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ON A FAMILY OF FUNCTIONAL EQUATIONS WITH ONE PARAMETER

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Dedicated to my friend Imre Kátai on his 70th birthday

Abstract. Let $I \subset \mathbb{R}$ be a non-void open interval and let $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$ be a given parameter. The functions $f, g : I \to \mathbb{R}_+$ are solutions of the functional equation

$$f\left(\frac{x+y}{2}\right)\Big(2\alpha g(y) - g(x)\Big) = \alpha f(x)g(y) - (1-\alpha)f(y)g(x)$$

$(x, y \in I)$, if and only if f and g are constant functions on I.

1. Introduction

We consider a one-parameter family of functional equations which plays important role in the solution of Matkowski-Sutô type problem ([1, 2, 3, 4, 8, 5, 6, 7]). These equations are interesting themselves independently of the original problem.

Let $I \subset \mathbb{R}$ be a non-void open interval and let $0 < \alpha < 1$ be a given parameter. We investigate the functional equation

$$f\left(\frac{x+y}{2}\right)\Big(2\alpha g(y) - g(x)\Big) = \alpha f(x)g(y) - (1-\alpha)f(y)g(x) \tag{1}$$

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for all $x, y \in I$, where $f, g : I \to \mathbb{R}_+$ ($\mathbb{R}_+$ is the set of positive real numbers) are unknown functions.

For the case $\alpha = \frac{1}{2}$ we have the following proposition ([1]).

Theorem 1 *If $\alpha = \frac{1}{2}$ and the functions $f, g : I \to \mathbb{R}_+$ are continuous solutions of the functional equation (1), then there exist constants $a, b \in \mathbb{R}$ and $c \in \mathbb{R}_+$, such that*

$$f(x) = ax + b > 0 \qquad \text{and} \qquad g(x) = \frac{c}{ax + b}$$

for all $x \in I$.

We remark, that in the case $I = \mathbb{R}$ the constant a is 0 necessarily (and $b \in \mathbb{R}_+$).

In this paper we discuss the following problems:

1. Is the continuity in Theorem 1 necessary, i.e. there exist solutions $f, g : I \to \mathbb{R}_+$ of (1) in the case $\alpha = \frac{1}{2}$ which are non continuous at the interval I or not.
2. What is the situation in the case $\alpha \neq \frac{1}{2}$.

2. On the non continuous solutions in the case $\alpha = \frac{1}{2}$

Theorem 2 *There exist non continuous solutions $f, g : I \to \mathbb{R}_+$ of the functional equation (1) in the case $\alpha = \frac{1}{2}$.*

Proof. Let $t \in I$ be fix. We define

$$g(x) := \begin{cases} a_1 & \text{if } x = t \\ a & \text{if } x \neq t \end{cases} \qquad (x \in I),$$

and

$$f(x) := \begin{cases} b_1 & \text{if } x = t \\ b & \text{if } x \neq t \end{cases} \qquad (x \in I),$$

where a_1, a, b_1, b are positive constants and $a_1 \neq a$, $b_1 \neq b$. These functions $f, g : I \to \mathbb{R}_+$ are non continuous at I and solutions of (1) in the case $\alpha = \frac{1}{2}$, if

$$2 = \frac{a_1 b - a b_1}{a(b - b_1)}. \tag{2}$$

This assertion is trivial if $x \neq t\,,\ y \neq t$.

If $x = t$ and $y \neq t$, then (1) yields

$$a(b - b_1) = \frac{1}{2}(a_1 b - a b_1)\,,$$

i.e. by (2) f and g are solutions of (1). By the symmetry of (1) we have the proof of our assertion. For the equation (2) is an example the following one: $a = b = 3\,,\ a_1 = 4$ and $b_1 = 2$. ■

2. Main result about the equation (1) in the case $\alpha \neq \frac{1}{2}$

Our main result of this paper is the following surprising

Theorem 3 *Let $\alpha \in]0,1[$ and $\alpha \neq \frac{1}{2}$. The functions $f, g : I \to \mathbb{R}_+$ are solutions of the functional equation (1) if and only if there exist constants $a, b \in \mathbb{R}_+$ such that*

$$f(x) = a \qquad \textit{and} \qquad g(x) = b$$

for all $x \in I$.

To prove this theorem we need the following lemmas.

Lemma 1 *Let $0 < \alpha < 1\,,\ \alpha \neq \frac{1}{2}$. If the pair (f, g) $(f, g : I \to \mathbb{R}_+)$ satisfies the functional equation (1) then the following equations*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)g(y) + f(y)g(x)}{g(x) + g(y)}, \tag{3}$$

$$\alpha g(y) - (\alpha + 1)g(x) \neq 0\,, \tag{4}$$

$$\frac{f(x)g(y)}{f(y)g(x)} = \frac{\alpha g(x) - (\alpha + 1)g(y)}{\alpha g(y) - (\alpha + 1)g(x)} \tag{5}$$

for all $x, y \in I$ are true.

Proof. In equation (1) we interchange x and y, then

$$f\left(\frac{x+y}{2}\right)[2\alpha g(x) - g(y)] = \alpha f(y)g(x) - (1 - \alpha)f(x)g(y). \tag{6}$$

We add equations (1) and (6), then we have

$$f\left(\frac{x+y}{2}\right)(2\alpha-1)[g(x)+g(y)]=(2\alpha-1)[f(x)g(y)+f(y)g(x)]\,.$$

From this equation by $2\alpha-1\neq 0$ it follows (3). From (3) by (1) we obtain

$$\frac{f(x)g(y)+f(y)g(x)}{g(x)+g(y)}[2\alpha g(y)-g(x)]=\alpha f(x)g(y)-(1-\alpha)f(y)g(x)$$

for all $x,y\in I$. From the above equation by short computation we have

$$f(x)g(y)[\alpha g(y)-(\alpha+1)g(x)]=f(y)g(x)[\alpha g(x)-(\alpha+1)g(y)] \qquad (7)$$

for all $x,y\in I$. If $x=y$, then $\alpha g(x)-(\alpha+1)g(x)=-g(x)<0$, therefore assertion (4) is true. If $x\neq y$ in (7) and $\alpha g(y)-(\alpha+1)g(x)=0$, then from (7) we have $\alpha g(x)-(\alpha+1)g(y)=0$, i.e.

$$\frac{g(x)}{g(y)}=\frac{\alpha}{1+\alpha} \qquad \text{and} \qquad \frac{g(y)}{g(x)}=\frac{\alpha}{1+\alpha},$$

which is impossible. Hence (4) is true for all $x,y\in I$. From (7) by (4) we have (5) for all $x,y\in I$. ■

Lemma 2 *Let $0<\alpha<1$, $\alpha\neq\frac{1}{2}$ be a fixed number. If the functions $f,g:I\to\mathbb{R}_+$ with the property $f(y_0)=g(y_0)=1$ $(y_0\in I)$ satisfy functional equation (1), then*

$$[g(x)-g(y)][1-g(x)][1-g(y)]=0 \qquad (8)$$

for all $x,y\in I$.

Proof. By 1 we know that (4) and (5) are true. From (5) with $y=y_0\in I$ we have

$$f(x)=g(x)\frac{\alpha g(x)-(\alpha+1)}{\alpha-(\alpha+1)g(x)}$$

for all $x\in I$. We substitute this form of f in equation (5), then we obtain

$$\frac{g(x)\frac{\alpha g(x)-(\alpha+1)}{\alpha-(\alpha+1)g(x)}g(y)}{g(y)\frac{\alpha g(y)-(\alpha+1)}{\alpha-(\alpha+1)g(y)}g(x)}=\frac{\alpha g(x)-(\alpha+1)g(y)}{\alpha g(y)-(\alpha+1)g(x)}$$

for all $x, y \in I$. From this equation with the notation

$$F(x,y) = [\alpha g(x) - (\alpha+1)][\alpha - (\alpha+1)g(y)][\alpha g(y) - (\alpha+1)g(x)]$$

we have

$$F(x,y) = F(y,x)$$

for all $x, y \in I$. From this equation with an easy computation and with the notation $A := \alpha^2(\alpha+1) + \alpha(\alpha+1)^2 > 0$ it follows

$$Ag(x) - Ag(y) + Ag(y)g^2(x) - Ag(x)g^2(y) + Ag^2(y) - Ag^2(x) = 0,$$

i.e.

$$[g(x) - g(y)][1 + g(x)g(y) - g(x) - g(y)] = 0\,.$$

But this is (8) for all $x, y \in I$. ∎

4. Proof of the Theorem 3

In this section we give a complete proof of Theorem 3.

Proof.

1. First we suppose that the functions $f, g : I \to \mathbb{R}_+$ are solutions of the functional equation (1) (where $0 < \alpha < 1\,,\ \alpha \neq \frac{1}{2}$) and $f(y_0) = g(y_0) = 1$ for $y_0 \in I$. We assert, that in this case $f(x) = g(x) = 1$ for all $x \in I$. Contrary, we suppose that there exists $y_1 \in I$ $(y_1 \neq y_0)$, such that

   $$g(y_1) = c \neq 1 \qquad \text{and} \qquad c > 0\,.$$

   With the substitution $y = y_1$ in (8) we have

   $$[g(x) - c][1 - g(x)] = 0 \tag{9}$$

   for all $x \in I$. We define

   $$E := \{\ x \mid x \in I\,,\ g(x) = 1\ \} \neq \emptyset$$

   and

   $$E^* := \{\ x \mid x \in I\,,\ g(x) = c\ \} \neq \emptyset\,.$$

By equation (9) any $x \in I$ is in E or in E^*, i.e. $E \cap E^* = \emptyset$ and $I = E \cup E^*$. By Lemma 2

$$f(x) = g(x)^{\frac{\alpha g(x)-(\alpha+1)}{\alpha-(\alpha+1)g(x)}} = \begin{cases} 1 & \text{if } x \in E, \\ c^{\frac{\alpha c-(\alpha+1)}{\alpha-(\alpha+1)c}} & \text{if } x \in E^*. \end{cases}$$

If $x \in E$ and $y \in E^*$ then by equation (3) we have

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)c + f(y)}{c+1} = \frac{c + c^{\frac{\alpha c-(\alpha+1)}{\alpha-(\alpha+1)c}}}{c+1}\,.$$

Now, $\frac{x+y}{2} \in E$ or $\frac{x+y}{2} \in E^*$. In the first case we have

$$\frac{c + c^{\frac{\alpha c-(\alpha+1)}{\alpha-(\alpha+1)c}}}{c+1} = 1\,,$$

or in the second case

$$\frac{c + c^{\frac{\alpha c-(\alpha+1)}{\alpha-(\alpha+1)c}}}{c+1} = c^{\frac{\alpha c-(\alpha+1)}{\alpha-(\alpha+1)c}}\,.$$

In both cases we obtain $c^2 = c$, i.e. $c = 1$, which is a contradiction. Then $g(x) = 1$ for all $x \in I$ and by (10) it follows $f(x) = 1$ for all $x \in I$.

2. If the pair (f, g) $(f, g : I \to \mathbb{R}_+)$ is a solution of (1) $(0 < \alpha < 1\,,\ \alpha \neq \frac{1}{2})$ then the pair $\left(\frac{f}{f(y_0)}, \frac{g}{g(y_0)}\right)$ $(y_0 \in I)$ is a solution of (1), too, and

$$\frac{f(y_0)}{f(y_0)} = 1, \qquad \frac{g(y_0)}{g(y_0)} = 1.$$

By (i) we have

$$\frac{f(x)}{f(y_0)} = 1, \qquad \frac{g(x)}{g(y_0)} = 1$$

for all $x \in I$. With $f(y_0) := a > 0$ and $g(y_0) := b > 0$ we obtain the assertion of Theorem 3.

■

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