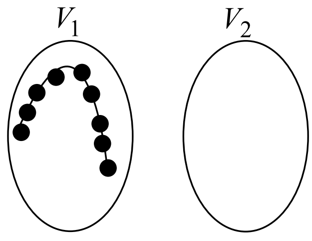


Fig. 1.21

Theorem 1.10 A graph and its complement are not both disconnected.

Proof Let G be a disconnected graph and u, v be any two vertices of $\overline{G}$, the complement of G. If u and v belong to different components of G, then u and v are adjacent in $\overline{G}$, by definition. If u and v belong to the same component, say G_i of G, then let w be a vertex of some other component, say G_j of G. By definition both u and w, and v and w are adjacent in $\overline{G}$. In either case u is connected to v by a path in $\overline{G}$. Thus $\overline{G}$ is connected. ❑

Theorem 1.11 For any graph G with six vertices, G or $\overline{G}$ contains a triangle.

Proof Let v be a vertex of a graph G with six vertices. Since v is adjacent either in G or $\overline{G}$ to the other five vertices of G, we assume without loss of generality, that there are three vertices u_1, u_2 and u_3 adjacent to v in G. If any two of three vertices are adjacent, then they are two vertices of a triangle whose third vertex is v. If no two of them are adjacent in G, then u_1, u_2 and u_3 are the vertices of a triangle in $\overline{G}$ (Fig. 1.22). ❑

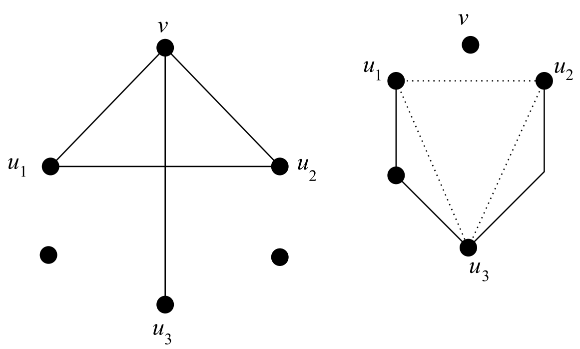


Fig. 1.22

Theorem 1.12 For any graph G of order n, $\Delta(G) \leq n-1$.

Proof Since a vertex in a graph can be joined to at most $n-1$ other vertices, the maximum degree a vertex can have is $n-1$. So, $\Delta(G) \le n-1$. ❑

Theorem 1.13 Every graph G contains a bipartite spanning subgraph whose size is at least half the size of G.

Proof Let B be a bipartite spanning subgraph of G with B being of maximum size and the bipartition of its vertex set be $V = V_1 \cup V_2$. Clearly, all the edges of G with one end in V_1 and the other in V_2 must be in B.

Now, if any vertex $u \in V_1$ is joined to p vertices of V_1, then u is joined to at least p vertices of V_2. That is, the number of vertices of V_2 joined to u is greater or equal to the number of vertices of V_1 joined to u (Fig. 1.23).

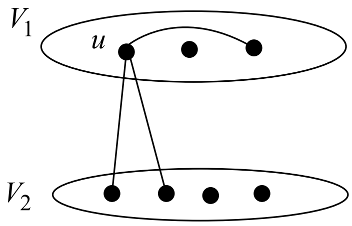


Fig. 1.23

For if $u \in V_1$ is joined to more vertices of V_1 than of V_2, then $V_1 - \{u\}$, $V_2 \cup \{u\}$ is a bipartition of V giving a bipartite subgraph of G with larger size, which is a contradiction to our assumption (Fig. 1.24).

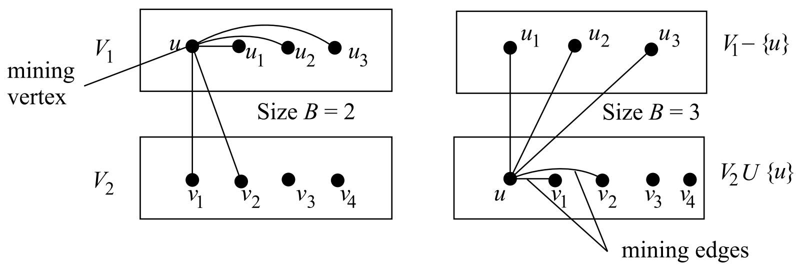


Fig. 1.24

Therefore, $d(u|B) \geq \frac{1}{2}d(u|B)$, for each $u \in V$.

Summing over all $u \in V$, we get $m(B) \geq m(G)$. ❑

Theorem 1.14 If G is a graph with at least n_0 vertices and at least $n_0.n(G) - \binom{n_0+1}{2} + 1$ edges, then G contains a subgraph H with $\delta(H) \geq n_0 + 1$, where $n(G)$ is the number of vertices in G and $n(G) \geq n_0$.

Proof Let G_{n_0} be the set of all such graphs G with

$$G_{n_0} = \left\{ G : n(G) > n_0,\ m(G) \geq n_0.n(G) - \binom{n_0+1}{2} + 1 \right\}.$$

If $G \in G_{n_0}$, then $n(G) > no$, because if $n(G) = no$, then

$$m(G) \geq n_0.n(G) - \binom{n_0+1}{2} + 1 \text{ gives } m(G) \geq n_0.n_0 - \binom{n_0+1}{2} + 1.$$

Therefore, $m(G) \geq n_0^2 - \frac{(n_0+1)n_0}{2} + 1 = n_0^2 - \left\{ n_0^2 - \frac{n_0(n_0-1)}{2} \right\} + 1$

$$= \frac{(n_0-1)n_0}{2} + 1 = \binom{n_0}{2} + 1.$$

Thus, $m(G) \geq \binom{n_0}{2} + 1$, which is a contradiction, as the number of edges can be at most $\binom{n_0}{2}$.

Now, if $\delta(G) \leq n_0$, let u be a vertex with $d(u) \leq n_0$. Then it is easily verified that $G-u$ also belongs to G_{n_0}. By repeating this process of removing vertices with degree smaller than n_0+1 (i.e., $\leq n_0$), we arrive at a graph $H \in G_{n_0}$ with $\delta(H) = n_0+1$. ❑

Example Consider the graph G of Figure 1.25. Here $n(G) = 6$. Let $n_0 = 3$. Clearly, $m(G) \geq 3 \times 6 - \binom{4}{2} + 1 = 13$. In $G-u$, we have $n(G-u) = 5$ and again let $n_0 = 3$. Then $m(G-u) \geq 3 \times 5 - \binom{4}{2} + 1 = 10$, and thus satisfies given conditions. Hence $G-u \in G_{n_0}$.

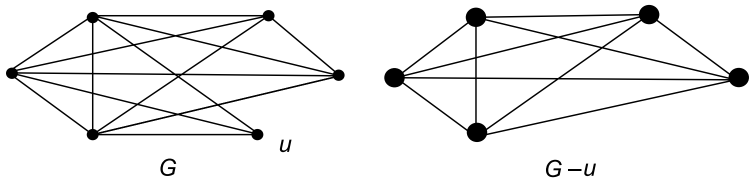


Fig. 1.25

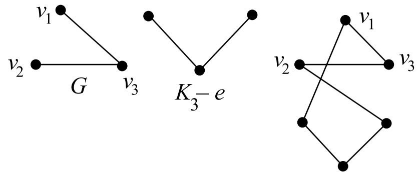
Theorem 1.15 Any graph G has a regular supergraph H of degree $\Delta(G)$ such that G is an induced subgraph of H.

Proof We have to prove that for a graph G, there exists a graph H which is regular and the degree of every vertex of H is $\Delta(G)$, and H is the supergraph of G, $\Delta(G)$ being the maximum degree in G. If G is regular, there is nothing to prove.

Now let G be not regular and $\Delta = \Delta(G)$ be the maximum degree in G. Let d_i be the degree of the vertex v_i in G. Let $d = \sum_{i=1}^{n} (\Delta - d_i)$.

Case 1 If d is even, take $\frac{d}{2}$ disjoint copies of the graph $K_{\Delta+1}-e$ (where e is an edge). For $1 \le i \le n$, join v_i to $\Delta - d_i$ different vertices from these copies having degree $\Delta - 1$. We get a graph H which is Δ-regular. Clearly, G is an induced subgraph of H.

Example Consider the graph of Figure 1.26. Here $\Delta = 2$, $d = 2-1+2-1+2-2 = 2$ (even). We take $\frac{d}{2} = \frac{2}{2} = 1$ copy of $K_{2+1} - e = K_3 - e$. Join v_i to $2 - d_i$ different vertices from this copy having degree $2-1 = 1$. Join v_1 to $2-1 = 1$ vertex of this copy having degree 1. Join v_2 to $2-1 = 1$ vertex.

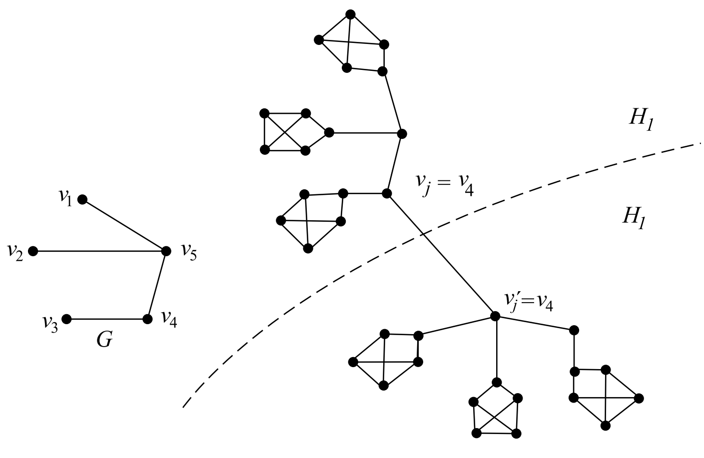


Graph H, which is 2 – regular

Fig. 1.26

Case 2 If d is odd, take $\frac{(d-1)}{2}$ disjoint copies of the graph $K_{\Delta+1} - e$. For $1 \le i \le n$, join v_i to $\Delta - d_i$ different vertices of degree $\Delta - 1$ from these copies. This can be done for all but one vertex of G, say v_j. Then $d(v_j) = \Delta - 1$ in the graph H_1 so constructed. Take another copy of H_1 say H_1' with $d(v_j') = \Delta - 1$. Then H is obtained by joining v_j and v_j' by an edge. Hence H is $\Delta -$ regular and G is an induced subgraph of H. ❑

Example Consider the graph G of Figure 1.27. Here $\Delta = 3$, $d = 3-1+3-1+3-1+3-2 = 2+2+2+1 = 7$, $\frac{(d-1)}{2} = \frac{(7-1)}{2} = 3$. We take 3 copies of $K_{3+1} - e = K_4 - e$.



Graph H, which is 3 – regular

Fig. 1.27

Theorem 1.16 If $G(X, Y, E)$ is a bipartite graph with maximum degree Δ, then G has a Δ-regular bipartite supergraph $H(U, V, F)$ with $X \subseteq U$, $Y \subseteq V$, $E \subseteq F$.

Proof Let Δ be the maximum degree in $G(X, Y, E)$. Take Δ copies of G (each isomorphic to G) say $G_i(X_i, Y_i, E_i)$, $1 \leq i \leq \Delta$. For each $x \in X_1$ with $d(x) < \Delta$, take $r = \Delta - d(x)$ vertices say $x_1, x_2, \ldots, x_r$. Let Y_o be the set of all such vertices as x ranges over X_1 and let $V = Y_o \cup Y_1 \cup \ldots \cup Y_\Delta$. Join each vertex x_i to one vertex $x(d(x) < \Delta)$ from each of the copies X_1, $X_2, \ldots, X_\Delta$.

Similarly, for each $y \in Y_1$ with $d(y) \leq \Delta$, take $s = \Delta - d(y)$ vertices say $y_1, y_2, \ldots, y_s$. Let X_o be that set of all such vertices as y ranges over Y_1 and let $U = X_0 \cup X_1 \cup \ldots \cup X_\Delta$. Join each vertex y_j to one vertex $y(d(y) < \Delta)$ from each of the copies $Y_1, Y_2, \ldots, Y_\Delta$. The resulting bipartite graph H has clearly Δ induced bipartite subgraphs G_i each isomorphic to G (Fig. 1.28). ❑

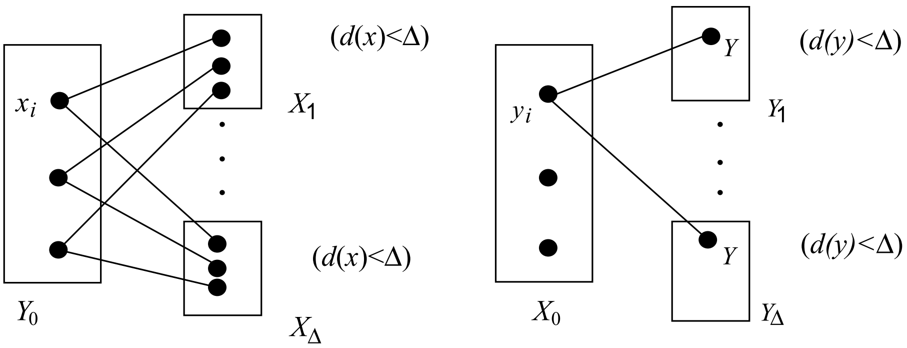


Fig. 1.28

Example 1 Consider the graph G of Figure 1.29. Here $\Delta = 3$, $d(u) = 2 < 3$, $d(x) = 2 < 3$, $d(y) = 2 < 3$, $d(z) = 1 < 3$. Now, $r = \Delta - d(u) = 3 - 2 = 1$ and $s = \Delta - d(y) + \Delta - d(y) + \Delta - d(z) = 3 - 2 + 3 - 2 + 3 - 1 = 4$.

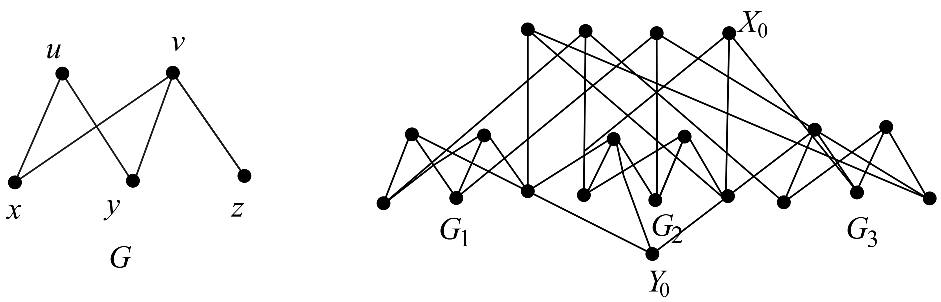


Fig. 1.29

Example 2 In the graph G of Figure 1.30, $\Delta = 2$ and $d(v) = 0 < 2$, $d(x) = 1 < 2$, $d(y) = 1 < 2$ so that $r = \Delta - d(v) = 2 - 0 = 2$ and $s = 2 - 1 + 2 - 1 = 2$.

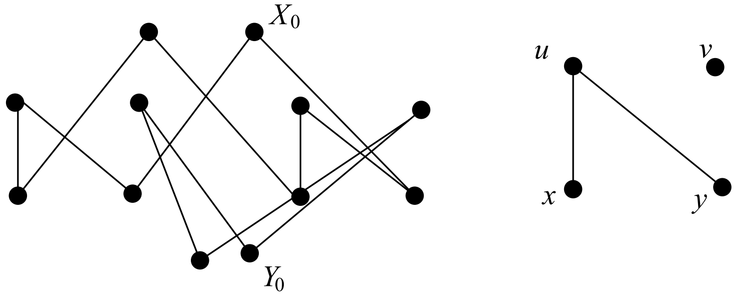


Fig. 1.30

1.7 Operations on Graphs

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. The union of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. The *intersection* of G_1 and G_2, denoted by $G_1 \cap G_2$, is the graph consisting only of those vertices and edges that are both in G_1 and G_2. Clearly, $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

The *ring sum* of G_1 and G_2, denoted by $G_1 \oplus G_2$, is a graph whose vertex set is $V_1 \cup V_2$ and whose edges are that of either G_1 or G_2, but not of both. Examples of union, intersection and ring sum are given in Figure 1.31.

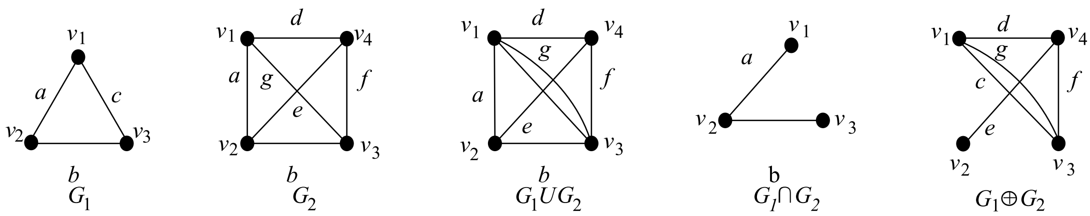


Fig. 1.31

We observe that these three operations are commutative, that is,

$$G_1 \cup G_2 = G_2 \cup G_1,\ G_1 \cap G_2 = G_2 \cap G_1,\ G_1 \oplus G_2 = G_2 \oplus G_1.$$

If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph and $G_1 \oplus G_2 = G_1 \cup G_2$. If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty. For any graph G, $G \cup G = G \cap G = G$ and $G \oplus G$ = null graph.

If H is a subgraph of G, then $G \oplus H$ is by definition, that subgraph of G which remains after all the edges in H have been removed from G. We write, $G \oplus H = G - H$, whenever $H \subseteq G$. $G \oplus H = G - H$ is also called complement of H in G. Figure 1.32 illustrates this operation.

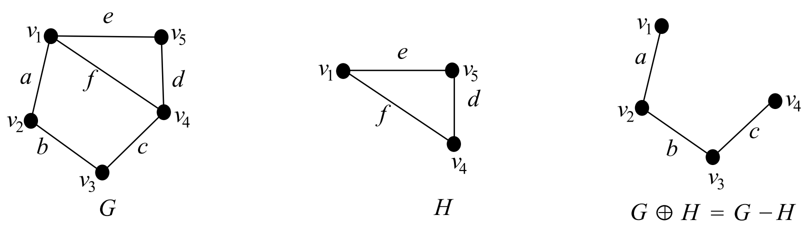


Fig. 1.32

Decomposition: A graph G is said to be decomposed into two subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 =$ a null graph. In other words, every edge of G occurs either in G_1 or in G_2, but not in both, while as some of the vertices can occur in both G_1 and G_2. In decomposition, isolated vertices are disregarded. In Figure 1.33, graph G has been decomposed into subgraph G_1 and G_2.

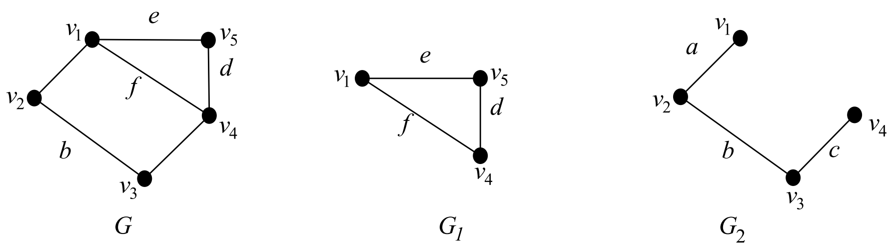


Fig. 1.33

Deletion: Let G be a graph and v be any vertex in G. Then $G - v$ denotes the subgraph of G by deleting vertex v, and all the edges of G which are incident with v. An example of deletion of a vertex is given in Figure 1.34.

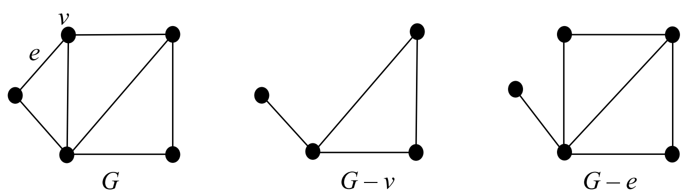


Fig. 1.34

If e is any edge of G, then $G - e$ is a subgraph of G obtained by deleting e from G (Fig. 1.34). Deletion of an edge does not imply deletion of its end vertices. Therefore $G - e = G \oplus e$.

Fusion: A pair of vertices u and v in a graph are said to be fused (merged or identified) if u and v are replaced by a single new vertex such that every edge incident on u or v is incident on this new vertex. Therefore, fusion of vertices does not alter the number of edges, but reduces the number of vertices by one. In Figure 1.35, vertices u and v are fused to a single vertex w.

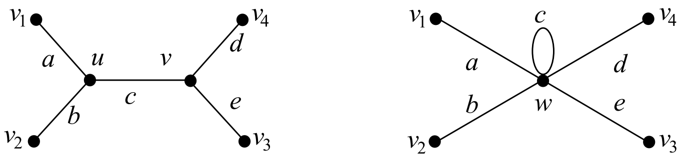


Fig. 1.35

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \Phi$ and $E_1 \cap E_2 = \Phi$. Let G be the compound graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup E_3$, where E_3 is a subset of the set of edges $\{v_i v_j : v_i \in V_1,\ v_j \in V_2\}$. We represent E_3 by the symmetric binary relation $\pi \subseteq V_1 \times V_2$, and the compound graph G by $G_1\ \pi\ G_2$. If $\pi = \Phi$, then we get the union of G_1 and G_2, denoted by $G_1 \cup G_2$. If $\pi = V_1 \times V_2$ (that is all edges between V_1 and V_2), we get the join of G_1 and G_2, denoted by $G_1 \vee G_2$. If π is a function from V_1 to V_2, we have the *function graph* G_1 f G_2. If π defines a homomorphism Φ from G_1 to G_2, we have the *homomorphism graph* $G_1\ \Phi\ G_2$. If $G \cong G_2 = G$ (say) and π is a bijection α of V_1 to V_2, we have a *permutation* graph $G\ \alpha\ G$. This is also denoted by $P_\alpha(G)$. If $G_1 \cong G_2 = G$ (say) and π is an automorphism α of G, then $G\alpha G$ is an automorphism graph. These operations are illustrated in Figure 1.36.

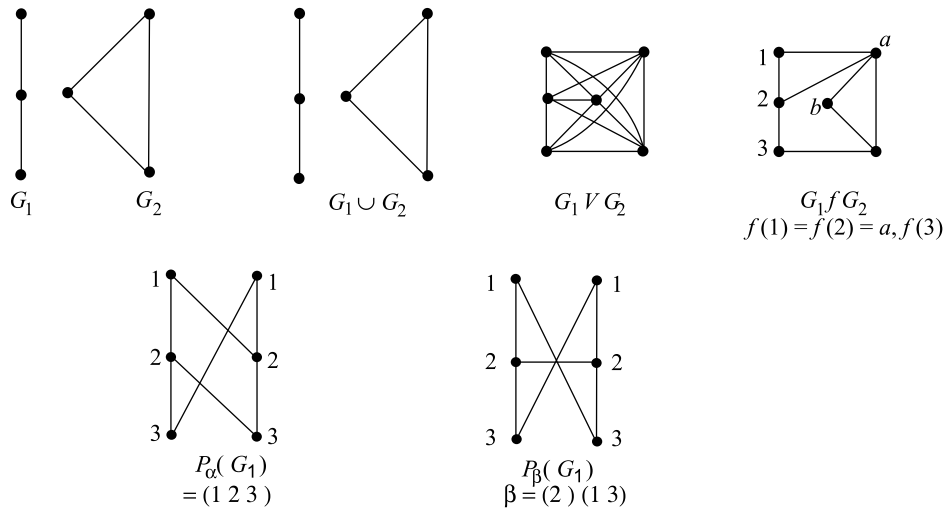


Fig. 1.36

Let G_1 (V_1, E_1) and G_2 (V_2, E_2) be two graphs, and let the vertex sets V_1 and V_2 be labelled by the same set of labels. We note here that each of V_1 and V_2 need not have all the labels.

The *sum* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph $G = (V, E)$, where $V = V_1 \cup V_2$ and an edge $v_iv_j \in E$ if and only if v_iv_j is an edge of G_1 or G_2 or both. The *direct sum* denoted by $G_1(+)G_2$ is the graph $G(V, E)$ with $V = V_1 \cup V_2$ and an edge $v_iv_j \in E$ if and only if v_iv_j is an edge of G_1 or G_2, but not of both. The *superposition* graph $G = G_1$ s G_2 has vertex set $V = V_1 \cup V_2$ and the edge set E contains all the edges of G_1 and G_2 with the identity of edges of G_1 and G_2 in G being preserved by assigning two different labels to these edges. So, if v_iv_j is an edge in both G_1 and G_2, then there are two edges v_iv_j in G with different labels. For illustration of these operations, see Figure 1.37. We can extend these operations to a finite number of graphs.

We form the compound graphs G from G_1 and G_2 with vertex sets $V_1 \times V_2$ as follows.

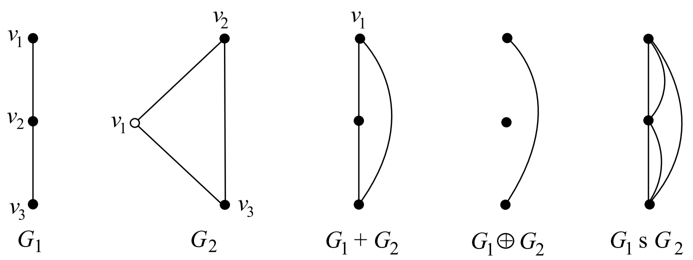


Fig. 1.37

The cartesian product of G_1 and G_2, denoted by $G = G_1 \times G_2$, is the graph whose vertex set is $V = V_1 \times V_2$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V, $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, there is an edge $w_1w_2 \in E(G)$ if and only if either (a) $u_1 = u_2$ and $v_1v_2 \in E_2$, or (b) $v_1 = v_2$ and $u_1u_2 \in E_1$.

The *tensor product* (conjunction), denoted by $G = G_1 \wedge G_2$, is the graph with vertex set $V = V_1 \times V_2$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V; $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, there is an edge $w_1w_2 \in E(G)$ if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

The *normal product* (strong product), denoted by $G = G_1 \circ G_2$, is the graph with vertex set $V = V_1 \times V_2$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in V; $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, there is an edge $w_1w_2 \in E(G)$ if and only if one of the following holds:

(a) $u_1 = u_2$ and $v_1v_2 \in E_2$ (b) $v_1 = v_2$ and $u_1u_2 \in E_1$ (c) $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

It is easy to see that $G_1 \times G_2$ and $G_1 \wedge G_2$ are spanning subgraphs of $G_1 \circ G_2$ and also $G_1 \circ G_2 = (G_1 \times G_2) \oplus (G_1 \wedge G_2)$. Figure 1.38 illustrates these operations.

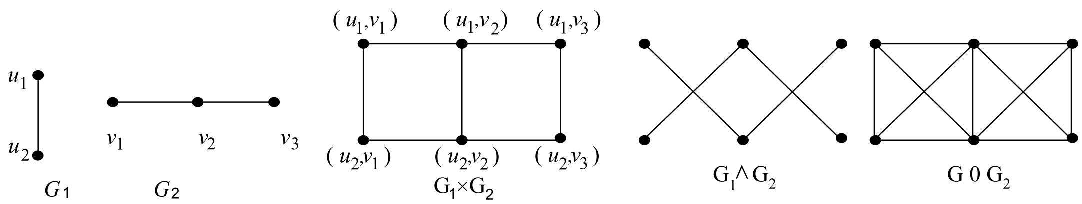


Fig. 1.38

The *composition* of G_1 and G_2, denoted by $G = G_1[G_2]$, is the graph with vertex set $V = V_1 \times V_2$, and for any two vertices $w_1 = (u_1,\ v_1)$ and $w_2 = (u_2,\ v_2)$ in V; $u_1,\ u_2 \in V_1$ and $v_1,\ v_2 \in V_2$, there is an edge $w_1w_2 \in E(G)$ if and only if either (a) $u_1u_2 \in E_1$ or (b) $u_1 = u_2$ and $v_1v_2 \in E_2$ (Fig. 1.39).

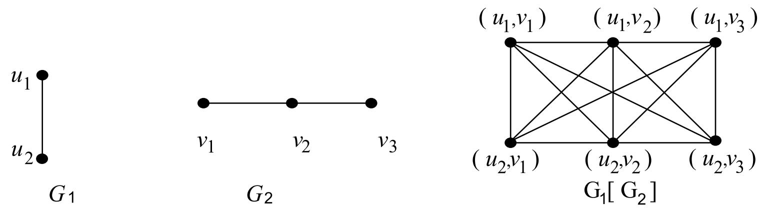


Fig. 1.39

1.8 Topological Operations

Definition: A *subdivision* of the edge $e = uv$ of a graph G is the replacement of the edge e by a new vertex w and two new edges uw and wv. This operation is also called an *elementary subdivision* of G (Fig. 1.40).

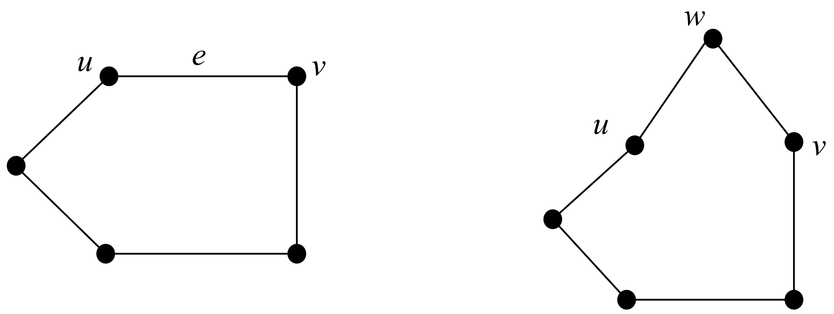


Fig. 1.40

A graph H obtained by a sequence of elementary subdivisions from a graph G is said to be a *subdivision graph* of G, or to be homeomorphic (or a homeomorph) of G. Two graphs H_1 and H_2 which are homeomorphs of the same graph G are said to be *homeomorphic* to each other (or homeomorphically reducible to G). A graph G is *homeomorphically irreducible* if whenever a graph H is homeomorphic to G, then H is homeomorphic from G. In Figure 1.41, H_1 and H_2 are homeomorphs of K_4 and K_4 itself is homeomorphically irreducible.

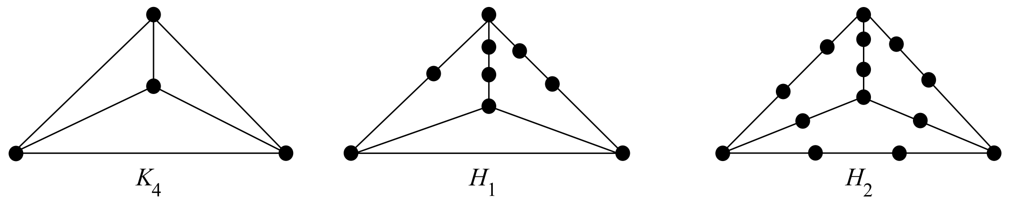


Fig. 1.41

The relation being homeomorphic to each other is an equivalence relation and each equivalence class contains a unique homeomorphically irreducible graph which is taken as the representative of the class.

Identification of the vertices: Let $G(V, E)$ be a general graph and let U be a subset of the vertex set V. Let H be the general graph obtained from G by the following operations.

i. Replace the set of vertices U by a single new vertex u.
ii. Replace the edge $e = ab$ with $a \in U$ and $b \in V - U$ by a corresponding edge $e' = ub$.
iii. Replace each edge $e = ab$ with $a, b \in U$ (b possibly being the same as a) by a loop at u.

Let K be the multigraph obtained from H by dropping all loops and L be the graph obtained from K by replacing each multiple edge by a single edge. Then H, K, L are respectively said to be obtained from G by a *general, multiple, simple identification* of the vertices of U. We write $H = G * U$, $K = G : U$, $L = G.U$. This operation is illustrated in Figure 1.42.

Remarks

1. The operation in (ii) may result in the generation of multiple edges and that in (iii), in the generation of loops even when G is a simple graph.

2. For $b \neq u$ in H, $q_H[a, b] = \sum\{q_G[a, b] : a \in U\}$.

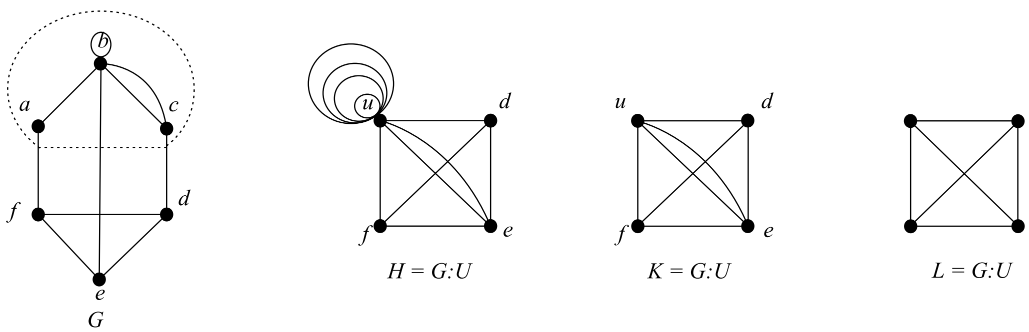


Fig. 1.42

Coalescence of G: Let $G(V, E)$ be a general graph and $V = U_1 \cup U_2 \cup \ldots \cup U_r$ be a partition π of the vertex set V with $U_i \cap U_j \neq \Phi$, $i \neq j$. Let $H(U, E')$ be the general graph defined by replacing

i. each U_i by a single new vertex u_i,

ii. each edge $e = v_i v_j$ with $v_i \in U_i$ and $v_j \in U_j$ $(i \neq j)$ by a corresponding edge $e' = u_i u_j$ and

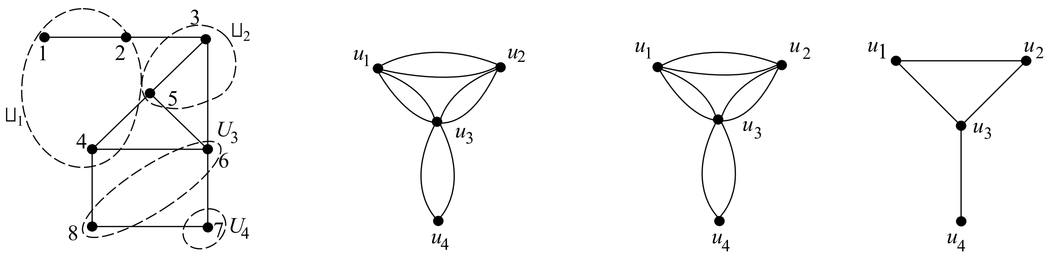
iii. each edge $e = ab$ with $a, b \in U_i$ by a loop at u_i.

Let K be the multigraph obtained from H by dropping the loops and L be the graph obtained from K by replacing the multiple edges by single edges. Then H, K, L are respectively called the *general/multiple/simple coalescence* of G and we write $H = G*\Pi, K = G:\Pi$ and $L = G.\Pi$. Coalescence of a graph is illustrated in Figure 1.43.

Remarks

1. Clearly H is obtained from G by sequentially identifying the vertices in U_i, $1 \leq i \leq r$ in any order.

2. To coalescence a partition π_1 of a proper subset V_1 of V, we may adopt the above definition by augmenting π_1 to a partition of V by adjoining the singleton sets corresponding to the vertices in $V - V_1$.



G G : π G : π G : π

Fig. 1.43

Let $G(V, E)$ be a general graph and $f : V \to W$ be a map of V onto $W = \{w_1, w_2, \ldots, w_r\}$. Let $U_i = f^{-1}(w)$ and $V = \bigcup_{i=1}^{r} U_i$ be the partition of V induced by f. Then the general/multiple/simple coalescences of G are denoted by $f_g(G), f_m(G), f_s(G)$ or $G*f, G:f, G.f$, respectively, and are referred to as a *general/multiple/simple homomorph* or homomorphic image of G. We will use $f(G)$ as a common symbol for these and may often abbreviate this to fG.

Let $M(W', E')$ be a general/multiple/simple graph with $W \subseteq W'$ such that $f(G)$ is a general/multiple/simple subgraph of M. Then f is said to be a *homomorphism* of G into M. If $W' = W$, then f is said to be an *onto (surjective) homomorphism*. If $F(G) =< f(V) >$, then f is called a *full homomorphism.* In case $f(G) = M$, then f is called a *full onto homomorphism.*

Remarks

1. An injective homomorphism (i.e., f is injective) is a *monomorphism.* A full onto monomorphism is an *isomorphism*.

2. If every U_i is an independent set in G, then f is a discrete homomorphism. If $U_1 = \{u,\, v\}$ and other U_i's are singleton sets of V, then f is called an *elementary homomorphism*, and if $uv \notin E$, then f is a *discrete elementary homomorphism*.

3. If every $< U_i >$ is a connected subgraph of G, then f is called *connected homomorphism.* An elementary homomorphism with $uv \in E$ is called a *connected elementary homomorphism*.

4. A mapping $f : V \to W'$, where $G(V,\, E)$ and $M(W',\, E')$ are simple graphs, is a homomorphism of G into M if and only if $uv \in E$ implies that $f(u)f(v) \in E'$.

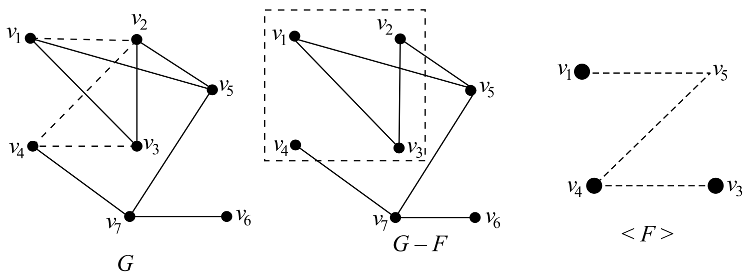
The following result is due to Ore [178].

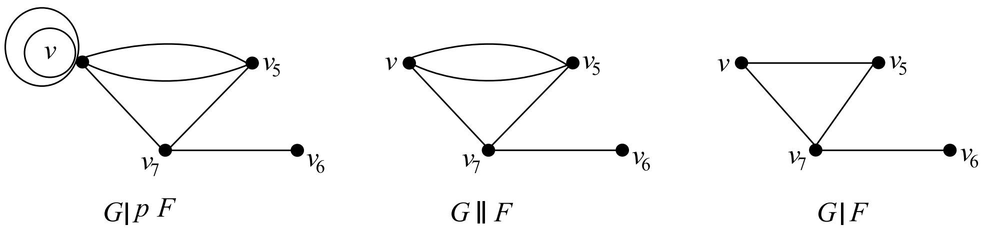
Theorem 1.17 Any homomorphism is the product of a connected and a discrete homomorphism.

Proof Let f be a homomorphism of $G(V,\, E)$ into $M(W',\, E')$ with $f(V) = W$ (i.e., $f(V_i) = W_i$). So, $V = f^{-1}(W)$ and $V_i = f^{-1}(W_i)$. Then each $< f^{-1}(W_i) >$ can consist of a number of components (which are, in fact, connected) $C_{i_1},\, C_{i_2},\, \ldots,\, C_{i_j},\, \ldots,\, C_{i_{n_i}}$ with vertex sets $V_{i_1},\, V_{i_2},\, \ldots,\, V_{i_j},\, \ldots,\, V_{i_{n_i}}$.

The coalescence of $V_i = f^{-1}(W_i)$ can be done in two stages. First, we perform a multiple coalescence of $V_i = V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_j} \cup \ldots \cup V_{i_{n_i}}$ sending vertex set V_{i_j} to a single vertex W_{i_j}. This, when done for all $V_i = f^{-1}(W_i)$'s, corresponds to a connected homomorphism Φ of G. Next, identify the vertices W_{i_j}, $1 \le j \le n_i$ of $\Phi(G)$ into a single vertex W_i for $1 \le i \le |W|$. This corresponds to a discrete homomorphism θ of $\Phi(G)$ into M. Clearly, $f(G) = \theta(\Phi(G))$. ❑

Contractions: Let $G(V,\, E)$ be a general graph and F be a subset of E such that the edge-induced subgraph $< F >$ is connected on the vertex subset $U = V(< F >)$. Then the general/multiple/simple graphs obtained by the identification of the vertex set U' in the general graph $G - F$ are called the *general/multiple/simple contractions* of F in G and are denoted by $G|\,\text{þ}\,F$, $G||F$ and $G|F$ respectively. Consider the graph of Figure 1.44. The vertices of $U = \{v_1,\, v_2,\, v_3,\, v_4\}$ in $G - F$ have been identified.





Identification of U = $\{v_1\ v_2, v_3, v_4\}$ in G–F

Fig. 1.44

Let $G(V,\ E)$ be a general graph and let F be a subset of E such that the edge-induced subgraph $< F >$ has components $< F_i >$ on vertex sets $V(< F_i >) = U_i$, $1 \le i \le r$. Let π be the partition of V induced by the U_i, vertices in U_i being taken as singleton sets of the partition.

The general/multiple/simple coalescences in the general graph $G - F$ are called the *general/multiple/simple contractions* of F in G and are denoted by $G|\,\text{þ}F$, $G||F$ and $G|F$ respectively.

Let $G(V,\ E)$ be a general graph and let $W = \{w_1,\ w_2,\ \ldots,\ w_r\}$. Suppose $f : V \to W$ is a map such that $< U_i >$ is a connected general subgraph of G, where $U_i = f^{-1}(w_i)$, $1 \le i \le r(f(U_i) = w_i)$. Let $F_i \subseteq E(U_i)$ be such that F_i induces a spanning connected subgraph of $< U_i >$ and let $F = \bigcup_{i=1}^{r} F_i$. Then $G|\,\text{þ}F$, $G||F$ and $G|F$ are respectively called general/ multiple/simple contraction induced by f and F and are denoted by $f_F(G)$. Here f is called a *contraction mapping* of G.

If $f_F(G) = H$, then G is said to be $g/m/s$- contractible to H and H is called a $g/m/s$-*contraction* of G, according as H is a $g/m/s$-graph. We call *simple contractible* and *simple contraction* as *contractible* and *contraction*.

If a subgraph of G is contractible to H (equivalently, if H is a subgraph of a contraction of G), then H is called a *subcontraction* of G, and we also say that G is *subcontractible* to H. We denote this by $G\}H$.

Remarks

1. It can be observed that corresponding to every general contraction there exists a unique contraction mapping f but given a mapping f from $V(G)$ to $V(H)$, with G and H being general graphs, there can be various contractions of G into H defined by suitable edge subsets of $\bigcup_{i=1}^{k} < f^{-1}(w_i) >$. Any two such contractions of the same graph G corresponding to the same function f differ only in the number of loops at $w_i \in W = V(H)$. Thus all such contractions correspond to a unique simple contraction induced by f.

2. We note the distinction between a contraction and a connected homomorphism. A connected homomorphism f is a particular case of a contraction induced by f when the edge subsets F_i are the null subsets of $E < f^{-1}(w_i) >$. This is called the *null con-*

traction corresponding to f and is denoted by $f_0(G)$. Also, the connected homomorphism induced by f differs from any proper contraction induced by f only in the number of loops at $w_i \in W$. The contraction induced by f in which each $F_i = E < f^{-1}(w_i) >$ is called the *full contraction* induced by f.

3. If $e = uv \in E$ and $F = \{e\}$, then $G|$ þ e is called an *elementary contraction* of G and differs from the corresponding connected elementary homomorphism $G * \{v,\ u\}$ by the absence of a loop at the common vertex $w = \{u,\ v\}$ in H.

The following result is due to Behzad and Chartrand [15].

Theorem 1.18 If a graph H is homeomorphic from a graph G, then G is a contraction of H.

Proof If $G = H$, then obviously G is the null contraction of H corresponding to the identity mapping $f : V(H) \to V(G)$.

Let $G \neq H$. Then H is obtained from G by a sequence of elementary subdivisions, say $\in_1,\ \in_2,\ \ldots,\ \in_k$, through the intermediate graphs $G_i = \in_i (G_{i-1})$, $G_0 = G$, $G_k = H$. Let G_i be obtained from G_{i-1} by subdividing the edge uv, introducing a new vertex w. Then G_{i-1} is obtainable from G_i by the elementary contraction with $f = \{uw\}$. That is, $G_{i-1} = G_i|uw = \Phi(G_i)$, say. Then, clearly we have $G = \Phi_1(\Phi_2 \ldots (\Phi_k(H) \ldots)$. Therefore G can be obtained from H by a sequence of elementary contractions and is thus a contraction of H. ❑

The converse of the above theorem is not true in general. To see this, consider the graphs in Fig. 1.45. Here the graph H is a contraction of the graph G by using the mapping $f(a) = f(b) = f(c) = a'$, $f(g) = g'$, $f(h) = h'$, $f(d) = f(e) = f(f) = d'$. But G is not homeomorphic from H.

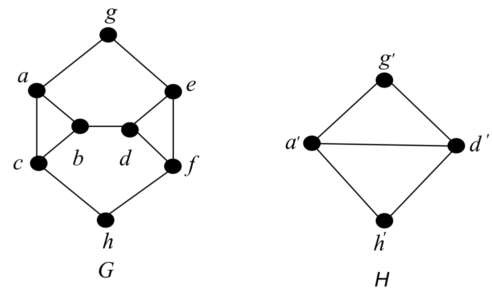


Fig. 1.45

Corollary 1.1 If a graph H contains a subgraph K homeomorphic from a non-trivial connected graph G, then G has a sub contraction of H.

The next result due to Halin can be found in Ore [177].

Theorem 1.19 If a graph G is contractible to a graph H and $\Delta(H) \leq 3$, then G has a subgraph homeomorphic from H.

Proof Let G be a graph contractible to the graph H. Hence there is a mapping $f : V(G) \to V(H)$ inducing the contraction H. Therefore, $G_i = < f^{-1}(v_i) >$ for $v_i \in V(H)$ are connected induced subgraphs of G which have been contracted to the vertices v_i of H. Let $U_i = f^{-1}(v_i)$.

As $\Delta(H) \leq 3$, for given U_1 there are at most three U_i, say U_2, U_3, U_4 such that there is an edge from U_1 to U_i. Let the end vertices in U_1 of three such edges from U_1 to U_2, U_3, U_4, respectively be u_2, u_3, u_4. Since G_1 is connected, there is $u_2 - u_3$ path P and a $u_4 - u_5$ path Q in G_1, where u_5 is the first vertex that Q has in common with P. (If there are only two vertices, we consider only the path P.) For each edge v_iv_j in H, we choose one edge between U_i and U_j in G and then the set of subgraphs PUQ in G_i. The graph H_1 so formed is a subgraph of G and clearly from the construction it is homeomorphic from H (Fig. 1.46). ❑

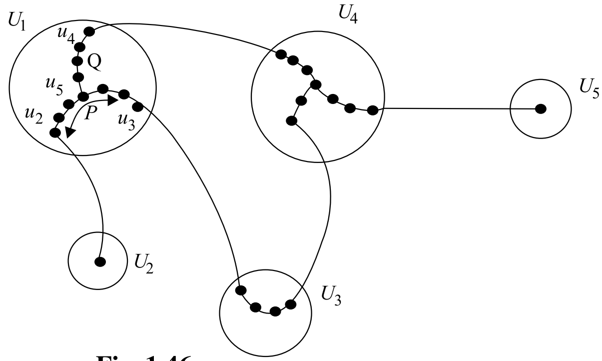


Fig. 1.46

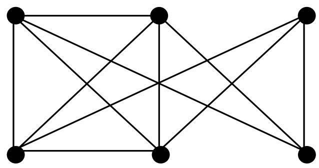
Corollary 1.2 If the graph G has a sub contraction H with $\Delta(H) \leq 3$, then G has a subgraph homeomorphic from H.

Theorem 1.20 If G is subcontractible to K_5, then either G has a subgraph homeomorphic from K_5, or G has a subcontraction to the graph L.

Proof We assume without loss of generality, that the given graph G is contractible to K_5. Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and let $U_i = f^{-1}(v_i)$, $1 \leq i \leq 5$, where $f : V(G) \to V(K_5)$ is the contraction mapping. Then $G_i = < U_i >$ are connected subgraphs of G and there is an edge in G between U_i and U_j for each pair (i, j). Let u_2, u_3, u_4, u_5 be the end vertices in U_1, of four such edges selected from U_1 to the other $U_i (2 \leq i \leq 5)$. Since G_1 is connected, there is a $u_2 - u_5$ path P in G, and there are $u_4 - u_4'$, $u_5 - u_5'$ paths Q and R, where u_4' and u_5' are the first vertices that these paths have in common with P.

If $u_4' \neq u_5'$, we can contract each U_i, $2 \leq i \leq 5$ to a vertex to get a K_4 and this together with the selected edges from U_1 to the U_i, and the path P, Q, R is a graph homeomorphic to L, so that G has a subcontraction to L.

If $u_4' = u_5'$, for any choice of P and for every set U_i, G has a subgraph homeomorphic from K_5. The same is the case when in each U_i, the four end vertices u_j coincide (Fig. 1.47). ❑



L

Fig. 1.47

1.9 Distance and eccentricity

Definition: Distance between two vertices u and v of a connected graph G is the length of the shortest path between u and v. It is denoted by $d(u,\ v)$. Clearly, by convection, $d(u,\ u) = 0$. The shortest path is called a $u-v$ *geodesic*, or a $u-v$ distance path.

Theorem 1.21 The distance function for a connected graph is a metric defined on its vertex set.

Proof Let G be a connected graph and $d(u,\ v)$ be the distance function, with u and v being vertices of G.

1. Clearly, $d(u,\ v) \geq 0$, with $d(u,\ u) = 0$.
2. Also, $d(u,\ v) = d(v,\ u)$, because the length of the shortest path from u to v is same as the length of the shortest path from v to u.
3. For any vertices $u,\ v,\ w$ in G, we have $d(u,\ v) \leq d(u,\ w) + d(w,\ v)$

Thus d is a metric. ❑

Note The distance between two vertices u and v in different components of a disconnected graph is infinite and this distance function does not induce a metric on the vertex set.

Neighbourhood: Let v be any vertex of a connected graph G. The ith neighbourhood of v is $N_i(v) = \{u \in V : d(v,\ u) = i\}$. We set $N_0(v) = \{v\}$ and denote $N_1(v)$ simply by $N(v)$, and call $N(v)$ as the neighbourhood of v, or the neighbours of v.

The s-ball at v is $B_s(v) = \bigcup_{j=0}^{s} N_j(v)$.

Eccentricity of a vertex: Let G be a connected graph. The eccentricity of a vertex v in G is the distance of the vertex u farthest from v. It is denoted by $e(v)$. That is, $e(v) = \max\{d(u,\ v) : u \in V\}$.

The minimum eccentricity is called the *radius* of G and the maximum eccentricity is called the *diameter* of G. The radius is denoted by r and diameter by d.

Therefore, $r = \min\{e(v) : v \in V\}$ and $d = \max\{e(v) : v \in V\}$.

When the graph is to be mentioned, we use the notations, $e(v|G)$, $r(G)$ and $d(G)$ for eccentricity, radius and diameter respectively.

The vertices of minimum eccentricity in G are called the *centres* of G and the vertices of maximum eccentricity are called the *periphery* of G. That is, the centre of G is $C(G) = \{v \in V : e(v) = r\}$ and periphery of G is $P(G) = \{v \in V; e(v) = d\}$.

A vertex in $C(G)$ is called a *central vertex* and a vertex in $P(G)$ is called a *peripheral vertex*. A graph having $C(G) = V(G)$ is called a *self-centered graph*.

A longest geodesic (maximum among the shortest paths between any two vertices) of a graph is called a *diametral path* of G. Note that this is not same as a longest path in G. The diameter is the length of any diametral path. A graph G is said to be *geodetic* if any two of its vertices are joined by a unique geodesic.

Theorem 1.22 The radius and diameter of a graph are related as $r \le d \le 2r$.

Proof Clearly, $r \le d$ follows from the definition of r and d. Let u and v be the ends of a diametral path and w be a central vertex. Then,

$d = d(u, v) \le d(u, w) + d(w, v) \le r + r = 2r$,

by using the triangle inequality of the metric and definition of r. ❑

Example Consider the graph shown in Figure 1.48. The eccentricities are 5, 4, 3, 3, 3, 4, 5, 4, 5; $r(G) = 3$, $d(G) = 5$, $C(G) = \{v_3, v_4, v_5\}$, $d(G) = \{v_1, v_7, v_9\}$. Diametral paths are $v_1v_2v_4v_5v_6v_7$, $v_1v_2v_3v_5v_6v_7$, $v_1v_2v_4v_5v_8v_9$ and $v_1v_2v_3v_5v_8v_9$.

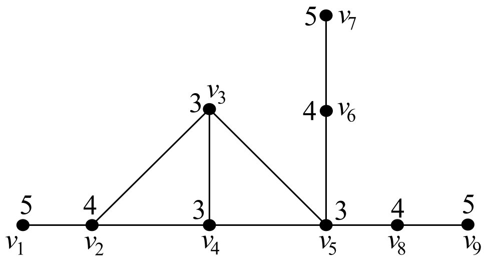


Fig. 1.48

Detours: A *detour path* between the vertices u and v in a graph G is the path of maximum length between u and v. The length of detour path is called *detour distance* and is denoted by $\partial(u, v)$.

The *detour eccentricity* of a vertex v is defined by $\partial(v) = \max\{\partial(v, u) : u \in V\}$.

The *detour diameter*, or detour number of G is defined as $\partial(G) = \max\{\partial(v) : v \in V\}$ and the *detour radius* is defined by $r_\partial(G) = \min\{\partial(v) : v \in V\}$.

A longest path in G is called a *detour path* in G and its length is $\partial(G)$. The *detour center* of G is $C_\partial(G) = \{v \in V : \partial(v) = r_\partial(G)\}$.

Theorem 1.23 If a connected graph G of order n has minimum degree δ, then

$$\partial(G) = \min(n-1, 2\delta). \tag{1.23.1}$$

Proof Clearly, $\partial(G) \leq n-1$. If $\partial(G) = n-1$, then (1.23.1) is obviously true, as min $(n-1,\ 2\delta) = 2$, or $n-1$ according as $\delta = 1$, or $\delta = n-1$. Now, let $\partial(G) < n-1$. Assume the contrary,

$$\partial(G) < \min(n-1,\ 2\delta). \tag{1.23.2}$$

Let $P = v_0 v_1 \ldots v_\partial$ be a path of length $\partial = \partial(G)$ and H be the subgraph induced by the vertex set of P. Then H does not contain a spanning cycle, because otherwise some vertex $v \in V(G) - V(H)$ will be adjacent to some vertex v_i of $V(H)$ giving a path of length $\partial + 1$. Thus $v_0 v_\partial \notin E$, and for similar reasons v_0 and v_∂ are adjacent only to vertices in H.

Let $S = \{v_i \in V(H) : v_0 v_i \in E\}$ and $T = \{v_i \in V(H) : v_{i-1} v_\partial \in E\}$.

Since H does not contain a spanning cycle, $S \cap T = \Phi$. Also, $v_0 \notin SUT$, so that $|S \cup T| \leq \partial$. Also, $|S| \geq \delta$ and $|T| \geq \delta$, therefore $\partial \geq |S \cup T| = |S| + |T| \geq \delta + \delta = 2\delta$. That is,

$$\partial \geq 2\delta. \tag{1.23.3}$$

Therefore, $2\delta \leq \partial < n-1$, so that $2\delta < n-1$.

Thus (1.23.2) gives $\partial < \min(n-1,\ 2\delta)$ implying $\partial < 2\delta$ (because min $(n-1,\ 2\delta) = 2\delta$ as $2\delta < n-1$), which contradicts (1.23.3) and so our assumption (1.23.2) is wrong.

Hence, $\partial(G) \geq \min(n-1,\ 2\delta)$. ❑

1.9 Exercises

1. If G is a graph with n vertices and m edges and if k is the smallest positive integer such that $k \geq 2m/n$, then prove G has a vertex of degree at least k.

2. If G is a graph with n vertices and if r vertices out of n have degree k and the others have degree $k+1$, then prove $r = (k+1)n - 2m$, where m is the number of edges in G.

3. If G is a k-regular graph (k being odd), prove that the number of edges in G is a multiple of k.

4. If G is a graph with n vertices and exactly $n-1$ edges, then prove that G has either a vertex of degree 1 or a vertex of degree zero.

5. If in a graph $G,\ \delta \geq \frac{n-1}{2}$, then show that G is connected.

6. If a graph G is not connected, then show that $\overline{G}$ is connected.

7. Show that if a self-complementary graph contains pendent vertex, then it must have at least another pendent vertex.

8. Prove that any two connected graphs with n vertices, all of degree 2, are isomorphic.
9. Prove that if a connected graph G is decomposed into two subgraphs G_1 and G_2, there must be at least one vertex common between G_1 and G_2.
10. Prove that a connected graph G remains connected after removing an edge e from G if and only if e is in some cycle of G.
11. If the intersection of two paths is a disconnected graph, show that the union of the two paths has at least one cycle.
12. Show that the order of a self-complementary graph is of the form $4n$ or $4n+1$, where n is a positive integer.
13. Draw all the non-isomorphic self complementary graphs on four vertices.
14. Show that the bipartition of a connected graph is unique. Prove that a graph with $n(n>1)$ vertices has at least $(n-1)([n/2])$ cycles.
15. Prove that a graph with n vertices must be connected if it has more than $(n-1)(n-2)/2$ edges.
16. Prove that a graph with n vertices $(n>2)$ cannot be bipartite if it has more than $n^2/4$ edges.
17. Prove that every graph with n vertices is isomorphic to a subgraph of K_n.
18. Prove that if a graph has more edges than vertices then it must possesses at least one cycle.