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11. Recurrences

The recursive definition of the Fibonacci numbers is well-known: if F_n is the n^{th} Fibonacci number then

$$F_0 = 0, \quad F_1 = 1 \; ,$$

$$F_{n+2} = F_{n+1} + F_n, \qquad \text{if } \; n \ge 0 \; .$$

We are interested in an explicit form of the numbers F_n for all natural numbers n. Actually, the problem is to solve an equation where the unknown is given recursively, in which case the equation is called a ***recurrence equation.*** The solution can be considered as a function over natural numbers, because F_n is defined for all n. Such recurrence equations are also known as ***difference equations,*** but could be named as ***discrete differential equations*** for their similarities to differential equations.

Definition 11.1 *A* ***k^{th}th order recurrence equation*** $(k \ge 1)$ *is an equation of the form*

$$f(x_n, x_{n+1}, \ldots, , x_{n+k}) = 0, \qquad n \ge 0 \; , \tag{11.1}$$

where x_n has to be given in an explicit form.

For a unique determination of x_n, k initial values must be given. Usually these values are $x_0, x_1, \ldots, x_{k-1}$. These can be considered as ***initial conditions.*** In case of the equation for Fibonacci-numbers, which is of second order, two initial values must be given.

A sequence $x_n = g(n)$ satisfying equation (11.1) and the corresponding initial conditions is called a ***particular solution.*** If all particular solutions of equation (11.1) can be obtained from the sequence $x_n = h(n, C_1, C_2, \ldots, C_k)$, by adequately choosing of the constants $C_1, C_2, \ldots, C_k$, then this sequence x_n is a ***general solution.***

Solving recurrence equations is not an easy task. In this chapter we will discuss methods which can be used in special cases. For simplicity of writing we will use the notation x_n instead of $x(n)$ as it appears in several books (sequences can be considered as functions over natural numbers).

The chapter is divided into three sections. In Section 11.1 we deal with solving linear recurrence equations, in Section 11.2 with generating functions and their use in solving recurrence equations and in Section 11.3 we focus our attention on the numerical solution of recurrence equations.

11.1. Linear recurrence equations

If the recurrence equation is of the form

$$f_0(n)x_n + f_1(n)x_{n+1} + \cdots + f_k(n)x_{n+k} = f(n), \quad n \geq 0 ,$$

where $f, f_0, f_1, \ldots, f_k$ are functions defined over natural numbers, $f_0, f_k \neq 0$, and x_n has to be given explicitly, then the recurrence equation is ***linear.*** If f is the zero function, then the equation is ***homogeneous,*** otherwise ***nonhomogeneous.*** If all the functions $f_0, f_1, \ldots, f_k$ are constant, the equation is called a ***linear recurrence equation with constant coefficients.***

11.1.1. Linear homogeneous equations with constant coefficients

Let the equation be

$$a_0x_n + a_1x_{n+1} + \cdots + a_kx_{n+k} = 0, \qquad n \geq k , \tag{11.2}$$

where $a_0, a_1, \ldots, a_k$ are real constants, $a_0, a_k \neq 0,\ k \geq 1$. If k initial conditions are given (usually $x_0, x_1, \ldots, x_{k-1}$), then the general solution of this equation can be uniquely given.

To solve the equation let us consider its ***characteristic equation***

$$a_0 + a_1r + \cdots + a_{k-1}r^{k-1} + a_kr^k = 0 , \tag{11.3}$$

a polynomial equation with real coefficients. This equation has k roots in the field of complex numbers. It can easily be seen after a simple substitution that if r_0 is a real solution of the characteristic equation, then $C_0r_0^n$ is a solution of (11.2), for arbitrary C_0.

The general solution of equation (11.2) is

$$x_n = C_1x_n^{(1)} + C_2x_n^{(2)} + \cdots + C_kx_n^{(k)} ,$$

where $x_n^{(i)}$ $(i = 1, 2, \ldots, k)$ are the linearly independent solutions of equation (11.2). The constants $C_1, C_2, \ldots, C_k$ can be determined from the initial conditions by solving a system of k equations.

The linearly independent solutions are supplied by the roots of the characteristic equation in the following way. A ***fundamental solution*** of equation (11.2) can be associated with each root of the characteristic equation. Let us consider the following cases.

Distinct real roots. Let $r_1, r_2, \ldots, r_p$ be distinct real roots of the characteristic equation. Then

$$r_1^n,\ r_2^n,\ \ldots,\ r_p^n$$

are solutions of equation (11.2), and

$$C_1r_1^n + C_2r_2^n + \cdots + C_pr_p^n \tag{11.4}$$

is also a solution, for arbitrary constants $C_1,\ C_2,\ \ldots,\ C_p$. If $p = k$, then (11.4) is the general solution of the recurrence equation.

Example 11.1 Solve the recurrence equation

$$x_{n+2} = x_{n+1} + x_n, \quad x_0 = 0, \; x_1 = 1 \; .$$

The corresponding characteristic equation is

$$r^2 - r - 1 = 0 \; ,$$

with the solutions

$$r_1 = \frac{1+\sqrt{5}}{2} \; , \quad r_2 = \frac{1-\sqrt{5}}{2} \; .$$

These are distinct real solutions, so the general solution of the equation is

$$x_n = C_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + C_2 \left(\frac{1-\sqrt{5}}{2} \right)^n \; .$$

The constants C_1 and C_2 can be determined using the initial conditions. From $x_0 = 0$, $x_1 = 1$ the following system of equations can be obtained.

$$\begin{aligned} C_1 + C_2 &= 0 \; , \\ C_1 \frac{1+\sqrt{5}}{2} + C_2 \frac{1-\sqrt{5}}{2} &= 1 \; . \end{aligned}$$

The solution of this system of equations is $C_1 = 1/\sqrt{5}, \; C_2 = -1/\sqrt{5}$. Therefore the general solution is

$$x_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n ,$$

which is the nth Fibonacci number F_n.

Multiple real roots. Let r be a real root of the characteristic equation with multiplicity p. Then

$$r^n, \; nr^n, \; n^2r^n, \; \ldots, \; n^{p-1}r^n$$

are solutions of equation (11.2) (fundamental solutions corresponding to r), and

$$\big(C_0 + C_1 n + C_2 n^2 + \cdots + C_{p-1} n^{p-1}\big) r^n \tag{11.5}$$

is also a solution, for any constants C_0, C_1, $\ldots$, C_{p-1}. If the characteristic equation has no other solutions, then (11.5) is a general solution of the recurrence equation.

Example 11.2 Solve the recurrence equation

$$x_{n+2} = 4x_{n+1} - 4x_n, \quad x_0 = 1, \; x_1 = 3 \; .$$

The characteristic equation is

$$r^2 - 4r + 4 = 0 \; ,$$

with $r = 2$ a solution with multiplicity 2. Then

$$x_n = (C_0 + C_1 n)2^n$$

is a general solution of the recurrence equation.

From the initial conditions we have

$$\begin{aligned} C_0 &= 1\,, \\ 2C_0 + 2C_1 &= 3\,. \end{aligned}$$

From this system of equations $C_0 = 1$, $C_1 = 1/2$, so the general solution is

$$x_n = \left(1 + \frac{1}{2}n\right) 2^n \quad \text{or} \quad x_n = (n+2)2^{n-1}\,.$$

Distinct complex roots. If the complex number $a(\cos b + i \sin b)$, written in trigonometric form, is a root of the characteristic equation, then its conjugate $a(\cos b - i \sin b)$ is also a root, because the coefficients of the characteristic equation are real numbers. Then

$$a^n \cos bn \quad \text{and} \quad a^n \sin bn$$

are solutions of equation (11.2) and

$$C_1 a^n \cos bn + C_2 a^n \sin bn \tag{11.6}$$

is also a solution, for any constants C_1 and C_2. If these are the only solutions of a second order characteristic equation, then (11.6) is a general solution.

Example 11.3 Solve the recurrence equation

$$x_{n+2} = 2x_{n+1} - 2x_n, \quad x_0 = 0, \; x_1 = 1\,.$$

The corresponding characteristic equation is

$$r^2 - 2r + 2 = 0\,,$$

with roots $1+i$ and $1-i$. These can be written in trigonometric form as $\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$ and $\sqrt{2}(\cos(\pi/4) - i \sin(\pi/4))$. Therefore

$$x_n = C_1(\sqrt{2})^n \cos\frac{n\pi}{4} + C_2(\sqrt{2})^n \sin\frac{n\pi}{4}$$

is a general solution of the recurrence equation. From the initial conditions

$$\begin{aligned} C_1 &= 0\,, \\ C_1\sqrt{2}\cos\frac{\pi}{4} + C_2\sqrt{2}\sin\frac{\pi}{4} &= 1\,. \end{aligned}$$

Therefore $C_1 = 0$, $C_2 = 1$. Hence the general solution is

$$x_n = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}\,.$$

Multiple complex roots. If the complex number written in trigonometric form as $a(\cos b + i \sin b)$ is a root of the characteristic equation with multiplicity p, then its conjugate $a(\cos b - i \sin b)$ is also a root with multiplicity p.

Then

$$a^n \cos bn, \; na^n \cos bn, \; \ldots, \; n^{p-1}a^n \cos bn$$

and

$$a^n \sin bn,\ na^n \sin bn,\ \ldots,\ n^{p-1}a^n \sin bn$$

are solutions of the recurrence equation (11.2). Then

$$(C_0 + C_1 n + \cdots + C_{p-1}n^{p-1})a^n \cos bn + (D_0 + D_1 n + \cdots + D_{p-1}n^{p-1})a^n \sin bn$$

is also a solution, where $C_0, C_1, \ldots, C_{p-1}, D_0, D_1, \ldots, D_{p-1}$ are arbitrary constants, which can be determined from the initial conditions. This solution is general if the characteristic equation has no other roots.

Example 11.4 Solve the recurrence equation

$$x_{n+4} + 2x_{n+2} + x_n = 0,\ \ x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3\ .$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = 0\ ,$$

which can be written as $(r^2+1)^2 = 0$. The complex numbers i and $-i$ are double roots. The trigonometric form of these are

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2},\quad \text{and}\quad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

respectively. Therefore the general solution is

$$x_n = (C_0 + C_1 n)\cos\frac{n\pi}{2} + (D_0 + D_1 n)\sin\frac{n\pi}{2}\ .$$

From the initial conditions we obtain

$$\begin{aligned} C_0 &= 0\ , \\ (C_0 + C_1)\cos\frac{\pi}{2} + (D_0 + D_1)\sin\frac{\pi}{2} &= 1\ , \\ (C_0 + 2C_1)\cos\pi + (D_0 + 2D_1)\sin\pi &= 2\ , \\ (C_0 + 3C_1)\cos\frac{3\pi}{2} + (D_0 + 3D_1)\sin\frac{3\pi}{2} &= 3\ , \end{aligned}$$

that is

$$\begin{aligned} C_0 &= 0\ , \\ D_0 + D_1 &= 1\ , \\ -2C_1 &= 2\ , \\ -D_0 - 3D_1 &= 3\ . \end{aligned}$$

Solving this system of equations $C_0 = 0$, $C_1 = -1$, $D_0 = 3$ and $D_1 = -2$. Thus the general solution is

$$x_n = (3 - 2n)\sin\frac{n\pi}{2} - n\cos\frac{n\pi}{2}\ .$$

Using these four cases all linear homogeneous equations with constant coefficients can be solved, if we can solve their characteristic equations.

Example 11.5 Solve the recurrence equation

$$x_{n+3} = 4x_{n+2} - 6x_{n+1} + 4x_n,\ \ x_0 = 0,\ x_1 = 1,\ x_2 = 1\ .$$

The characteristic equation is

$$r^3 - 4r^2 + 6r - 4 = 0 \, ,$$

with roots 2, $1+i$ and $1-i$. Therefore the general solution is

$$x_n = C_1 2^n + C_2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} + C_3\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4} \, .$$

After determining the constants we obtain

$$x_n = -2^{n-1} + \frac{\left(\sqrt{2}\right)^n}{2}\left(\cos\frac{n\pi}{4} + 3\sin\frac{n\pi}{4}\right) \, .$$

The general solution. The characteristic equation of the kth order linear homogeneous equation (11.2) has k roots in the field of complex numbers, which are not necessarily distinct. Let these roots be the following:

r_1 real, with multiplicity p_1 ($p_1 \geq 1$),
r_2 real, with multiplicity p_2 ($p_2 \geq 1$),
...
r_t real, with multiplicity p_t ($p_t \geq 1$),
$s_1 = a_1(\cos b_1 + i\sin b_1)$ complex, with multiplicity q_1 ($q_1 \geq 1$),
$s_2 = a_2(\cos b_2 + i\sin b_2)$ complex, with multiplicity q_2 ($q_2 \geq 1$),
...
$s_m = a_m(\cos b_m + i\sin b_m)$ complex, with multiplicity q_m ($q_m \geq 1$).

Since the equation has k roots, $p_1 + p_2 + \cdots + p_t + 2(q_1 + q_2 + \cdots + q_m) = k$.
In this case the general solution of equation (11.2) is

$$\begin{aligned} x_n &= \sum_{j=1}^{t}\left(C_0^{(j)} + C_1^{(j)} n + \cdots + C_{p_j-1}^{(j)} n^{p_j-1}\right) r_j^n \\ &+ \sum_{j=1}^{m}\left(D_0^{(j)} + D_1^{(j)} n + \cdots + D_{q_j-1}^{(j)} n^{q_j-1}\right) a_j^n \cos b_j n \\ &+ \sum_{j=1}^{m}\left(E_0^{(j)} + E_1^{(j)} n + \cdots + E_{q_j-1}^{(j)} n^{q_j-1}\right) a_j^n \sin b_j n \, , \end{aligned} \tag{11.7}$$

where
$C_0^{(j)},\ C_1^{(j)},\ \ldots,\ C_{p_j-1}^{(j)},\ j = 1, 2, \ldots, t$,
$D_0^{(l)},\ E_0^{(l)},\ D_1^{(l)},\ E_1^{(l)},\ \ldots,\ D_{p_l-1}^{(l)},\ E_{p_l-1}^{(l)},\ l = 1, 2, \ldots, m$ are constants, which can be determined from the initial conditions.

The above statements can be summarised in the following theorem.

Theorem 11.2 *Let $k \geq 1$ be an integer and $a_0, a_1, \ldots, a_k$ real numbers with $a_0,\ a_k \neq 0$. The general solution of the linear recurrence equation (11.2) can be obtained as a linear combination of the terms $n^j r_i^n$, where r_i are the roots of the characteristic equation (11.3) with multiplicity p_i ($0 \leq j < p_i$) and the coefficients of the linear combination depend on the initial conditions.*

The proof of the theorem is left to the Reader (see Exercise 11.1-5).

The algorithm for the general solution is the following.

Linear-Homogeneous

1 determine the characteristic equation of the recurrence equation
2 find all roots of the characteristic equation with their multiplicities
3 find the general solution (11.7) based on the roots
4 determine the constants of (11.7) using the initial conditions, if these exists.

11.1.2. Linear nonhomogeneous recurrence equations

Consider the linear nonhomogeneous recurrence equation with constant coefficients

$$a_0x_n + a_1x_{n+1} + \cdots + a_kx_{n+k} = f(n) , \tag{11.8}$$

where $a_0, a_1, \ldots, a_k$ are real constants, $a_0, a_k \neq 0$, $k \geq 1$, and f is not the zero function.

The corresponding linear homogeneous equation (11.2) can be solved using Theorem 11.2. If a particular solution of equation (11.8) is known, then equation (11.8) can be solved.

Theorem 11.3 *Let $k \geq 1$ be an integer, a_0, a_1, $\ldots, a_k$ real numbers, a_0, $a_k \neq 0$. If $x_n^{(1)}$ is a particular solution of the linear nonhomogeneous equation (11.8) and $x_n^{(0)}$ is a general solution of the linear homogeneous equation (11.2), then*

$$x_n = x_n^{(0)} + x_n^{(1)}$$

is a general solution of the equation (11.8).

The proof of the theorem is left to the Reader (see Exercise 11.1-6).

Example 11.6 Solve the recurrence equation

$$x_{n+2} + x_{n+1} - 2x_n = 2^n, \quad x_0 = 0, \ x_1 = 1 .$$

First we solve the homogeneous equation

$$x_{n+2} + x_{n+1} - 2x_n = 0 ,$$

and obtain the general solution

$$x_n^{(0)} = C_1(-2)^n + C_2 ,$$

since the roots of the characteristic equation are -2 and 1. It is easy to see that

$$x_n = C_1(-2)^n + C_2 + 2^{n-2}$$

is a solution of the nonhomogeneous equation. Therefore the general solution is

$$x_n = -\frac{1}{4}(-2)^n + 2^{n-2} \quad \text{or} \quad x_n = \frac{2^n - (-2)^n}{4} ,$$

$f(n)$	$x_n^{(1)}$
$n^p a^n$	$(C_0 + C_1 n + \cdots + C_p n^p)a^n$
$a^n n^p \sin bn$	$(C_0 + C_1 n + \cdots + C_p n^p)a^n \sin bn + (D_0 + D_1 n + \cdots + D_p n^p)a^n \cos bn$
$a^n n^p \cos bn$	$(C_0 + C_1 n + \cdots + C_p n^p)a^n \sin bn + (D_0 + D_1 n + \cdots + D_p n^p)a^n \cos bn$

Figure 11.1 The form of particular solutions.

The constants C_1 and C_2 can be determined using the initial conditions. Thus,

$$x_n = \begin{cases} 0, & \text{if } n \text{ is even },\\ 2^{n-1}, & \text{if } n \text{ is odd }. \end{cases}$$

A particular solution can be obtained using the *method of variation of constants.* However, there are cases when there is an easier way of finding a particular solution. In Figure 11.1 we can see types of functions $f(n)$, for which a particular solution $x_n^{(1)}$ can be obtained in the given form in the table. The constants can be obtained by substitutions.

In the previous example $f(n) = 2^n$, so the first case can be used with $a = 2$ and $p = 0$. Therefore we try to find a particular solution of the form $C_0 2^n$. After substitution we obtain $C_0 = 1/4$, thus the particular solution is

$$x_n^{(1)} = 2^{n-2}\ .$$

Exercises

11.1-1 Solve the recurrence equation

$$H_n = 2H_{n-1} + 1, \ \text{ if } n \ge 1, \text{ and } \quad H_0 = 0\ .$$

(Here H_n is the optimal number of moves in the problem of the Towers of Hanoi.)

11.1-2 Analyse the problem of the Towers of Hanoi if n discs have to be moved from stick A to stick C in such a way that no disc can be moved *directly* from A to C and vice versa.

Hint. Show that if the optimal number of moves is denoted by M_n, and $n \ge 1$, then $M_n = 3M_{n-1} + 2$.

11.1-3 Solve the recurrence equation

$$(n+1)R_n = 2(2n-1)R_{n-1}, \ \text{ if } n \ge 1, \text{ and } \quad R_0 = 1\ .$$

11.1-4 Solve the linear nonhomogeneous recurrence equation

$$x_n = 2^n - 2 + 2x_{n-1}, \quad \text{if } n \geq 2, \qquad \text{and} \qquad x_1 = 0 \; .$$

Hint. Try to find a particular solution of the form $C_1 n 2^n + C_2$.
11.1-5⋆ Prove Theorem 11.2.
11.1-6 Prove Theorem 11.3.

11.2. Generating functions and recurrence equations

Generating functions can be used, among others, to solve recurrence equations, count objects (e.g. binary trees), prove identities and solve partition problems. Counting the number of objects can be done by stating and solving recurrence equations. These equations are usually not linear, and generating functions can help us in solving them.

11.2.1. Definition and operations

Associate a series with the infinite sequence $(a_n)_{n\geq 0} = \langle a_0, a_1, a_2, \ldots, a_n, \ldots \rangle$ the following way

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = \sum_{n\geq 0} a_n z^n \; .$$

This is called the ***generating function*** of the sequence $(a_n)_{n\geq 0}$.

For example, in case of the Fibonacci numbers this generating function is

$$F(z) = \sum_{n\geq 0} F_n z^n = z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + 13z^7 + \cdots \; .$$

Multiplying both sides of the equation by z, then by z^2, we obtain

$$\begin{aligned} F(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots + F_n z^n + \cdots \; , \\ zF(z) &= F_0 z + F_1 z^2 + F_2 z^3 + \cdots + F_{n-1} z^n + \cdots \; , \\ z^2 F(z) &= F_0 z^2 + F_1 z^3 + \cdots + F_{n-2} z^n + \cdots \; . \end{aligned}$$

If we subtract the second and the third equations from the first one term by term, then use the defining formula of the Fibonacci numbers, we get

$$F(z)(1 - z - z^2) = z \; ,$$

that is

$$F(z) = \frac{z}{1 - z - z^2} \; . \tag{11.9}$$

The correctness of these operations can be proved mathematically, but here we do not want to go into details. The formulae obtained using generating functions can usually also be proved using other methods.

Let us consider the following generating functions

$$A(z) = \sum_{n\geq 0} a_n z^n \text{ and } B(z) = \sum_{n\geq 0} b_n z^n.$$

The generating functions $A(z)$ and $B(z)$ are *equal*, if and only if $a_n = b_n$ for all n natural numbers.

Now we define the following operations with the generating functions: addition, multiplication by real number, shift, multiplication, derivation and integration.

Addition and multiplication by real number.

$$\alpha A(z) + \beta B(z) = \sum_{n\geq 0} (\alpha a_n + \beta b_n) z^n \ .$$

Shift. The generating function

$$z^k A(z) = \sum_{n\geq 0} a_n z^{n+k} = \sum_{n\geq k} a_{n-k} z^n$$

represents the sequence $< \underbrace{0, 0, \ldots, 0}_{k}, a_0, a_1, \ldots >$, while the generating function

$$\frac{1}{z^k}(A(z) - a_0 - a_1 z - a_2 z^2 - \cdots - a_{k-1} z^{k-1}) = \sum_{n\geq k} a_n z^{n-k} = \sum_{n\geq 0} a_{k+n} z^n$$

represents the sequence $< a_k, a_{k+1}, a_{k+2}, \ldots >$.

Example 11.7 Let $A(z) = 1 + z + z^2 + \cdots$. Then

$$\frac{1}{z}\Big(A(z) - 1\Big) = A(z) \quad \text{and} \quad A(z) = \frac{1}{1-z} \ .$$

Multiplication. If $A(z)$ and $B(z)$ are generating functions, then

$$\begin{aligned} A(z)B(z) &= (a_0 + a_1 z + \cdots + a_n z^n + \cdots)(b_0 + b_1 z + \cdots + b_n z^n + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots \\ &= \sum_{n\geq 0} s_n z^n, \end{aligned}$$

where $s_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Special case. If $b_n = 1$ for all natural numbers n, then

$$A(z)\frac{1}{1-z} = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k \right) z^n \ . \tag{11.10}$$

If, in addition, $a_n = 1$ for all n, then

$$\frac{1}{(1-z)^2} = \sum_{n\geq 0} (n+1) z^n \; . \qquad (11.11)$$

Derivation.

$$A'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n\geq 0} (n+1) a_{n+1} z^n \; .$$

Example 11.8 After differentiating both sides of the generating function

$$A(z) = \sum_{n\geq 0} z^n = \frac{1}{1-z},$$

we obtain

$$A'(z) = \sum_{n\geq 1} n z^{n-1} = \frac{1}{(1-z)^2} \; .$$

Integration.

$$\int_0^z A(t)dt = a_0 z + \frac{1}{2} a_1 z^2 + \frac{1}{3} a_2 z^3 + \cdots = \sum_{n\geq 1} \frac{1}{n} a_{n-1} z^n \; .$$

Example 11.9 Let

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

After integrating both sides we get

$$\ln \frac{1}{1-z} = z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \cdots = \sum_{n\geq 1} \frac{1}{n} z^n \; .$$

Multiplying the above generating functions we obtain

$$\frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{n\geq 1} H_n z^n \; ,$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ $\quad (H_0 = 0, \;\; H_1 = 1)$ are the so-called *harmonic numbers*.

Changing the arguments. Let $A(z) = \sum_{n\geq 0} a_n z^n$ represent the sequence $\langle a_0, a_1, a_2, \ldots \rangle$, then $A(cz) = \sum_{n\geq 0} c^n a_n z^n$ represents the sequence $\langle a_0, ca_1, c^2 a_2, \ldots c^n a_n, \ldots \rangle$. The following statements holds

$$\frac{1}{2}\Big(A(z) + A(-z)\Big) = a_0 + a_2 z^2 + \cdots + a_{2n} z^{2n} + \cdots \; ,$$

$$\frac{1}{2}\Big(A(z) - A(-z)\Big) = a_1 z + a_3 z^3 + \cdots + a_{2n-1} z^{2n-1} + \cdots \; .$$

Example 11.10 Let $A(z) = 1 + z + z^2 + z^3 + \cdots = \dfrac{1}{1-z}$. Then

$$1 + z^2 + z^4 + \cdots = \frac{1}{2}\big(A(z) + A(-z)\big) = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right) = \frac{1}{1-z^2} ,$$

which can also be obtained by substituting z by z^2 in $A(z)$. We can obtain the sum of the odd power terms in the same way,

$$z + z^3 + z^5 + \cdots = \frac{1}{2}\big(A(z) - A(-z)\big) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{1+z}\right) = \frac{z}{1-z^2} .$$

Using generating functions we can obtain interesting formulae. For example, let $A(z) = 1/(1-z) = 1 + z + z^2 + z^3 + \cdots$. Then $zA\big(z(1+z)\big) = F(z)$, which is the generating function of the Fibonacci numbers. From this

$$zA\big(z(1+z)\big) = z + z^2(1+z) + z^3(1+z)^2 + z^4(1+z)^3 + \cdots .$$

The coefficient of z^{n+1} on the left-hand side is F_{n+1}, that is the $(n+1)$th Fibonacci number, while the coefficient of z^{n+1} on the right-hand side is

$$\sum_{k\ge 0} \binom{n-k}{k} ,$$

after using the binomial formula in each term. Hence

$$F_{n+1} = \sum_{k\ge 0} \binom{n-k}{k} = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k} . \tag{11.12}$$

Remember that the binomial formula can be generalised for all real r, namely

$$(1+z)^r = \sum_{n\ge 0} \binom{r}{n} z^n ,$$

which is the generating function of the binomial coefficients for a given r. Here $\binom{r}{n}$ is a generalisation of the number of combinations for any real number r, that is

$$\binom{r}{n} = \begin{cases} \dfrac{r(r-1)(r-2)\ldots(r-n+1)}{n(n-1)\ldots 1}, & \text{if } n > 0 , \\ 1, & \text{fi } n = 0 , \\ 0, & \text{if } n < 0 . \end{cases}$$

We can obtain useful formulae using this generalisation for negative r. Let

$$\frac{1}{(1-z)^m} = (1-z)^{-m} = \sum_{k\ge 0} \binom{-m}{k} (-z)^k .$$

Since, by a simple computation, we get

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k} ,$$

the following formula can be obtained

$$\frac{1}{(1-z)^{m+1}} = \sum_{k\ge 0} \binom{m+k}{k} z^k \ .$$

Then

$$\frac{z^m}{(1-z)^{m+1}} = \sum_{k\ge 0} \binom{m+k}{k} z^{m+k} = \sum_{k\ge 0} \binom{m+k}{m} z^{m+k} = \sum_{k\ge 0} \binom{k}{m} z^k \ ,$$

and

$$\sum_{k\ge 0} \binom{k}{m} z^k = \frac{z^m}{(1-z)^{m+1}} \ , \qquad (11.13)$$

where m is a natural number.

11.2.2. Solving recurrence equations by generating functions

If the generating function of the general solution of a recurrence equation to be solved can be expanded in such a way that the coefficients are in closed form, then this method is successful.

Let the recurrence equation be

$$F(x_n, x_{n-1}, \ldots, x_{n-k}) = 0 \ . \qquad (11.14)$$

To solve it, let us consider the generating function

$$X(z) = \sum_{n\ge 0} x_n z^n.$$

If (11.14) can be written as $G\big(X(z)\big) = 0$ and can be solved for $X(z)$, then $X(z)$ can be expanded into series in such a way that x_n can be written in closed form, equation (11.14) can be solved.

Now we give a general method for solving linear nonhomogeneous recurrence equations. After this we give three examples for the nonlinear case. In the first two examples the number of elements in some sets of binary trees, while in the third example the number of leaves of binary trees is computed. The corresponding recurrence equations (11.15), (11.17) and (11.18) will be solved using generating functions.

Linear nonhomogeneous recurrence equations with constant coefficients. Multiply both sides of equation (11.8) by z^n. Then

$$a_0 x_n z^n + a_1 x_{n+1} z^n + \cdots + a_k x_{n+k} z^n = f(n) z^n \ .$$

Summing up both sides of the equation term by term we get

$$a_0 \sum_{n\ge 0} x_n z^n + a_1 \sum_{n\ge 0} x_{n+1} z^n + \cdots + a_k \sum_{n\ge 0} x_{n+k} z^n = \sum_{n\ge 0} f(n) z^n \ .$$

Then

$$a_0 \sum_{n\geq 0} x_n z^n + \frac{a_1}{z} \sum_{n\geq 0} x_{n+1} z^{n+1} + \cdots + \frac{a_k}{z^k} \sum_{n\geq 0} x_{n+k} z^{n+k} = \sum_{n\geq 0} f(n) z^n .$$

Let

$$X(z) = \sum_{n\geq 0} x_n z^n \quad \text{and} \quad F(z) = \sum_{n\geq 0} f(n) z^n .$$

The equation can be written as

$$a_0 X(z) + \frac{a_1}{z}\Big(X(z) - x_0\Big) + \cdots + \frac{a_k}{z^k}\Big(X(z) - x_0 - x_1 z - \cdots - x_{k-1} z^{k-1}\Big) = F(z) .$$

This can be solved for $X(z)$. If $X(z)$ is a rational fraction, then it can be decomposed into partial (elementary) fractions which, after expanding them into series, will give us the general solution x_n of the original recurrence equation. We can also try to use the expansion into series in the case when the function is not a rational fraction.

Example 11.11 Solve the following equation using the above method

$$x_{n+1} - 2x_n = 2^{n+1} - 2, \text{ if } n \geq 0 \quad \text{and } x_0 = 0 .$$

After multiplying and summing we have

$$\frac{1}{z} \sum_{n\geq 0} x_{n+1} z^{n+1} - 2 \sum_{n\geq 0} x_n z^n = 2 \sum_{n\geq 0} 2^n z^n - 2 \sum_{n\geq 0} z^n ,$$

and

$$\frac{1}{z}\Big(X(z) - x_0\Big) - 2X(z) = \frac{2}{1-2z} - \frac{2}{1-z} .$$

Since $x_0 = 0$, after decomposing the right-hand side into partial fractions[1]), the solution of the equation is

$$X(z) = \frac{2z}{(1-2z)^2} + \frac{2}{1-z} - \frac{2}{1-2z} .$$

After differentiating the generating function

$$\frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

term by term we get

$$\frac{2}{(1-2z)^2} = \sum_{n\geq 1} n 2^n z^{n-1} .$$

Thus

$$X(z) = \sum_{n\geq 0} n 2^n z^n + 2 \sum_{n\geq 0} z^n - 2 \sum_{n\geq 0} 2^n z^n = \sum_{n\geq 0} \Big((n-2)2^n + 2\Big) z^n ,$$

therefore

$$x_n = (n-2)2^n + 2 .$$

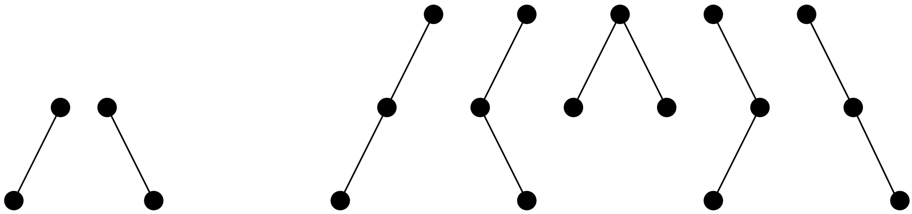

Figure 11.2 Binary trees with two and three vertices.

The number of binary trees. Let us denote by b_n the number of binary trees with n vertices. Then $b_1 = 1$, $b_2 = 2$, $b_3 = 5$ (see Figure 11.2). Let $b_0 = 1$. (We will see later that this is a good choice.)

In a binary tree with n vertices, there are altogether $n-1$ vertices in the left and right subtrees. If the left subtree has k vertices and the right subtree has $n-1-k$ vertices, then there exists $b_k b_{n-1-k}$ such binary trees. Summing over $k = 0, 1, \ldots, n-1$, we obtain exactly the number b_n of binary trees. Thus for any natural number $n \geq 1$ the recurrence equation in b_n is

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0 \ . \tag{11.15}$$

This can also be written as

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k} \ .$$

Multiplying both sides by z^n, then summing over all n, we obtain

$$\sum_{n \geq 1} b_n z^n = \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} b_k b_{n-1-k} \right) z^n \ . \tag{11.16}$$

Let $B(z) = \sum\limits_{n \geq 0} b_n z^n$ be the generating function of the numbers b_n. The left-hand side of (11.16) is exactly $B(z) - 1$ (because $b_0 = 1$). The right-hand side looks like a product of two generating functions. To see which functions are in consideration, let us use the notation

$$A(z) = zB(z) = \sum_{n \geq 0} b_n z^{n+1} = \sum_{n \geq 1} b_{n-1} z^n \ .$$

Then the right-hand side of (11.16) is exactly $A(z)B(z)$, which is $zB^2(z)$. Therefore

$$B(z) - 1 = zB^2(z), \quad B(0) = 1 \ .$$

---

[1] For decomposing the fraction into partial fractions we can use the Method of Undetermined Coefficients.

Solving this equation for $B(z)$ gives

$$B(z) = \frac{1 \pm \sqrt{1-4z}}{2z} .$$

We have to choose the negative sign because $B(0) = 1$. Thus

$$\begin{aligned}
B(z) &= \frac{1}{2z}\left(1-\sqrt{1-4z}\right) = \frac{1}{2z}\left(1-(1-4z)^{1/2}\right) \\
&= \frac{1}{2z}\left(1-\sum_{n\geq 0}\binom{1/2}{n}(-4z)^n\right) = \frac{1}{2z}\left(1-\sum_{n\geq 0}\binom{1/2}{n}(-1)^n 2^{2n} z^n\right) \\
&= \frac{1}{2z} - \binom{1/2}{0}\frac{2^0 z^0}{2z} + \binom{1/2}{1}\frac{2^2 z}{2z} - \cdots - \binom{1/2}{n}(-1)^n \frac{2^{2n} z^n}{2z} + \cdots \\
&= \binom{1/2}{1} 2 - \binom{1/2}{2} 2^3 z + \cdots - \binom{1/2}{n}(-1)^n 2^{2n-1} z^{n-1} + \cdots \\
&= \sum_{n\geq 0}\binom{1/2}{n+1}(-1)^n 2^{2n+1} z^n = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n} z^n .
\end{aligned}$$

Therefore $b_n = \dfrac{1}{n+1}\dbinom{2n}{n}$. The numbers b_n are also called the Catalan numbers.

Remark. In the previous computation we used the following formula that can be proved easily

$$\binom{1/2}{n+1} = \frac{(-1)^n}{2^{2n+1}(n+1)}\binom{2n}{n} .$$

The number of leaves of all binary trees of n vertices. Let us count the number of leaves (vertices with degree 1) in the set of all binary trees of n vertices. Denote this number by f_n. We remark that the root is not considered leaf even if it is of degree 1. It is easy to see that $f_2 = 2$, $f_3 = 6$. Let $f_0 = 0$ and $f_1 = 1$, conventionally. Later we will see that this choice of the initial values is appropriate.

As in the case of numbering the binary trees, consider the binary trees of n vertices having k vertices in the left subtree and $n-k-1$ vertices in the right subtree. There are b_k such left subtrees and b_{n-1-k} right subtrees. If we consider such a left subtree and all such right subtrees, then together there are f_{n-1-k} leaves in the right subtrees. So for a given k there are $b_{n-1-k}f_k + b_k f_{n-1-k}$ leaves. After summing we have

$$f_n = \sum_{k=0}^{n-1}\left(f_k b_{n-1-k} + b_k f_{n-1-k}\right) .$$

By an easy computation we get

$$f_n = 2(f_0 b_{n-1} + f_1 b_{n-2} + \cdots + f_{n-1} b_0), \quad n \geq 2 . \tag{11.17}$$

This is a recurrence equation, with solution f_n. Let

$$F(z) = \sum_{n\geq 0} f_n z^n \quad \text{and} \quad B(z) = \sum_{n\geq 0} b_n z^n .$$

Multiplying both sides of (11.17) by z^n and summing gives

$$\sum_{n\geq 2} f_n z^n = 2\sum_{n\geq 2}\left(\sum_{k=0}^{n-1} f_k b_{n-1-k}\right) z^n .$$

Since $f_0 = 0$ and $f_1 = 1$,

$$F(z) - z = 2zF(z)B(z) .$$

Thus

$$F(z) = \frac{z}{1-2zB(z)} ,$$

and since

$$B(z) = \frac{1}{2z}\left(1-\sqrt{1-4z}\,\right) ,$$

we have

$$F(z) = \frac{z}{\sqrt{1-4z}} = z(1-4z)^{-1/2} = z\sum_{n\geq 0}\binom{-1/2}{n}(-4z)^n .$$

After the computations

$$F(z) = \sum_{n\geq 0}\binom{2n}{n} z^{n+1} = \sum_{n\geq 1}\binom{2n-2}{n-1} z^n ,$$

and

$$f_n = \binom{2n-2}{n-1} \quad \text{or} \quad f_{n+1} = \binom{2n}{n} = (n+1)b_n .$$

The number of binary trees with n vertices and k leaves. A bit harder problem: how many binary trees are there with n vertices and k leaves? Let us denote this number by $b_n^{(k)}$. It is easy to see that $b_n^{(k)} = 0$, if $k > \lfloor (n+1)/2 \rfloor$. By a simple reasoning the case $k=1$ can be solved. The result is $b_n^{(1)} = 2^{n-1}$ for any natural number $n \geq 1$. Let $b_0^{(0)} = 1$, conventionally. We will see later that this choice of the initial value is appropriate. Let us consider, as in case of the previous problems, the left and right subtrees. If the left subtree has i vertices and j leaves, then the right subtree has $n-i-1$ vertices and $k-j$ leaves. The number of these trees is $b_i^{(j)} b_{n-i-1}^{(k-j)}$. Summing over k and j gives

$$b_n^{(k)} = 2b_{n-1}^{(k)} + \sum_{i=1}^{n-2}\sum_{j=1}^{k-1} b_i^{(j)} b_{n-i-1}^{(k-j)} . \tag{11.18}$$

For solving this recurrence equation the generating function

$$B^{(k)}(z) = \sum_{n\geq 0} b_n^{(k)} z^n, \qquad \text{where } k \geq 1$$

will be used. Multiplying both sides of equation (11.18) by z^n and summing over $n = 0, 1, 2, \ldots$, we get

$$\sum_{n\geq 1} b_n^{(k)} z^n = 2\sum_{n\geq 1} b_{n-1}^{(k)} z^n + \sum_{n\geq 1}\left(\sum_{i=1}^{n-2}\sum_{j=1}^{k-1} b_i^{(j)} b_{n-i-1}^{(k-j)}\right) z^n \, .$$

Changing the order of summation gives

$$\sum_{n\geq 1} b_n^{(k)} z^n = 2\sum_{n\geq 1} b_{n-1}^{(k)} z^n + \sum_{j=1}^{k-1}\sum_{n\geq 1}\left(\sum_{i=1}^{n-2} b_i^{(j)} b_{n-i-1}^{(k-j)}\right) z^n \, .$$

Thus

$$B^{(k)}(z) = 2zB^{(k)}(z) + z\left(\sum_{j=1}^{k-1} B^{(j)}(z)B^{(k-j)}(z)\right)$$

or

$$B^{(k)}(z) = \frac{z}{1-2z}\left(\sum_{j=1}^{k-1} B^{(j)}(z)B^{(k-j)}(z)\right) \, . \tag{11.19}$$

Step by step, we can write the following:

$$B^{(2)}(z) = \frac{z}{1-2z}\left(B^{(1)}(z)\right)^2 ,$$

$$B^{(3)}(z) = \frac{2z^2}{(1-2z)^2}\left(B^{(1)}(z)\right)^3 ,$$

$$B^{(4)}(z) = \frac{5z^3}{(1-2z)^3}\left(B^{(1)}(z)\right)^4 .$$

Let us try to find the solution in the form

$$B^{(k)}(z) = \frac{c_k z^{k-1}}{(1-2z)^{k-1}}\left(B^{(1)}(z)\right)^k ,$$

where $c_2 = 1$, $c_3 = 2$, $c_4 = 5$. Substituting in (11.19) gives a recursion for the numbers c_k

$$c_k = \sum_{i=1}^{k-1} c_i c_{k-i} \, .$$

We solve this equation using the generating function method. If $k = 2$, then $c_2 = c_1 c_1$, and so $c_1 = 1$. Let $c_0 = 1$. If $C(z) = \sum_{n\geq 0} c_n z^n$ is the generating function of the numbers c_n, then, using the formula of multiplication of the generating functions we obtain

$$C(z) - 1 - z = (C(z)-1)^2 \quad \text{or} \quad C^2(z) - 3C(z) + z + 2 = 0 \, ,$$

thus

$$C(z) = \frac{3 - \sqrt{1-4z}}{2} \ .$$

Since $C(0) = 1$, only the negative sign can be chosen. After expanding the generating function we get

$$\begin{aligned} C(z) &= \frac{3}{2} - \frac{1}{2}(1-4z)^{1/2} = \frac{3}{2} - \frac{1}{2}\sum_{n\ge 0} \frac{-1}{2n-1}\binom{2n}{n} z^n \\ &= \frac{3}{2} + \sum_{n\ge 0} \frac{1}{2(2n-1)}\binom{2n}{n} z^n = 1 + \sum_{n\ge 1} \frac{1}{2(2n-1)}\binom{2n}{n} z^n \ . \end{aligned}$$

From this

$$c_n = \frac{1}{2(2n-1)}\binom{2n}{n}, \quad n \ge 1 \ .$$

Since $b_n^{(1)} = 2^{n-1}$ for $n \ge 1$, it can be proved easily that $B^{(1)} = z/(1-2z)$. Thus

$$B^{(k)}(z) = \frac{1}{2(2k-1)}\binom{2k}{k} \frac{z^{2k-1}}{(1-2z)^{2k-1}} \ .$$

Using the formula

$$\frac{1}{(1-z)^m} = \sum_{n\ge 0} \binom{n+m-1}{n} z^n \ ,$$

therefore

$$\begin{aligned} B^{(k)}(z) &= \frac{1}{2(2k-1)}\binom{2k}{k} \sum_{n\ge 0} \binom{2k+n-2}{n} 2^n z^{2k+n-1} \\ &= \frac{1}{2(2k-1)}\binom{2k}{k} \sum_{n\ge 2k-1} \binom{n-1}{n-2k+1} 2^{n-2k+1} z^n \ . \end{aligned}$$

Thus

$$b_n^{(k)} = \frac{1}{2k-1}\binom{2k}{k}\binom{n-1}{2k-2} 2^{n-2k}$$

or

$$b_n^{(k)} = \frac{1}{n}\binom{2k}{k}\binom{n}{2k-1} 2^{n-2k} \ .$$

11.2.3. The Z-transform method

When solving linear nonhomogeneous equations using generating functions, the solution is usually done by the expansion of a rational fraction. The Z-transform method can help us in expanding such a function. Let $P(z)/Q(z)$ be a rational fraction, where the degree of $P(z)$ is less, than the degree of $Q(z)$. If the roots of the denominator are known, the rational fraction can be expanded into partial fractions using the Method of Undetermined Coefficients.

Let us first consider the case when the denominator has distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then

$$\frac{P(z)}{Q(z)} = \frac{A_1}{z-\alpha_1} + \cdots + \frac{A_i}{z-\alpha_i} + \cdots + \frac{A_k}{z-\alpha_k} \ .$$

It is easy to see that

$$A_i = \lim_{z\to\alpha_i} (z-\alpha_i)\frac{P(z)}{Q(z)}, \qquad i = 1, 2, \ldots, k \ .$$

But

$$\frac{A_i}{z-\alpha_i} = \frac{A_i}{-\alpha_i\left(1 - \dfrac{1}{\alpha_i}z\right)} = \frac{-A_i\beta_i}{1-\beta_i z} \ ,$$

where $\beta_i = 1/\alpha_i$. Now, by expanding this partial fraction, we get

$$\frac{-A_i\beta_i}{1-\beta_i z} = -A_i\beta_i\left(1 + \beta_i z + \cdots + \beta_i^n z^n + \cdots\right).$$

Denote the coefficient of z^n by $C_i(n)$, then $C_i(n) = -A_i\beta_i^{n+1}$, so

$$C_i(n) = -A_i\beta_i^{n+1} = -\beta_i^{n+1}\lim_{z\to\alpha_i}(z-\alpha_i)\frac{P(z)}{Q(z)} \ ,$$

or

$$C_i(n) = -\beta_i^{n+1}\lim_{z\to\alpha_i}\frac{(z-\alpha_i)P(z)}{Q(z)} \ .$$

After the transformation $z \to 1/z$ and using $\beta_i = 1/\alpha_i$ we obtain

$$C_i(n) = \lim_{z\to\beta_i}\left((z-\beta_i)z^{n-1}\frac{p(z)}{q(z)}\right) \ ,$$

where

$$\frac{p(z)}{q(z)} = \frac{P(1/z)}{Q(1/z)} \ .$$

Thus in the expansion of $X(z) = \dfrac{P(z)}{Q(z)}$ the coefficient of z^n is

$$C_1(n) + C_2(n) + \cdots + C_k(n) \ .$$

If α is a root of the polynomial $Q(z)$, then $\beta = 1/\alpha$ is a root of $q(z)$. E.g. if

$$\frac{P(z)}{Q(z)} = \frac{2z^z}{(1-z)(1-2z)} \ , \quad \text{then} \quad \frac{p(z)}{q(z)} = \frac{2}{(z-1)(z-2)} \ .$$

If case of multiple roots, e.g. if β_i has multiplicity p, their contribution to the solution is

$$C_i(n) = \frac{1}{(p-1)!}\lim_{z\to\beta_i}\frac{d^{p-1}}{dz^{p-1}}\left((z-\beta_i)^p z^{n-1}\frac{p(z)}{q(z)}\right) \ .$$

Here $\dfrac{d^p}{dz^p}f(z)$ is the derivative of order p of the function $f(z)$.

All these can be summarised in the following algorithm. Suppose that the coefficients of the equation are in array A, and the constants of the solution are in array C.

Linear-Nonhomogeneous(A, k, f)

let $a_0x_n + a_1x_{n+1} + \cdots + a_kx_{n+k} = f(n)$ be the equation, where $f(n)$
  is a rational fraction; multiply both sides by z^n, and sum over all n
transform the equation into the form $X(z) = P(z)/Q(z)$, where
  $X(z) = \sum_{n\geq 0} x_n z^n$, $P(z)$ and $Q(z)$ are polynomials
use the transformation $z \to 1/z$, and let the result be
  $p(z)/q(z)$, where $p(z)$ are $q(z)$ are polynomials
denote the roots of $q(z)$ by
  β_1, with multiplicity p_1, $p_1 \geq 1$,
  β_2, with multiplicity p_2, $p_2 \geq 1$,
  ...
  β_k, with multiplicity p_k, $p_k \geq 1$;
then the general solution of the original equation is
$x_n = C_1(n) + C_2(n) + \cdots + C_k(n)$, where
$C_i(n) = 1/((p_i - 1)!) \lim_{z\to\beta_i} \frac{d^{p_i-1}}{dz^{p_i-1}} \left((z-\beta_i)^{p_i} z^{n-1}(p(z)/q(z))\right), i = 1, 2, \ldots, k.$
return C

If we substitute z by $1/z$ in the generating function, the result is the so-called Z-transform, for which similar operations can be defined as for the generating functions. The residue theorem for the Z-transform gives the same result. The name of the method is derived from this observation.

Example 11.12 Solve the recurrence equation

$$x_{n+1} - 2x_n = 2^{n+1} - 2, \quad \text{if} \quad n \geq 0, \quad x_0 = 0 \; .$$

Multiplying both sides by z^n and summing we obtain

$$\sum_{n\geq 0} x_{n+1}z^n - 2\sum_{n\geq 0} x_n z^n = \sum_{n\geq 0} 2^{n+1}z^n - \sum_{n\geq 0} 2z^n \; ,$$

or

$$\frac{1}{z}X(z) - 2X(z) = \frac{2}{1-2z} - \frac{2}{1-z}, \quad \text{where } X(z) = \sum_{n\geq 0} x_n z^n \; .$$

Thus

$$X(z) = \frac{2z^2}{(1-z)(1-2z)^2} \; .$$

After the transformation $z \to 1/z$ we get

$$\frac{p(z)}{q(z)} = \frac{2z}{(z-1)(z-2)^2} \; ,$$

where the roots of the denominator are 1 with multiplicity 1 and 2 with multiplicity 2.

Thus

$$C_1 = \lim_{z\to 1} \frac{2z^n}{(z-2)^2} = 2 \qquad \text{and}$$

$$C_2 = \lim_{z\to 2} \frac{d}{dz}\left(\frac{2z^n}{z-1}\right) = 2\lim_{z\to 2} \frac{nz^{n-1}(z-1)-z^n}{(z-1)^2} = 2^n(n-2)\ .$$

Therefore the general solution is

$$x_n = 2^n(n-2)+2, \quad n \geq 0\ .$$

Example 11.13 Solve the recurrence equation

$$x_{n+2} = 2x_{n+1} - 2x_n, \quad \text{if} \quad n \geq 0, \quad x_0 = 0, \quad x_1 = 1\ .$$

Multiplying by z^n and summing gives

$$\frac{1}{z^2}\sum_{n\geq 0} x_{n+2}z^{n+2} = \frac{2}{z}\sum_{n\geq 0} x_{n+1}z^{n+1} - 2\sum_{n\geq 0} x_n z^n\ ,$$

so

$$\frac{1}{z^2}\Big(F(z)-z\Big) = \frac{2}{z}F(z) - 2F(z)\ ,$$

that is

$$F(z)\left(\frac{1}{z^2}-\frac{2}{z}+2\right) = -\frac{1}{z}\ .$$

Then

$$F(1/z) = \frac{-z}{z^2-2z+2}\ .$$

The roots of the denominator are $1+i$ and $1-i$. Let us compute $C_1(n)$ and $C_2(n)$:

$$C_1(n) = \lim_{z\to 1+i} \frac{-z^{n+1}}{z-(1-i)} = \frac{i(1+i)^n}{2} \quad \text{and}$$

$$C_2(n) = \lim_{z\to 1-i} \frac{-z^{n+1}}{z-(1+i)} = \frac{-i(1-i)^n}{2}\ .$$

Since

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \quad 1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\ ,$$

raising to the nth power gives

$$(1+i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right), \quad (1-i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)\ ,$$

$$x_n = C_1(n) + C_2(n) = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}\ .$$

Exercises

11.2-1 How many binary trees are there with n vertices and no empty left and right subtrees?

11.2-2 How many binary trees are there with n vertices, in which each vertex which is not a leaf, has exactly two descendants?

11.2-3 Solve the following recurrent equation using generating functions.

$$H_n = 2H_{n-1} + 1, \quad H_0 = 0\ .$$

(H_n is the number of moves in the problem of the Towers of Hanoi.)

11.2-4 Solve the following recurrent equation using the Z-transform method.

$$F_{n+2} = F_{n+1} + F_n + 1, \text{ if } n \geq 0, \quad \text{and} \quad F_0 = 0, F_1 = 1 .$$

11.2-5 Solve the following system of recurrence equations:

$$\begin{aligned} u_n &= v_{n-1} + u_{n-2} , \\ v_n &= u_n + u_{n-1} , \end{aligned}$$

where $u_0 = 1, u_1 = 2, v_0 = 1$.

11.3. Numerical solution

Using the following function we can solve the linear recurrent equations numerically. The equation is given in the form

$$a_0 x_n + a_1 x_{n+1} + \cdots + a_k x_{n+k} = f(n) ,$$

where $a_0, a_k \neq 0, k \geq 1$. The coefficients $a_0, a_1, \ldots, a_k$ are kept in array A, the initial values $x_0, x_1, \ldots, x_{k-1}$ in array X. To find x_n we will compute step by step the values $x_k, x_{k+1}, \ldots, x_n$, keeping in the previous k values of the sequence in the first k positions of X (i.e. in the positions with indices $0, 1, \ldots, k-1$).

Recurrence(A, X, k, n, f)

```
1  for j ← k to n
2      do v ← A[0] · X[0]
3         for i ← 1 to k − 1
4             do v ← v + A[i] · X[i]
5         v ← (f(j − k) − v)/A[k]
6         if j ≠ n
7            then for i ← 0 to k − 2
8                      do X[i] ← X[i + 1]
9                 X[k − 1] ← v
10 return v
```

Lines 2–5 compute the values x_j $(j = k, k+1, \ldots, n)$ (using the previous k values), denoted by v in the algorithm. In lines 7–9, if n is not yet reached, we copy the last k values in the first k positions of X. In line 10 x_n is obtained. It is easy to see that the computation time is $\Theta(kn)$, if we disregard the time to compute the values of the function.

Exercises

11.3-1 How many additions, subtractions, multiplications and divisions are required using the algorithm Recurrence, while it computes x_{1000} using the data given in Example 11.4?

Problems

11-1 Existence of a solution of a homogeneous equation using generating function

Prove that a linear homogeneous equation cannot be solved using generating functions (because $X(z) = 0$ is obtained) if and only if $x_n = 0$ for all n.

11-2 Complex roots in case of Z-transform

What happens if the roots of the denominator are complex when applying the Z-transform method? The solution of the recurrence equation must be real. Does the method ensure this?

Chapter Notes

Recurrence equations are discussed in details by Agarwal [1], Elaydi [3], Flajolet and Sedgewick [11], Greene and Knuth [6], Mickens [10], and also in the recent books written by Drmota [2], further by Flajolet and Sedgewick [4].

Knuth [7] and Graham, Knuth and Patashnik [5] deal with generating functions. In the book of Vilenkin [12] there are a lot of simple and interesting problems about recurrences and generating functions.

In [9] Lovász also presents problems on generating functions.

Counting the binary trees is from Knuth [7], counting the leaves in the set of all binary trees and counting the binary trees with n vertices and k leaves are from Zoltán Kása [8].

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