# The Tight Bound of First Fit Decreasing Bin-Packing Algorithm Is $FFD(I) \leq 11/9OPT(I) + 6/9$

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**Abstract.** First Fit Decreasing is a classical bin packing algorithm: the items are ordered into their nonincreasing order, and then in this order the next item is always packed into the first bin where it fits. For an instance $I$ let $FFD(I)$ and $OPT(I)$ denote the number of the used bins by algorithm FFD, and an optimal algorithm, respectively. We show in this paper that

$$FFD(I) \leq 11/9OPT(I) + 6/9, \quad (1)$$

and that this bound is tight. The tight bound of the additive constant was an open question for many years.

**Keywords:** first fit decreasing, tight bound.

## 1 Introduction

We can define the classical bin packing problem in the next way: There are items with sizes $y_1, y_2, ..., y_n$, which are positive real numbers, we are looking for minimum number of unit capacity bins, so that each item is packed into exactly one of the bins, and the sum of the sizes of the items being packed into a bin can not be more than 1. The problem is $NP$-hard, and FFD (First Fit Decreasing) is a classical bin packing algorithm: the items are ordered into their nonincreasing order, and then in this order the next item is always placed into the first bin, where it fits. For an instance $I$ let $FFD(I)$ and $OPT(I)$ denote the number of the used bins by algorithm FFD, and an optimal algorithm, respectively.

D.S. Johnson in his doctoral thesis in 1973 showed [5] that $FFD(I) \leq 11/9OPT(I)+4$. Later, B.S. Baker [2], in 1985 gave a slightly simpler proof, and showed that the additive constant is not more than 3. Then, in 1991, Yue Minyi [4] proved that $FFD(I) \leq 11/9OPT(I) + 1$, but his proof has some problems. It is not easy to understand, and leaves many gaps to be verified by the reader. Later, in 1997, Li Rongheng and Yue Minyi [1] again tightened the additive constant to be 7/9, but they do not prove the statement, only give a draft about it. In that paper the authors also conjectured that the tight additive constant is 5/9. What is the least value of the additive constant was an open question for many-many years. Now we show that

$$FFD(I) \leq 11/9OPT(I) + 6/9, \quad (2)$$

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and prove that this bound is already the tight one, the additive constant can not be further decreased. We follow mainly (but not completely) the proof of [4].

Since the tight upper bound was not known previously, even the next question, which seems to be quite trivial, could not be answered: Is there an instance $I$ for the problem for which $OPT(I) = 5$ and $FFD(I) = 7$ hold? In the recent work [6] we give a short proof that there is not such instance. (From statement (2) follows that there can not be such instance, but if we know only $FFD(I) \leq 11/9OPT(I) + 1$, the question can not be answered.) The next example shows that the additive constant can not be less than 6/9: (this example is a simple modification of that being in [3], Chapter 2, page 16). The example consists of six fully packed optimal bins, an the items are packed by FFD into eight bins. Then the upper bound (2) is proven to be tight, since $11/9 \cdot 6 + 6/9 = 72/9 = 8$. The example is as follows: Let $B_1 = \{1/2 + \varepsilon, 1/4 + \varepsilon, 1/4 - 2\varepsilon\}$, and $B_2 = \{1/4 + 2\varepsilon, 1/4 + 2\varepsilon, 1/4 - 2\varepsilon, 1/4 - 2\varepsilon\}$. If there are 4 copies from $B_1$ and 2 copies from $B_2$ in the optimal packing, the number of FFD bins will be 8. We also get the tight bound, if the number of the previous optimal bins are $6k + 4$ and $3k + 2$, respectively.

Can we now give (supposing that (2) really holds) for **all** integer $m$ the biggest possible number $n$, such that $OPT(I) = m$ and $FFD(I) = n$? We need for this purpose one more example: If there are $6k + 1$ copies from $B_1$ and $3k + 1$ copies from $B_2$ in the optimal packing, the number of FFD bins will be $11k + 3$. Then we get the next table for the maximum possible values of $n$, (we denote the difference of the $n$ and $m$ by $d$):

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| OPT(I)=m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| FFD(I)=n | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 | 12 |
| n-m=d | | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| m | | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| n | | 14 | 15 | 16 | 17 | 19 | 20 | 21 | 22 | 23 |
| d | | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 |
| m | | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| n | | 25 | 26 | 27 | 28 | 30 | 31 | 32 | 33 | 34 |
| d | | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 | 6 |

For higher values of $m$, $n$, and $d$ the table follows in the same way.

For example, there exists such instance, for which the number of the bins in an optimal packing is 5, and the number of bins used by FFD is 6, but this latter value can not be bigger. Without the tight upper bound (2) and previous examples we could not know the maximum value of $n$ in infinite many cases. Then now, it remained only to see the proof for the upper bound. Since the complete detailed proof requires more than 30 pages and we have a page limit, some details can not be treated here, but the author gladly send the manuscript with the whole proof to the interested reader.

## 2 Preliminaries

In this paper we show the next theorem:

**Theorem 1.** $9FFD(L) \leq 11OPT(L) + 6$.

Proof: It is trivial, that this statement is equivalent with (2), and since $FFD(I)$ and $OPT(I)$ are integers, it suffices to show that there is not such instance for which

$$9FFD(I) \geq 11OPT(I) + 7 \tag{3}$$

holds. Suppose to the contrary that $I$ is a minimal counterexample, i.e. contains minimum number of items, and (3) holds. It is trivial that then $OPT(I) \geq 2$ and $FFD(I) \geq 3$ must hold.

Let us denote the optimal bins as $B_i^*$ for $i = 1, ..., OPT(I)$, and the FFD bins as $B_i$ for $i = 1, ..., FFD(I)$. The sum of the sizes of items being packed into a bin will be denoted as $Y(B_i)$, and $Y(B_i^*)$, respectively. From the minimality of the counterexample follows that the last FFD bin contains only one, i.e. the last item. Let this item be denoted as $X$. (The size of this item also will be denoted simply as $X$.) Let the size of the items be $y_k$, for $k = 1, ..., n$, we also denote the items as $y_k$. We suppose w.l.o.g. that the size of the items are nonincreasing, i.e. $y_1 \geq y_2 \geq ... \geq y_n = X$.

We also use the next denotations: Let the $j$-th item of the $i$-th optimal bin be denoted as $A_{i,j}^*$ for every $i = 1, ..., OPT(I)$, and let the $j$-th item of the $i$-th FFD bin be denoted as $A_{i,j}$ for every $i = 1, ..., FFD(I)$. (We use different denotations for the same items). We assume without loss of the generality that for every $i$ and every $j_1 < j_2$ holds that $A_{i,j_1}^* \geq A_{i,j_2}^*$, and $A_{i,j_1}^*$ comes before $A_{i,j_2}^*$ in the nonincreasing order of the items, and similarly, follows from the FFD rule that for every $i$ and every $j_1 < j_2$ holds that $A_{i,j_1} \geq A_{i,j_2}$, and $A_{i,j_1}$ comes before $A_{i,j_2}$ in the nonincreasing order of the items. A bin is called as i-bin, if it contains exactly $i$ items.

Because all items fit in the optimal packing into $OPT(I)$ optimal bins, follows that $\sum_{k=1}^{n} y_k \leq OPT(I)$. Note that item $X$ does not fit into any previous FFD bin, thus we get

$$Y(B_i) + X > 1, i = 1, ..., FFD(I) - 1. \tag{4}$$

**Lemma 1.** $X > \frac{FFD(I)-OPT(I)-1}{FFD(I)-2} \geq 2/11$.

*Proof.* The second inequality is equivalent by (3). From (4) follows that $Y(B_i) > 1-X$ for all $1 \leq i \leq FFD(I)-1$, and $X+Y(B_1) > 1$. Applying these inequalities we get

$$OPT(I) \geq \sum_{k=1}^{OPT(I)} y_k = X + Y(B_1) + \sum_{i=2}^{FFD(I)-1} Y(B_i) > 1 + (1 - X)(FFD(I) - 2),$$

from which the first inequality follows.

**Corollary 1.** $X > \frac{\lceil 11/9OPT(I)+7/9\rceil - OPT(I)-1}{\lceil 11/9OPT(I)+7/9\rceil -2}$

*Proof.* We apply (3), the previous lemma, and the facts that $FFD(I)$ is integer, and the ratio $\frac{FFD(I)-OPT(I)-1}{FFD(I)-2}$ is increasing regarding $FFD(I)$.

From now we know that each optimal or FFD bin can contain at most five items.

**Definition 1.** *We say that bin $B_i$ dominates the optimal bin $B_j^*$ for some $i$ and $j$, if for every item $y_k$ being in $B_j^*$ there exists an item $y_l$ being in $B_i$ for which $y_k \leq y_l$ and these items in $B_i$ are different.*

**Lemma 2.** *There are no bins $B_i$ and $B_j^*$ such that $B_i$ dominates $B_j^*$.*

*Proof.* Swap one by one the items in $B_j^*$ by items of $B_i$ that dominate them. Then omitting the elements of this bin we get a smaller counterexample, that is a contradiction.

**Lemma 3.** *Each optimal bin contains at least three items.*

*Proof.* If an optimal bin contains one element, then by the domination lemma we get a contradiction. Suppose that an optimal bin contains two items, $Y$ and $Z$, and $Y \geq Z$. Consider the moment when $Y$ is packed. If this item is packed as a first item into an FFD bin, then $Z$ fits into this bin, thus at least one more item, not less than $Z$ will be packed into this bin, which again contradicts to Lemma 2. If $Y$ is not a first item, then the first item is not less than $Y$, and the second one (i.e. $Y$) is not less than $Z$, a contradiction again.

**Lemma 4.** *Every FFD bin but the last one contains at least two items.*

*Proof.* Suppose that $B_i$ contains one element, for some $1 \leq i \leq FFD(I)-1$, let this item be $Y$. Let this item be packed into the $B_j^*$ optimal bin, then there is an other item in this optimal bin, say $Z$. Then $Y + Z \leq 1$, follows that $Y + X \leq 1$, thus the last item $X$ fits into this bin, a contradiction.

**Lemma 5.** $X \leq 1/4$.

*Proof.* Suppose that $X > 1/4$, then every optimal bin contains at most three items. From Lemma 3 we get that every optimal bin contains exactly three items, thus the number of the items is $3OPT(I)$, and all items are less than $1/2$.

Suppose that there are two consecutive bins $B_i$ and $B_j$, (then $j = i+1$), and $B_i$ contains three, and $B_j$ contains two elements. If $A_{i1} + A_{i2} \geq A_{j1} + A_{j2}$, then because $A_{i3}$ fits into the $i$-th bin, and $A_{i3} \geq X$, follows that $X$ fits into the $j$-th bin, a contradiction. Thus $A_{i1} + A_{i2} < A_{j1} + A_{j2}$. Because $A_{i1} \geq A_{j1}$, follows that $A_{i2} < A_{j2}$. Thus $A_{j2}$ is packed before $A_{i2}$, and it did not fit into the $i$-th bin, thus $A_{i1} + A_{j2} > 1$, thus at least one of them is bigger than a half, a contradiction.

Follows that the first some FFD bins contain two items, the next FFD bins contain three items, and the last FFD bin contains only one item. Let $n_i$ be the number of the FFD $i$-bins, for $i = 2, 3$. Then $n_2 + n_3 + 1 = FFD(I)$, and the number of the items is $3FFD(I) - n_2 - 2$. Since the sum of the sizes of any two items is less than 1, the first $2n_2$ items (in the nonincreasing order) are packed pairwise into the first $n_2$ FFD bins, and follows that

$$y_{2n_2-1} + y_{2n_2} + X > 1. \tag{5}$$

On the other hand, consider the first item in the nonincreasing order, which is a second item in some optimal bin, i.e. consider the largest $A^*_{i,2}$ item. This cannot be later than the $(OPT(I)+1)$-th item, thus $A^*_{i,2} = y_{k_2}$ for some $k_2 \leq OPT(I)+1$. Let $A^*_{i,1} = y_{k_1}$ and $A^*_{i,3} = y_{k_3}$, then $k_1 < k_2 < k_3$, and

$$y_{OPT(I)} + y_{OPT(I)+1} + X \leq y_{k_1} + y_{k_2} + y_{k_3} \leq 1. \tag{6}$$

Comparing (5) and (6) follows that $OPT(I) \geq 2n_2$. We get that the number of items is

$$3OPT(I) = 3FFD(I) - n_2 - 2 \geq 3FFD(I) - OPT(I)/2 - 2 \tag{7}$$
$$\geq 3\,(11/9OPT(I) + 7/9) - OPT(I)/2 - 2 \tag{8}$$
$$= 19/6OPT(I) + 1/3 > 3OPT(I), \tag{9}$$

a contradiction.

At this point we already know that $X$ must lie in interval $(2/11; 1/4]$. In the remaining part of the paper we will divide our investigations into two parts, as $1/5 < X \leq 1/4$, or $2/11 < X \leq 1/5$.

**Lemma 6.** *For the value of optimum bins holds that $OPT(I) \geq 6$, and in case $X \leq 1/5$ a stronger inequality $OPT(I) \geq 10$ also holds.*

*Proof.* If $2 \leq OPT(I) \leq 4$, then from Corollary 1 we get that $X > 1/4$, which is contradiction. Reference [6] shows that in case $OPT(I) = 5$ follows that $FFD(I) \leq 6$, which contradicts to (3). Thus follows that $OPT(I) \geq 6$. Similarly we get that if $X \leq 1/5$, then $6 \leq OPT(I) \leq 9$ is not possible, thus $OPT(I) \geq 10$.

We call a bin as open bin if there is at least one item already packed into the bin. A bin is closed if no more item will be packed into this bin. An item which is packed into the last opened bin called as **regular item**, otherwise the item is called as a **fallback item**. Let $A$ be an arbitrary regular item. We say, that $B$ is a **further** item, if it comes (in the nonincreasing order of the items) after $A$, and will be packed into a later bin. An arbitrary bin is called as $(A, B, C)$ bin, if $A$, $B$, and $C$ are items, and exactly these items are packed into that bin. Similar denotations are also used if there are only two, or there are more than three items in a bin. We often will use the next lemma from [4].

**Lemma 7.** *Let $x_i$ be the last item in the nonincreasing order which is packed into a $(L, x_i)$ FFD-bin, where $L > 1/2$. Then (i), it can be supposed that there is not such item which has size bigger than $x_i$ and not bigger than $1 - L$, (ii) if there is another item $x_k$ with the same size as $x_i$, then it is packed into a $(L', x_k)$ FFD-bin where $L'$ precedes $L$ and they have equal sizes.*

*Proof.* Suppose to the contrary that there is an item $x_k$ for which $x_i < x_k \le 1-L$ holds. Let $x_k$ be the last such item. This is packed by FFD into an earlier bin, which contains exactly one more item $L' \ge L > 1/2$. Then decrease the size of this item to be equal to $x_i$. Then it will be packed into the same bin, and there will no more item be packed into this bin. This means that we get a new, appropriate counterexample. By induction we get a counterexample which meets property (i). Regarding (ii): If there exists an $x_k$ with the same size as $x_i$, then it is packed into a $(L', x_k)$ FFD-bin, where $L' \ge L$. Suppose that $L' > L$, and let $L'$ be the last such item. Then we can decrease the size of $L'$ to be equal to $L$, and we get a new counterexample again.

**Lemma 8.** *Let $x_i$ be the last item which is packed into a $(B_0, x_i)$ FFD-bin where $\frac{1-X}{2} < B_0 \le 1/2$, ($x_i$ comes after $B_0$). Then (i), each previous item with size between $\frac{1-X}{2}$ and $1/2$ has the same size as $B_0$, (ii) all $L$ items greater than half have the same size, (iii) all $L > 1/2$ is packed into some $(L, x_k)$ FFD-bin where $x_k = x_i$.*

*Proof.* The proof is similar to the previous one.

## 3 Case $1/5 < X \le 1/4$

We put the items into some classes according to their sizes. The classification used here is not the same, only similar to one used in Yue's paper. The classes are **large, big, medium, small, quite small,** and **very small** items, the items being in a class are denoted as $L, B, M, S, U, V$, respectively. We also add some weights to the items, as follows:

| Name | Class | Weight | Or simply |
|---|---|---|---|
| Large | $\frac{1}{2} < L$ | $23/36(1-X)$ | 23 |
| Big | $\frac{1-X}{2} < B \le \frac{1}{2}$ | $18/36(1-X)$ | 18 |
| Medium | $\frac{1}{3} < M \le \frac{1-X}{2}$ | $15/36(1-X)$ | 15 |
| Small | $\frac{1-X}{3} < S \le \frac{1}{3}$ | $12/36(1-X)$ | 12 |
| qUite small | $\frac{1}{4} < U \le \frac{1-X}{3}$ | $9/36(1-X)$ | 9 |
| Very small | $X \le V \le \frac{1}{4}$ | $9/36(1-X)$ | 9 |

The classification of the items in case $1/5 < X \le 1/4$.

Note, that the classes are well defined, furthermore $\frac{1-X}{2} < \frac{1+X}{3} < 2X$ holds since $X > 1/5$. Note, that since every optimal bin have at least three items, follows that $L + 2X \le 1$ holds for any L item, thus in the FFD packing an M item fits into an L-bin, if there is not other item packed yet into the bin. We use the denotation $c_1L + c_2B + c_3M + c_4S + c_5U + c_6V > c_7$, where $c_i$ are integers or 0 for $i = 1, ..., 7$, and the inequality holds substituting the sizes of any large, big, medium, small, quite small and very small items. For example $L + 2U > 1$ holds, since $L > 1/2$ and $U > 1/4$.

We denote the weight of an item $Z$ as $w(Z)$, and the weight of an optimal or FFD bin as $w(B^*)$, or $w(B)$, respectively. We define the **reserve** of an optimal bin as $r(B^*) = 44/36(1-X) - w(B^*)$.. When we define the weights of the classes, we do it in such a way, that no optimal bin will have weight more than $44/36(1-X)$, i.e. the reserve of all optimal bins are nonnegative, and almost all of the optimal bins have positive reserve. (This will not be true in one case, then we will modify the weights of the items.) Define the **surplus** of an FFD bin as $sur(B) = w(B) - (1-X)$, if this value is nonnegative. Otherwise, let $short(B) = (1-X) - w(B)$ be called as **shortage**, (similarly, as in Yue's paper). If the weight of every FFD bin was at least $1-X$, (i.e. in case when there is not shortage, also applying that the reserve of all optimal bin is nonnegative), we could easily get that

$$(1-X)FFD(I) \leq \sum_{k=1}^{FFD(I)} w\left(B_k\right) = w(I) = \sum_{k=1}^{OPT(I)} w\left(B_k^*\right) \leq 11/9(1-X)OPT(I),$$

and our proof would be ready. Unfortunately, such FFD bins that has less weight (i.e. has some shortage) may exist. But we prove that all shortage can be **covered** by the reserve of the optimal bins plus the surplus of the other FFD bins, plus the required additive constant $27/36(1-X)$. In this section the weight of the smallest class will is $w(V) = w(X) = 9/36(1-X)$. Thus, the shortage of the last FFD bin, which contains only the last item $X$, is just $(1-X) - w(X) = 27/36(1-X)$, thus the additive constant just covers the shortage of the last FFD bin. For the simplicity, we will say that the weight of a $V$ item is 9, (and similarly in case of other items), and the shortage of a bin containing only $X$ is 27, (instead than $9/36(1-X)$, and $27/36(1-X)$, respectively), i.e. we say simple only the numerator of the ratio.

Let $sur(I)$ and $res(I)$ be the total value of the surplus and reserve of all FFD and optimal bins, respectively, let the required additive constant $27/36(1-X)$ be denoted as $rex(I)$, and finally let $short(I)$ be the total value of the shortage given by all FFD bins. Then we have

$$w(I) = \textstyle\sum_{k=1}^{FFD(I)} w\left(B_k\right) = (1-X)FFD(I) + sur(I) - short(I), \qquad (10)$$

$$w(I) = \textstyle\sum_{k=1}^{OPT(I)} w\left(B_k^*\right) = 11/9(1-X)OPT(I) - res(I). \qquad (11)$$

Suppose that

$$res(I) + sur(I) + rex(I) \geq short(I) \qquad (12)$$

holds. Then then applying (10) and (11), we have

$$(1-X)FFD(I) = w(I) - sur(I) + short(I) \qquad (13)$$

$$\leq w(I) + res(I) + rex(I) \qquad (14)$$

$$= 11/9(1-X)OPT(I) + 27/36(1-X), \qquad (15)$$

and dividing by $(1-X)$, and considering that $27/36 < 7/9$ we get our main result. Thus in the remained part of this section we prove (12). First, let us see what bins are possible, according only to the definition of the classes. First we list all possible optimal bins, then all possible FFD bins. In the last rows $r$ means the reserve of the optimal bins, while $s$ denotes the value of the surplus or shortage of the FFD bins. If $s$ is positive, then it means surplus, if it is negative, for example if it is $-3$ , it means that the shortage of that FFD bin is three. The value of reserve, surplus of shortage of some bin can be easily computed by the weights of the classes. (In case of (L,U) or (L,V) FFD bins we define the weights of the classes in some other way.)

OPT

| | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | |
| B | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | |
| M | | | | 1 | 1 | 1 | | | | | | | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| S | 1 | | | 1 | | | 2 | 1 | 1 | | | | 1 | | | 2 | 1 | 1 | | | | | |
| U | | 1 | | | 1 | | | 1 | | 2 | 1 | | | 1 | | | 1 | | 2 | 1 | 1 | | |
| V | 1 | 1 | 2 | | | 1 | | | 1 | | 1 | 2 | | | 1 | | | 1 | | 2 | 1 | 3 | 2 |
| r | 0 | 3 | 3 | -1 | 2 | 2 | 2 | 5 | 5 | 8 | 8 | 8 | 2 | 5 | 5 | 5 | 8 | 8 | 11 | 2 | 11 | 2 | 11 |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| S | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | |
| U | | 1 | 1 | | | 2 | 2 | 1 | 1 | | | 3 | 3 | 2 | 2 | 1 | 1 | | |
| V | | 1 | | 2 | 1 | 1 | | 2 | 1 | 3 | 2 | 1 | | 2 | 1 | 3 | 2 | 4 | 3 |
| r | 8 | 2 | 11 | 2 | 11 | 5 | 14 | 5 | 14 | 5 | 14 | 8 | 17 | 8 | 17 | 8 | 17 | 8 | 17 |

FFD

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| L | 1 | 1 | 1 | 1 | 1 | | | |
| B | 1 | | | | | 2 | 1 | 1 |
| M | | 1 | | | | | 1 | |
| S | | | 1 | | | | | 1 |
| U | | | | 1 | | | | |
| V | | | | | 1 | | | |
| s | 5 | 2 | -1 | | | 0 | -3 | -6 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L | 1 | 1 | 1 | | | | | | | | | |
| B | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| M | | | | 1 | 1 | 1 | | | | | | |
| S | 1 | | | 1 | | | 2 | 1 | 1 | | | |
| U | | 1 | | | 1 | | | 1 | | 2 | 1 | |
| V | 1 | 1 | 2 | | | 1 | | | 1 | | 1 | 2 |
| s | 8 | 5 | 5 | 9 | 6 | 6 | 6 | 3 | 3 | 0 | 0 | 0 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| L | | | | | | | | | | | |
| B | | | | | | | | | | | |
| M | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| S | 1 | | | 2 | 1 | 1 | | | | | |
| U | | 1 | | | 1 | | 2 | 1 | 1 | | |
| V | | | 1 | | | 1 | | 2 | 1 | 3 | 2 |
| s | 6 | 3 | 3 | 3 | 0 | 0 | -3 | 6 | -3 | 6 | -3 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| S | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | |
| U | | 1 | 1 | | | 2 | 2 | 1 | 1 | | | 3 | 2 | 1 | | |
| V | | 1 | | 2 | 1 | 1 | | 2 | 1 | 3 | 2 | 1 | 2 | 3 | 4 | 1 |
| s | 0 | 6 | -3 | 6 | -3 | 3 | -6 | 3 | -6 | 3 | -6 | 0 | 0 | 0 | 0 | -27 |

**Lemma 9.** *There can not be other optimal, nor FFD bins.*

*Proof.* Each optimal bin contains three or four items, and each FFD bin can contain at least two, and at most four items. Furthermore each bin has items with total size at most one, and each FFD has items with size more than $1-X$. Using the next inequalities: $L+2U>1$, $2B+V>1$, $B+3V>1$, $M+S+2V>1$, $3S+V>1$, $B+U+X\le 1$, $3U+X\le 1$, (all of them can be shown easily), we get that exactly the above bins are possible.

We emphasize, that all types of the above FFD bins can not occur at the same time! For example if there exists a (B,2S) FFD bin, then there can not be other FFD bin which contains a big item and no large item, since then these two B items would fit into one bin.

We use the next notation: If L is packed into (L,A) FFD bin, and (L,B,C) optimal bin, where A, B, and C are items and L is a large item, we say that L is packed into the $\{(L,A),(L,B,C)\}$ **bbin** (to denote, that this is not really a bin, but a bin-pair). The first and second part of a bbin are the FFD and the optimal bins of item L, respectively. An arbitrary bin is denoted as (A,.) bin, where A is a class, and the bin contains at least one item from class A, but does not contain items from higher classes. For example, (M,.) denotes a bin, which contain at least one M item, but does not contain L or B items.

**Theorem 2.** *Suppose that there are not (L,U), (L,V) FFD bins, and there is not* $\{(L,S),(L,S,V)\}$ *bbin. Then statement (12) holds.*

*Proof.* First suppose that there is not (B,M,S) optimal bin. Since there is not $\{(L,S),(L,S,V)\}$ bbin, follows that by every L item we get 2 reserve at least, and the shortage caused by the (L,S) FFD bins are all covered. Also, since every optimal not L-bin has at least 2 reserve, we have totally at least 12 reserve, since there must be at least six optimal bins by Lemma 6. On the other hand, let us count the possible shortage caused by not (L,S) bins. If there is (B,S) FFD bin, then from Lemma 8 we know that there is not M item. Then it can not be (B,M) FFD bin, and we have at most 12 shortage, since it can not be at the same time two (S,.) FFD bins with shortage, and all shortage is covered. In the opposite case, if there is not (B,S) FFD bin, then the total not covered shortage is at most 3 (by a possible existing (B,M) FFD bin) plus 3 (by an (M,.) FFD bin) plus 6 by an (S,.) FFD bin, this is altogether again 12, and it is again covered.

Now suppose that there is at least one (B,M,S) optimal bin. Then,.follows that $\frac{1-X}{2}+\frac{1}{3}+S<B+M+S\leq 1$ holds for the previous B, M, and S items, from what we get that $S<\frac{1}{6}+\frac{X}{2}$ holds for the smallest S item. Let this smallest S be denoted as $S_0$. Again, since all S is greater than $\frac{1-X}{3}$, and all M is greater than $\frac{1}{3}$, follows that there is such B item which is less than $\frac{1+X}{3}$. Then we conclude the next things: (i), It can not be $\{(L,M),(L,S,V)\}$ bbin, since then, from Lemma 7 follows that all B items must be greater than the sum of these two S and V items being in the (L,S,V) bin, thus $B>\frac{1-X}{3}+X$ holds for all B items, contradiction. (ii), $M+S+S_0\leq\frac{1-X}{2}+\frac{1}{3}+\frac{1}{6}+\frac{X}{2}=1$ holds. From this second property follows, that if there is (M,S,U) or (M,S,V) FFD bin, then there is not further S item.

Now we are ready to finish the investigation of this case. We redefine the weight of exactly those M items what are packed into (B,M,S) optimal bins as 12, let these M items be called as small M items. Then, as before we have at least 12 reserve, since by all L items we get 2 reserve, and by all other optimal bins we have 2 more reserve, and all shortage caused by the (L,S) or (L,M) FFD bins are covered.

How many shortage can cause the not L FFD bins? Let us realize, that both M items in an (2M,U) or (2M,V) FFD bin can not be small M items, since a small M item, plus an appropriate B item and the smallest S item fit into one bin. Follows that by (2M,S), (2M,U) or (2M,V) FFD bins we get no shortage. By a (B,M) or (B,S) FFD bin (they can not exist at the same time), by an (M,.) FFD bin, and finally by an (S,.) FFD bin we can get at most $6+6+6=18$ shortage. If there is not (B,M) nor (B,S) bin, then all shortage is covered. If there is not (M,.) FFD bin with shortage, again, all shortage is covered, and this is also true for the (S,.) FFD bins. Finally we can consider the next things:

Suppose that there is an (M,.) FFD bin which has no S item, then there is not further S item, thus it can not be (S,.) bin with shortage, and all shortage is covered. Also, we have seen that if there is (M,S,U) or (M,S,V) FFD bin, then again, there is not further S item. Thus, if there is shortage caused by some (M,.) bin, then the total not covered shortage is at most $6+6+0=12$, otherwise, if there is, again it is at most $6+0+6=12$, and the statement is proved.

**Theorem 3.** *If there is (L,V) or (L,U) FFD bin, or there is* $\{(L,S),(L,S,V)\}$ *bbin, then statement (12) also holds.*

*Proof.* The proof can be made by case analysis. The details can not fit here, but the author gladly send it to the interested reader.

Thus we proved that in case $1/5 < X \leq 1/4$ our statement holds. It only remained the following case.

## 4 Case $2/11 < X \leq 1/5$

In this case we redefine the classes of the items and their weights, as follows:

| Class | Weight | Or simply |
|---|---|---|
| $\frac{1}{2} < L$ | $\frac{24}{36}(1-X)$ | 24 |
| $\frac{1-X}{2} < B1 \leq \frac{1}{2}$ | $\frac{18}{36}(1-X)$ | 18 |
| $\frac{3}{8}-\frac{X}{8} < B2 \leq \frac{1-X}{2}$ | $\frac{16}{36}(1-X)$ | 16 |
| $\frac{1}{3} < M1 \leq \frac{3}{8}-\frac{X}{8}$ | $\frac{15}{36}(1-X)$ | 15 |
| $\frac{1+X}{4} < M2 \leq \frac{1}{3}$ | $\frac{12}{36}(1-X)$ | 12 |
| $\frac{1}{4} < S \leq \frac{1+X}{4}$ | $\frac{10}{36}(1-X)$ | 10 |
| $\frac{1-X}{4} < U \leq \frac{1}{4}$ | $\frac{9}{36}(1-X)$ | 9 |
| $X \leq V \leq \frac{1-X}{4}$ | $\frac{8}{36}(1-X)$ | 8 |

The classification of the items in case $2/11 < X \leq 1/5$

Then the investigations are similar to that being in the previous section, but there are more than 200 optimal, and also more than 200 possible FFD bins.

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