Mathematical and Computer Modelling 50 (2009) 287–291

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Extending partial tournaments

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ARTICLE INFO

Article history:
Received 3 October 2008
Accepted 16 December 2008

Keywords:
Tournament digraph
Tournament matrix
Matrix completions

ABSTRACT

Let A be a $(0, 1, *)$-matrix with main diagonal all 0's and such that if $a_{i,j} = 1$ or $*$ then $a_{j,i} = *$ or 0. Under what conditions on the row sums, and or column sums, of A is it possible to change the $*$'s to 0's or 1's and obtain a tournament matrix (the adjacency matrix of a tournament) with a specified score sequence? We answer this question in the case of regular and nearly regular tournaments. The result we give is best possible in the sense that no relaxation of any condition will always yield a matrix that can be so extended.

1. Introduction

A matrix in which some of the entries are known and some are not is called a *partial matrix*. More specifically, A is a partial matrix and every entry of A is a real number or the symbol "$*$" which indicates the entry is an unknown real number. A problem that has been of interest in the past few decades, the *matrix completion problem*, is of the form: "Given a partial matrix, under what conditions is it possible to assign the unknown entries real fixed values so that the matrix has the property $\mathcal{P}$?" For example, if A is the matrix $\begin{bmatrix} 1 & 1 \\ 1 & * \end{bmatrix}$,is it possible to change the * to some number such that A is a real positive definite symmetric matrix? For a survey of results about matrix completions see [1]. Other matrix completion problems involve the rank of a matrix [2] or the adjacency matrix of a graph or digraph [3,4].

Another area of interest for more than the past few decades is that of tournaments, see the classic text [5] and the recent survey by the third author [6]. A tournament can be viewed as an orientation of the complete simple (loopless) graph; that is, a directed graph such that between any two distinct vertices i and j there is either the arc (i, j) or the arc (j, i) in the edge set, but not both. A tournament matrix is a square matrix which is the adjacency matrix of some tournament. In this case if A is a tournament matrix then $A + A^t = J - I$ where J is the matrix of all ones and I is the identity matrix.

Suppose T is a tournament with vertices labeled $v_1, v_2, \ldots, v_n$ such that $|\{x : v_i \to x\}| \geq |\{y : v_j \to y\}|$ if $i < j$. The number $|\{x : v_i \to x\}| = s_i$ is called the *score* of vertex v_i. The list $(s_1, s_2, \ldots, s_n)$ will be called the *score sequence* of T. If A is the adjacency matrix of T, that is $a_{ij} = 1$ if and only if $v_i \to v_j$ and 0 otherwise, then the score sequence is equivalent to the vector $A\mathbf{j}$ where $\mathbf{j}$ is the $n \times 1$ vector of all 1's.

In this article we investigate conditions on what can be thought of as a $(0, 1, *)$-partial matrix with essentially all off-diagonal zeros being $*$'s so that we are assured the matrix can be completed to a tournament matrix with a given score sequence. In Section 3 we give these conditions for regular tournament matrices (adjacency matrices for tournaments with all scores equal) and nearly regular tournament matrices (adjacency matrices for tournaments with $\max_{i \neq j}\{|s_i - s_j|\} = 1$). The conditions are in terms of the row sums of the $(0, 1)$-matrix to be completed and are best possible in the sense that no relaxation of any condition will always yield a matrix that can be so extended. The next section contains preliminary definitions and notation, and a theorem of Ford and Fulkerson from [7] which we will use.

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0895-7177/$ – see front matter
doi:10.1016/j.mcm.2008.12.015

2. Preliminaries

Definition 2.1. A *tournament of order n* is a directed graph which is an orientation of the complete, simple, undirected graph on n vertices. That is, a tournament is a loopless digraph in which any two distinct vertices are connected by exactly one arc.

Definition 2.2. A *regular tournament* is a tournament which has the same number of outgoing arcs for each vertex; that is, each vertex has the same score.

Remark 2.3. Note that each vertex in a regular tournament of order n has out-degree $\frac{(n-1)}{2}$. Thus, there is no regular tournament of positive even order. Tournaments of even order with scores nearly equal are defined next.

Definition 2.4. A *nearly regular tournament* of order n is a tournament that has $\frac{n}{2}$ vertices each of score $\frac{n}{2}$ and $\frac{n}{2}$ vertices each of score $\frac{(n-2)}{2}$.

Remark 2.5. Note that every nearly regular tournament of order n arises from the deletion of a vertex from a regular tournament of order $n+1$.

Definition 2.6. An *adjacency* matrix of a digraph is a $(0, 1)$-matrix $M = [m_{i,j}]$ such that $m_{i,j} = 1$ if and only if there is an arc with initial vertex i and terminal vertex j.

Remark 2.7. Let $A(T)$ be an adjacency matrix of the tournament T. Then, $A(T)$ is a $(0, 1)$-matrix such that $A(T)+A(T)^{t}+I = J$, where I denotes the identity matrix and J the matrix of all ones. That is, if M is an adjacency matrix of a tournament, M has a zero diagonal and for $i \neq j$, $m_{i,j} \neq 0$ if and only if $m_{j,i} = 0$.

Definition 2.8. Adjacency matrices of tournaments are usually called *tournament matrices*.

Remark 2.9. If n is odd, the adjacency matrices of regular tournaments (*regular tournament matrices*) are tournament matrices with the same number of entries equal to 1 in each row. If n is even, the adjacency matrices of nearly regular tournaments (*nearly regular tournament matrices*) are tournament matrices that have $\frac{n}{2}$ rows each with $\frac{n}{2}$ entries equal to 1, and $\frac{n}{2}$ rows each with $\frac{n-2}{2}$ entries equal to 1.

Definition 2.10. Let A and B be $m \times n$ matrices. The matrix A is said to *dominate* the matrix B, written $A \geq B$ or $B \leq A$, if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$.

Definition 2.11. A *proper mixed graph* is a triple, $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ such that for any vertex $x \in \mathcal{V}$, $\{x, x\} \notin \mathcal{E}$ and $(x, x) \notin \mathcal{A}$, and for any two distinct vertices x and y of $\mathcal{V}$ exactly one of the following conditions holds: (a) $\{x, y\}$ is an edge of $\mathcal{E}$, (b) (x, y) is an arc of $\mathcal{A}$, or (c) (y, x) is an arc of $\mathcal{A}$. In the sequel, we will refer to a proper mixed graph as simply a *mixed graph*.

Definition 2.12. A *closed route* in a mixed graph G is a sequence of vertices, edges and arcs of the form

$$x_1, e_1, x_2, e_2, \ldots, e_N, x_1$$

where x_i is a vertex in $\mathcal{V}$ and each e_i is either the edge $\{x_i, x_{i+1}\}$ in $\mathcal{E}$ or the arc (x_i, x_{i+1}) in $\mathcal{A}$. A mixed graph, G, is *unicursal* if there exists a closed route in G that contains each edge and each arc of G exactly once.

Definition 2.13. For $X \subseteq \mathcal{V}$ let $\overline{X}$ denote the complement of X in $\mathcal{V}$. We denote by $d(X)$ the number of edges in $\mathcal{E}$ of the form $\{x, y\}$ where $x \in X$ and $y \in \overline{X}$; by $d^+(X)$ the number of arcs in $\mathcal{A}$ of the form (x, y) where $x \in X$ and $y \in \overline{X}$; and by $d^-(X)$ the number of arcs in $\mathcal{A}$ of the form (x, y) where $y \in X$ and $x \in \overline{X}$.

We recall the following theorem of Ford and Fulkerson, [7, Theorem 7.1].

Theorem 2.14. *The mixed graph* $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ *is unicursal if and only if*

1. *G is connected;*
2. *every vertex in G is incident with an even number of edges and arcs; and*
3. *for every $X \subseteq \mathcal{V}$ the difference between the number of directed arcs from X to $\overline{X}$ and the number of directed arcs from $\overline{X}$ to X is less than or equal to the number of edges in $\mathcal{E}$ connecting vertices in X to vertices in $\overline{X}$. That is, $d(X) \geq |d^+(X) - d^-(X)|$.*

The following corollary is a special case of Theorem 2.14 which will suffice for our needs.

Corollary 2.15. *Let n be odd and G be a mixed graph such that between any two vertices there is exactly one arc or edge (i.e., a partially oriented complete graph). Then G is unicursal if and only if for every $X \subseteq \mathcal{V}$, $d(X) \geq |d^+(X) - d^-(X)|$.*

Definition 2.16. A directed graph D is *Eulerian* if there is an alternating sequence of vertices and arcs $x_1, e_1, x_2, e_2, \ldots, e_N, x_1$, where N is the number of arcs in D, each x_i is a vertex of D, each e_i is an arc of D, $e_i = (x_i, x_{i+1})$, for $1 \le i \le N-1$, $e_N = (x_N, x_1)$, and each arc of D appears exactly once in the sequence (vertices may repeat).

Remark 2.17. The following facts are easily established.

- A tournament T of order n is regular if and only if n is odd and T is Eulerian.
- A unicursal mixed graph has an Eulerian orientation.

3. Completions of regular and nearly regular tournaments

Given $A = [a_{i,j}]$, we denote the sum of the entries in row i of A by $r_i(A) = \sum_{j=1}^{n} a_{i,j}$. Instead of $r_i(A)$, we will use r_i if the context makes clear the matrix we are referring to. The following basic lemma will be used in the main theorem.

Lemma 3.1. *Let A be an $n \times n$, $(0, 1)$ matrix. For $1 \le i \le n$, if $r_i \le \max(\frac{n-1}{2} - i + 1, 0)$, then the sum of any j column sums, $1 \le j \le \frac{n-1}{2}$, is at most $\sum_{i=1}^{j} \left(\frac{n-1}{2} - i + 1\right)$.*

Proof. Consider the number of ones in any j columns of A. Note that the first $\frac{n-1}{2}$ row sums of A are at most $\frac{n-1}{2}, \frac{n-1}{2} - 1, \ldots, 2, 1$ and the others are all zero. So the last j rows of the j columns of A have at most $j + (j-1) + \cdots + 2 + 1$ ones. The first $\frac{n-1}{2} - j$ rows of these columns have at most $j \cdot (\frac{n-1}{2} - j)$ ones. Thus there are at most $j \cdot (\frac{n-1}{2} - j) + (\sum_{i=1}^{j} i)$ ones. But, $j \cdot (\frac{n-1}{2} - j) + (\sum_{i=1}^{j} i) = j \cdot (\frac{n-1}{2} - j) + \frac{j(j+1)}{2} = \frac{j}{2}(n-j)$ and $\sum_{i=1}^{j} \left(\frac{n-1}{2} - i + 1\right) = j(\frac{n-1}{2}) - (\sum_{i=1}^{j} i) + j = \frac{j}{2}(n-j)$. ■

Theorem 3.2. *Let n be odd and A be an $n \times n$, $(0, 1)$ matrix dominated by a tournament matrix such that the row sums of A are $r_1 \ge r_2 \ge \cdots \ge r_n$. For $1 \le i \le n$, if $r_i \le \max(\frac{n-1}{2} - i + 1, 0)$, then A is dominated by a regular tournament matrix.*

Proof. Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be the mixed graph with $\mathcal{V} = \{1, 2, \ldots, n\}$ obtained from the complete graph by orienting the edges corresponding to the nonzero entries of the matrix $A = [a_{i,j}]$; that is $\mathcal{A} = \{(i, j) : a_{i,j} = 1\}$. Let $X \subseteq \mathcal{V}$; we show $d(X) \ge |d^+(X) - d^-(X)|$.

Case 1: $d^+(X) \ge d^-(X)$. If $|X| \le \frac{n-1}{2}$ then

$$d^+(X) \le \sum_{i=1}^{|X|} r_i \le \sum_{i=1}^{|X|} \left(\frac{n-1}{2} - i + 1\right) = \frac{n - |X|}{2} |X|.$$

For $|X| > \frac{n-1}{2}$, let the multiset of $n - |X|$ column sums corresponding to the vertices in $\mathcal{V} \setminus X$ be denoted $\{c_{i_j} : j = 1, \ldots, n - |X|\}$. Then

$$d^+(X) \le \sum_{j=1}^{n-|X|} c_{i_j}.$$

By Lemma 3.1, the sum is at most

$$\sum_{j=1}^{n-|X|} \left(\frac{n-1}{2} - j + 1\right) = (n - |X|) \frac{n-1}{2} - \sum_{j=1}^{n-|X|-1} j,$$

which simplifies to $\frac{n-|X|}{2}|X|$. In either case we have $d^+(X) \le \frac{n-|X|}{2}|X|$.

Note $d(X) + d^-(X) = |X|(n - |X|) - d^+(X)$, and so

$$d(X) + d^-(X) \ge |X|(n - |X|) - \frac{n - |X|}{2}|X| = \frac{n - |X|}{2}|X| \ge d^+(X).$$

We have $d(X) \ge d^+(X) - d^-(X) = |d^+(X) - d^-(X)|$ by our assumption for this case. By Corollary 2.15, G is unicursal.

Case 2: $d^+(X) \le d^-(X)$. Note that $d^+(X) = d^-(\overline{X})$ and $d^-(X) = d^+(\overline{X})$ so our assumption in this case yields $d^+(\overline{X}) \ge d^-(\overline{X})$. Applying the argument in Case 1 with X replaced by $\overline{X}$ yields $d(\overline{X}) + d^-(\overline{X}) \ge d^+(\overline{X})$. Thus, $d(\overline{X}) \ge d^+(\overline{X}) - d^-(\overline{X})$. And, $d^+(\overline{X}) - d^-(\overline{X}) = d^-(X) - d^+(X) = |d^-(X) - d^+(X)| = |d^+(X) - d^-(X)|$. Since $d(\overline{X}) = d(X)$, we have $d(X) \ge |d^+(X) - d^-(X)|$ and by Corollary 2.15 G is unicursal.

By the above Remark 2.17, there is an orientation of G that is a regular tournament. Therefore, A is dominated by a regular tournament matrix. ■

Theorem 3.3. *Let n be even and A be a $(0, 1)$ matrix dominated by a tournament matrix such that the row sums of A are $r_1 \ge r_2 \ge \cdots \ge r_n$. For $1 \le i \le n$, if $r_i \le \max(\frac{n}{2} - i + 1, 0)$, then A is dominated by a nearly regular tournament matrix.*

Proof. Append an $(n+1)$th row and column of zeros to A to form A^+. Then, an easy check shows that A^+ satisfies the hypothesis of Theorem 3.2. Thus A^+ is dominated by a regular tournament matrix Z. Delete the last row and column of Z to obtain a nearly regular tournament matrix that dominates A. ■

We let K denote the (0, 1)-matrix that has a zero main diagonal and 1's elsewhere.

A mixed graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is said to be *determined by the matrix* A, where A is dominated by a tournament matrix, if the digraph whose adjacency matrix is A is $D = (\mathcal{V}, \mathcal{A})$ and $K - (A + A^t)$ is the adjacency matrix of the undirected graph $U = (\mathcal{V}, \mathcal{E})$.

Remark 3.4. Note that the hypotheses of the theorems are best possible in the sense that if any of the inequalities are relaxed there is an example of a (0, 1)-matrix dominated by a tournament matrix that is not dominated by a regular (or nearly regular) tournament matrix. For example, suppose $n = 2k+1$ and let A be the $(2k+1) \times (2k+1)(0,1)$-matrix so that for some j, $1 < j \le k$, the last $k - j + 2$ entries of the jth row are all 1's and all other entries in that row are 0's, and for each i, $1 \le i \le (j-1)$, the last $k - i + 1$ entries of the ith row are all 1's and all other entries in that row are 0's, and all of the entries in all of the other rows of A are 0's.

Suppose that a $(2k+1) \times (2k+1)(0,1)$ regular tournament matrix B dominates A. So, every 1 in A yields a 1 in B in the corresponding position, and, each row sum and each column sum of B is equal to k. Since the first row of A has k 1's, the first row of B is the same as the first row of A. Thus, the last k entries of the first column of B are all 0's. And, since the first entry of the first column of B is 0, we deduce that the 2nd through $(k+1)$th positions of the first column of B are all 1's. If $j = 2$, then the 2nd row of B has 1's in the last k positions (by the definition of A) as well as in position 1 (by the remark just made) for a total of $k+1$ 1's, a contradiction. So, we assume that $j > 2$. We use induction to show that for all i, $2 \le i \le j-1$, the ith row of B has 1s in the k positions $1, 2, \ldots, i-1$ and $i+k+1, i+k+2, \ldots, 2k+1$, resulting in 0's in all other $k+1$ positions $i, i+1, i+2, \ldots, i+k$. Of course, this will imply that the ith column of B has 0's in positions $1, 2, \ldots, i-1, i$ and $i+k+1, i+k+2, \ldots, 2k+1$, and 1s in all other k positions $i+1, i+2, \ldots, i+k$. Notice that this implies that the $(k+1-i) \times i$ submatrix of B consisting of rows $i+1, i+2, \ldots, k+1$ and columns $1, 2, \ldots, i$ consists entirely of 1s. Suppose that B is as described for some i, $2 \le i < j-1$. Consider row $(i+1)$ of B. As $i+1 < j$, the last $k-(i+1)+1 = k-i$ positions of this row are all 1s (by the definition of A). And, as noted, the induction hypothesis implies that the first i positions of this row are all 1s. This accounts for all of the k 1s in this row of B. So, in row $i+1$ there are 1s in positions $1, 2, \ldots, i-1, i$ and $i+k+2, i+k+3, \ldots, 2k+1$, and 0's in positions $i+1, i+2, \ldots, i+k+1$ (and, the $(i+1)$th column of B has 0's in positions $1, 2, \ldots, i, i+1$ and $i+k+3, i+k+2, \ldots, 2k+1$, and 1s in all other k positions $i+2, i+3, \ldots, i+k+2$), as required. This completes the induction. In particular, as each of the first $j-1$ columns of B have been determined, we see that the first $j-1$ entries of row j of B are 1s. But, by the definition of A, the last $k-j+2$ entries of row j of B are 1s. This means that B contains at least $(k+1)$ 1s, a contradiction. Consequently, there is no such matrix B.

An alternative explanation is: Let G be the mixed graph determined by A. Let $X = \{1, 2, \ldots, j\}$. Then $d^+(X) = \sum_{i=1}^{j} r_i = \sum_{i=1}^{j-1}\left(\frac{n-1}{2} - i + 1\right) + \frac{n-1}{2} - j + 2 = \frac{j(n-j)}{2} + 1$. Then, $d(X) + d^-(X) = j(n-j) - d^+(X) = j(n-j) - \left(\frac{j(n-j)}{2} + 1\right) = \frac{j(n-j)}{2} - 1 < d^+(X)$. By Theorem 2.14, G is not unicursal and hence A is not dominated by a regular tournament matrix. A similar example holds for the nearly regular case.

Theorem 3.3 does not guarantee any specific ordering of the scores. That is, the row sums of the resulting nearly regular tournament matrix might not be in decreasing order when listed from first row to last row. However, if the hypothesis is tightened, any permutation of the scores of a nearly regular tournament can be guaranteed:

Theorem 3.5. *Let n be even and A be a (0, 1) matrix dominated by a tournament matrix such that the row sums of A are $r_1 \ge r_2 \ge \cdots \ge r_n$. Let $\mathcal{I} \subseteq \mathcal{V}$ with $|\mathcal{I}| = \frac{n}{2}$. For $1 \le i \le n$, if $r_i \le \max(\frac{n}{2} - i, 0)$, then A is dominated by a nearly regular tournament matrix with scores precisely $s_i = \frac{n}{2}$ for $i \in \mathcal{I}$ and $s_j = \frac{n}{2} - 1$ for $j \notin \mathcal{I}$.*

Proof. Let B be the matrix whose entries are $b_{i+1,j+1} = a_{i,j}$ for $1 \le i, j \le n$; $b_{1,i+1} = 1$ for $i \in \mathcal{I}$; $b_{1,j+1} = 0$, for $j \notin \mathcal{I}$; and $b_{k,1} = 0$ for $k = 1, \ldots, n+1$. Then B satisfies the hypothesis of Theorem 3.2, and hence is dominated by a regular tournament matrix C. Delete the first row and column of C to get a nearly regular tournament M which dominates A. Because of the choice of the first row of B, the rows of M that have row sum $\frac{n}{2}$ are precisely the rows indexed by $\mathcal{I}$. ■

Corollary 3.6. *Let n be odd, $k \le \frac{n-1}{2}$, and let A be an $n \times n(0,1)$-matrix which is dominated by a tournament matrix. For $1 \le i \le n$, if $r_i(A) \le \max(\frac{n-1}{2} - i + 1, 0)$, then A is dominated by a tournament matrix with score sequence of k scores $\frac{n+1}{2}$, k scores $\frac{n-3}{2}$ and $n - 2k$ scores $\frac{n-1}{2}$.*

Proof. For $\mathcal{Q} \subseteq \{1, 2, \ldots, n-1\}$, let $\overline{\mathcal{Q}}$ denote the complement of $\mathcal{Q}$ in $\{1, 2, \ldots, n-1\}$. We may assume that the first row sum of A is $\frac{n-1}{2}$ by changing some 0's to 1's if needed. Let $\mathcal{I} = \{i | a_{1,i+1} = 1\}$, let $\mathcal{X} \subseteq \mathcal{I}$ such that $|\mathcal{X}| = k$, and let $\mathcal{Y} \subseteq \overline{\mathcal{I}}$ such that $|\mathcal{Y}| = k$, and let $\mathcal{Z} = \overline{\mathcal{X} \cup \mathcal{Y}}$, so that $|\mathcal{Z}| = (n-1) - 2k$. Let $B = (b_{i,j})$ where $b_{i,j} = a_{i+1,j+1}$, $1 \le i, j \le n-1$. Then the ith row sum of B is at most $\frac{n-1}{2} - (i+1) + 1 = \frac{(n-1)}{2} - i$. By Theorem 3.5 there is a nearly regular tournament matrix C which dominates B with row sums r_i, $i = 1, \ldots, n-1$ with $r_i = \frac{(n-1)}{2}$ for all $i \in \mathcal{Y} \cup (\mathcal{Z} \cap \overline{\mathcal{X}})$ and $r_j = \frac{(n-1)}{2} - 1$ for all

$j \in \mathcal{X} \cup (\mathcal{Z} \cap \overline{\mathcal{Y}})$. Let M be the matrix $\begin{bmatrix} 0 & \vec{a}^t \\ \vec{b} & C \end{bmatrix}$,where $[0, \vec{a}^t]$ is the first row of A and $\vec{b}$ is the $(0, 1)(n-1)$-vector with $b_i = 1$ if and only if $i \in \overline{\mathcal{I}}$. Then, M dominates A and has the desired score sequence. ■

Recently Brualdi and Kiernan [8] have established necessary and sufficient conditions for a partial tournament to be dominated by a tournament of fixed score sequence.

Acknowledgments

In 2006 at the Southeastern International Conference on Combinatorics, Graph Theory and Computing at Boca Raton, FL, the first author presented a talk which ended with a conjecture, essentially Theorem 3.2. The next morning, Dustin Stewart and the authors met and an initial proof using network flows was outlined by Dustin Stewart.

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