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Reconstruction of complete interval tournaments. II.

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Abstract. Let a, b ($b \geq a$) and n ($n \geq 2$) be nonnegative integers and let $\mathcal{T}(a, b, n)$ be the set of such generalized tournaments, in which every pair of distinct players is connected at most with b, and at least with a arcs. In [38] we gave a necessary and sufficient condition to decide whether a given sequence of nonnegative integers $D = (d_1, d_2, \ldots, d_n)$ can be realized as the out-degree sequence of a $T \in \mathcal{T}(a, b, n)$. Extending the results of [38] we show that for any sequence of nonnegative integers D there exist f and g such that some element $T \in \mathcal{T}(g, f, n)$ has D as its out-degree sequence, and for any (a, b, n)-tournament T' with the same out-degree sequence D hold $a \leq g$ and $b \geq f$. We propose a $\Theta(n)$ algorithm to determine f and g and an $O(d_n n^2)$ algorithm to construct a corresponding tournament T.

1 Introduction

Let a, b ($b \geq a$) and n ($n \geq 2$) be nonnegative integers and let $\mathcal{T}(a, b, n)$ be the set of such generalized tournaments, in which every pair of distinct players is connected at most with b, and at least with a arcs. The elements of $\mathcal{T}(a, b, n)$ are called (a, b, n)-*tournaments.* The vector $D = (d_1, d_2, \ldots, d_n)$ of the out-degrees of $T \in \mathcal{T}(a, b, n)$ is called *the score vector* of T. If the elements of D are in nondecreasing order, then D is called *the score sequence* of T.

An arbitrary vector $D = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is called *graphical vector,* iff there exists a loopless multigraph whose degree vector is

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D, and D is called *digraphical vector* (or *score vector*) iff there exists a loopless directed multigraph whose out-degree vector is D.

A nondecreasingly ordered graphical vector is called *graphical sequence*, and a nondecreasinly ordered digraphical vector is called *digraphical sequence* (or *score sequence*).

The number of arcs of T going from player P_i to player P_j is denoted by m_{ij} $(1 \leq i, j \leq n)$, and the matrix $\mathcal{M} = [1..n, 1..n]$ is called *point matrix* or *tournament matrix* of T.

In the last sixty years many efforts were devoted to the study of both types of vectors, resp. sequences. E.g. in the papers [14, 16, 17, 18, 19, 24, 28, 30, 31, 32, 34, 42, 64, 82, 79, 84, 91] the graphical sequences, while in the papers [1, 2, 3, 7, 9, 25, 26, 27, 29, 31, 35, 45, 46, 51, 54, 53, 56, 57, 58, 60, 61, 62, 65, 73, 74, 77, 88, 80, 93, 94] the score sequences were discussed.

Even in the last two years many authors investigated the conditions, when D is graphical (e.g. [4, 10, 11, 20, 21, 22, 23, 36, 37, 40, 44, 47, 48, 55, 71, 76, 87, 89, 90, 97]) or digraphical (e.g. [5, 15, 33, 38, 43, 50, 52, 59, 63, 66, 67, 68, 69, 70, 78, 81, 83, 95]).

In this paper we deal only with directed graphs and usually follow the terminology used by K. B. Reid [74, 75]. If in the given context a, b and n are fixed or non important, then we speak simply on *tournaments* instead of generalized or (a, b, n)-tournaments.

We consider the loopless directed multigraphs as generalized tournaments, in which the number of arcs from vertex/player P_i to vertex/player P_j is denoted by m_{ij}, where m_{ij} means the number of points won by player P_i in the match with player P_j.

The first question: how one can characterize the set of the score sequences of the (a, b, n)-tournaments. Or, with another words, for which sequences D of nonnegative integers does exist an (a, b, n)-tournament whose out-degree sequence is D. The answer is given in Section 2.

If T is an (a, b, n)-tournament with point matrix $\mathcal{M} = [1..n, 1..n]$, then let $E(T)$, $F(T)$ and $G(T)$ be defined as follows: $E(T) = \max_{1 \leq i,j \leq n} m_{ij}$, $F(T) = \max_{1 \leq i < j \leq n}(m_{ij} + m_{ji})$, and $g(T) = \min_{1 \leq i < j \leq n}(m_{ij} + m_{ji})$. Let $\Delta(D)$ denote the set of all tournaments having D as out-degree sequence, and let $e(D)$, $f(D)$ and $g(D)$ be defined as follows: $e(D) = \{\min\ E(T) \mid T \in \Delta(D)\}$, $f(D) = \{\min\ f(T) \mid T \in \Delta(D)\}$, and $g(D) = \{\max\ G(T) \mid T \in \Delta(D)\}$. In the sequel we use the short notations E, F, G, e, f, g, and Δ.

Hulett et al. [37, 92], Kapoor et al. [41], and Tripathi et al. [85, 87] investigated the construction problem of a minimal size graph having a prescribed degree set [72, 96]. In a similar way we follow a mini-max approach formulating the following questions: given a sequence D of nonnegative integers,

- How to compute e and how to construct a tournament $T \in \Delta$ characterized by e? In Section 3 a formula to compute e, and an algorithm to construct a corresponding tournament are presented.
- How to compute f and g? In Section 4 an algorithm to compute f and g is described.
- How to construct a tournament $T \in \Delta$ characterized by f and g? In Section 5 an algorithm to construct a corresponding tournament is presented and analysed.

We describe the proposed algorithms in words, by examples and by the pseudocode used in [12].

Researchers of these problems often mention different applications, e.g. in biology [51], chemistry Hakimi [30], and Kim et al. in networks [44].

2 Existence of a tournament with arbitrary degree sequence

Since the numbers of points m_{ij} are not limited, it is easy to construct a $(0, d_n, n)$-tournament for any D.

Lemma 1 *If* $n \geq 2$, *then for any vector of nonnegative integers* $D = (d_1, d_2, \ldots, d_n)$ *there exists a loopless directed multigraph* T *with out-degree vector* D *so, that* $E \leq d_n$.

Proof. Let $m_{n1} = d_n$ and $m_{i,i+1} = d_i$ for $i = 1, 2, \ldots, n-1$, and let the remaining m_{ij} values be equal to zero. ■

Using weighted graphs it would be easy to extend the definition of the (a, b, n)-tournaments to allow *arbitrary real values* of a, b, and D. The following algorithm Naive-Construct works without changes also for input consisting of real numbers.

We remark that Ore in 1956 [62] gave the necessary and sufficient conditions of the existence of a tournament with prescribed in-degree and out-degree vectors. Further Ford and Fulkerson [15, Theorem11.1] published in 1962 necessary and sufficient conditions of the existence of a tournament having prescribed lower and upper bounds for the in-degree and out-degree of the vertices. They results also can serve as basis of the existence of a tournament having arbitrary out-degree sequence.

2.1 Definition of a naive reconstructing algorithm

Sorting of the elements of D is not necessary.

Input. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: arbitrary sequence of nonnegative integer numbers.

Output. $\mathcal{M} = [1..n, 1..n]$: the point matrix of the reconstructed tournament.

Working variables. i, j: cycle variables.

```
Naive-Construct(n, D)
for i ← 1 to n
     for j ← 1 to n
          do m_ij ← 0
m_n1 ← d_n
for i ← 1 to n − 1
     do m_i,i+1 ← d_i
return M
```

The running time of this algorithm is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has n^2 elements, this algorithm is asymptotically optimal.

3 Computation of e

This is also an easy question. From here we suppose that D is a nondecreasing sequence of nonnegative integers, that is $0 \leq d_1 \leq d_2 \leq \ldots \leq d_n$. Let $h = \lceil d_n/(n-1) \rceil$.

Since $\Delta(D)$ is a finite set for any finite score vector D, $e(D) = \min\{E(T) | T \in \Delta(D)\}$ exists.

Lemma 2 *If $n \geq 2$, then for any sequence $D = (d_1, d_2, \ldots, d_n)$ there exists a $(0, b, n)$-tournament T such that*

$$E \leq h \qquad \textit{and} \quad b \leq 2h, \tag{1}$$

and h is the smallest upper bound for e, and $2h$ is the smallest possible upper bound for b.

Proof. If all players gather their points in a uniform as possible manner, that is

$$\max_{1 \leq j \leq n} m_{ij} - \min_{1 \leq j \leq n,\ i \neq j} m_{ij} \leq 1 \quad \text{for } i = 1,\ 2,\ \ldots,\ n, \tag{2}$$

then we get $E \leq h$, that is the bound is valid. Since player P_n has to gather d_n points, the pigeonhole principle [6, 13, 39] implies $E \geq h$, that is the bound is not improvable. $E \leq h$ implies $\max_{1 \leq i < j \leq n} m_{ij} + m_{ji} \leq 2h$. The score sequence $D = (d_1, d_2, \ldots, d_n) = (2n(n-1), 2n(n-1), \ldots, 2n(n-1))$ shows, that the upper bound $b \leq 2h$ is not improvable. ■

Corollary 1 *If* $n \geq 2$*, then for any sequence* $D = (d_1, d_2, \ldots, d_n)$ *holds* $e(D) = \lceil d_n/(n-1) \rceil$.

Proof. According to Lemma 2 $h = \lceil d_n/(n-1) \rceil$ is the smallest upper bound for e. ■

3.1 Definition of a construction algorithm

The following algorithm constructs a $(0, 2h, n)$-tournament T having $E \leq h$ for any D.

Input. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: arbitrary sequence of nonnegative integer numbers.

Output. $\mathcal{M} = [1..n, 1..n]$: the point matrix of the tournament.

Working variables. i, j, l: cycle variables;
k: the number of the "larger parts" in the uniform distribution of the points.

Pigeonhole-Construct(n, D)

```
for i ← 1 to n
    do m_ii ← 0
       k ← d_i − (n − 1)⌊d_i/(n − 1)⌋
    for j ← 1 to k
       do l ← i + j (mod n)
          m_il ← ⌈d_n/(n − 1)⌉
    for j ← k + 1 to n − 1
       do l ← i + j (mod n)
          m_il ← ⌊d_n/(n − 1)⌋
return M
```

The running time of Pigeonhole-Construct is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has n^2 elements, this algorithm is asymptotically optimal.

4 Computation of f and g

Let S_i $(i = 1, 2, \ldots, n)$ be the sum of the first i elements of D, B_i $(i = 1, 2, \ldots, n)$ be the binomial coefficient $n(n-1)/2$. Then the players together can have S_n points only if $fB_n \geq S_n$. Since the score of player P_n is d_n, the pigeonhole principle implies $f \geq \lceil d_n/(n-1) \rceil$.

These observations result the following lower bound for f:

$$f \geq \max\left(\left\lceil \frac{S_n}{B_n} \right\rceil, \left\lceil \frac{d_n}{n-1} \right\rceil \right). \tag{3}$$

If every player gathers his points in a uniform as possible manner then

$$f \leq 2 \left\lceil \frac{d_n}{n-1} \right\rceil. \tag{4}$$

These observations imply a useful characterization of f.

Lemma 3 *If $n \geq 2$, then for arbitrary sequence $D = (d_1, d_2, \ldots, d_n)$ there exists a (g, f, n)-tournament having D as its out-degree sequence and the following bounds for f and g:*

$$\max\left(\left\lceil \frac{S}{B_n} \right\rceil, \left\lceil \frac{d_n}{n-1} \right\rceil \right) \leq f \leq 2 \left\lceil \frac{d_n}{n-1} \right\rceil, \tag{5}$$

$$0 \leq g \leq f. \tag{6}$$

Proof. (5) follows from (3) and (4), (6) follows from the definition of f. ■

It is worth to remark, that if $d_n/(n-1)$ is integer and the scores are identical, then the lower and upper bounds in (5) coincide and so Lemma 3 gives the exact value of F.

In connection with this lemma we consider three examples. If $d_i = d_n = 2c(n-1)$ $(c > 0,\ i = 1, 2, \ldots, n-1)$, then $d_n/(n-1) = 2c$ and $S_n/B_n = c$, that is S_n/B_n is twice larger than $d_n/(n-1)$. In the other extremal case, when $d_i = 0$ $(i = 1, 2, \ldots, n-1)$ and $d_n = cn(n-1) > 0$, then $d_n/(n-1) = cn$, $S_n/B_n = 2c$, so $d_n/(n-1)$ is $n/2$ times larger, than S_n/B_n.

If $D = (0, 0, 0, 40, 40, 40)$, then Lemma 3 gives the bounds $8 \leq f \leq 16$. Elementary calculations show that Figure 1 contains the solution with minimal f, where $f = 10$.

In [38] we proved the following assertion.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_5	Score
P_1	—	0	0	0	0	0	0
P_2	0	—	0	0	0	0	0
P_3	0	0	—	0	0	0	0
P_4	10	10	10	—	5	5	40
P_5	10	10	10	5	—	5	40
P_6	10	10	10	5	5	—	40

Figure 1: Point matrix of a $(0,10,6)$-tournament with $f = 10$ for $D = (0,0,0,40,40,40)$.

Theorem 1 *For* $n \geq 2$ *a nondecreasing sequence* $D = (d_1, d_2, \ldots, d_n)$ *of nonnegative integers is the score sequence of some* (a, b, n)*-tournament if and only if*

$$aB_k \leq \sum_{i=1}^{k} d_i \leq bB_n - L_k - (n-k)d_k \quad (1 \leq k \leq n), \tag{7}$$

where

$$L_0 = 0, \text{ and } L_k = \max\left(L_{k-1},\ bB_k - \sum_{i=1}^{k} d_i\right) \quad (1 \leq k \leq n). \tag{8}$$

The theorem proved by Moon [57], and later by Kemnitz and Dolff [43] for (a, a, n)-tournaments is the special case $a = b$ of Theorem 1. Theorem 3.1.4 of [20] is the special case $a = b = 2$. The theorem of Landau [51] is the special case $a = b = 1$ of Theorem 1.

4.1 Definition of a testing algorithm

The following algorithm Interval-Test decides whether a given D is a score sequence of an (a, b, n)-tournament or not. This algorithm is based on Theorem 1 and returns $W =$ True if D is a score sequence, and returns $W =$ False otherwise.

Input. a: minimal number of points divided after each match;
b: maximal number of points divided after each match.

Output. W: logical variable ($W =$ True shows that D is an (a, b, n)-tournament.

Local working variable. i: cycle variable;
$L = (L_0, L_1, \ldots, L_n)$: the sequence of the values of the loss function.

Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores.

```
Interval-Test(a, b)
for i ← 1 to n
    do L_i ← max(L_{i−1}, bB_n − S_i − (n − i)d_i)
       if S_i < aB_i
          then W ← False
               return W
       if S_i > bB_n − L_i − (n − i)d_i
          then W ← False
               return W
return W
```

In worst case Interval-Test runs in $\Theta(n)$ time even in the general case $0 < a < b$ (n the best case the running time of Interval-Test is $\Theta(n)$). It is worth to mention, that the often referenced Havel–Hakimi algorithm [30, 34] even in the special case $a = b = 1$ decides in $\Theta(n^2)$ time whether a sequence D is digraphical or not.

4.2 Definition of an algorithm computing f and g

The following algorithm is based on the bounds of f and g given by Lemma 3 and the logarithmic search algorithm described by D. E. Knuth [49, page 410].

Input. No special input (global working variables serve as input).
Output. b: the minimal F.
a: the maximal G.
Local working variables. i: cycle variable;
l: lower bound of the interval of the possible values of F;
u: upper bound of the interval of the possible values of F.
Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores;
W: logical variable (its value is True, when the investigated D is a score sequence).

```
MinF-MaxG
01 B_0 ← 0                                  ▷ Initialization
02 S_0 ← 0
03 L_0 ← 0
04 for i ← 1 to n
05     do B_i ← B_{i−1} + i − 1
06        S_i ← S_{i−1} + d_i
07 l ← max(⌈S_n/B_n⌉, ⌈d_n/(n − 1)⌉)
08 u ← 2 ⌈d_n/(n − 1)⌉
09 W ← True                                 ▷ Computation of f
10 Interval-Test(0, l)
11 if W = True
12    then b ← l
13    go to 23
14 b ← ⌈(l + u)/2⌉
15 Interval-Test(0, f)
16 if W = True
17    then go to 19
18 l ← b
19 if u = l + 1
20    then b ← u
21    go to 39
22 go to 14
23 l ← 0                                    ▷ Computation of g
24 u ← f
25 Interval-Test(b, b)
26 if W = True
27    then a ← f
28         go to 39
29 a ← ⌈(l + u)/2⌉
30 Interval-Test(0, a)
31 if W = True
32    then l ← a
33         go to 35
34 u ← a
35 if u = l + 1
36    then a ← l
37         go to 39
38 go to 29
```

```
return a, b
```

MinF-MaxG determines f and g.

Lemma 4 *Algorithm* MinG-MaxG *computes the values* f *and* g *for arbitrary sequence* $D = (d_1, d_2, \ldots, d_n)$ *in* $O(n \log(d_n/(n))$ *time.*

Proof. According to Lemma 3 F is an element of the interval $[\lceil d_n/(n-1) \rceil, \lceil 2d_n/(n-1) \rceil]$ and g is an element of the interval $[0, f]$. Using Theorem B of [49, page 412] we get that $O(\log(d_n/n))$ calls of Interval-Test is sufficient, so the $O(n)$ run time of Interval-Test implies the required running time of MinF-MaxG. ■

4.3 Computing of f and g in linear time

Analysing Theorem 1 and the work of algorithm MinF-MaxG one can observe that the maximal value of G and the minimal value of F can be computed independently by Linear-MinF-MaxG.

Input. No special input (global working variables serve as input).
Output. b: the minimal F.
a: the maximal G.
Local working variables. i: cycle variable.
Global working variables. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores.

Linear-MinF-MaxG

```
B_0 ← 0                              ▷ Initialization
S_0 ← 0
L_0 ← 0
for i ← 1 to n
      do B_i ← B_{i−1} + i − 1
            S_i ← S_{i−1} + d_i
a ← 0)
b ← min 2 ⌈d_n/(n − 1)⌉
for i ← 1 to n                       ▷ Computation of f
do a_i ← ⌈(2S_i/(n^2 − n)⌉) < a
      if a_i > a
      a ← a_i
```

13 **for** $i \leftarrow 1 \text{to} n$ ▷ Computation of f
14 **do** $L_i \leftarrow \max(L_{i-1}, bB_n - S_i - (n-i)d_i$
15 $\quad b_i \leftarrow (S_i + (n-i)d_i + L_i)/B_i$
16 $\quad$ **if** $b_i < b$
17 $\quad$ **then** $b \leftarrow b_i$
18 **return** a, b

Lemma 5 *Algorithm* Linear-MinG-MaxG *computes the values* f *and* g *for arbitrary sequence* $D = (d_1, d_2, \ldots, d_n)$ *in* $\Theta(n)$ *time.*

Proof. Lines 01–03, 07, and 18 require only constant time, lines 04–06, 09–12, and 13–17 require $\Theta(n)$ time, so the total running time is $\Theta(n)$. ■

5 Tournament with f and g

The following reconstruction algorithm is based on balancing between additional points (they are similar to „excess", introduced by Brauer et al. [8]) and missing points introduced in [38]. The greediness of the algorithm Havel–Hakimi [30, 34] also characterizes this algorithm.

This algorithm is an extended version of the algorithm Score-Slicing proposed in [38].

5.1 Definition of the minimax reconstruction algorithm

The work of the slicing program is managed by the following program Mini-Max.

Input. n: the number of players ($n \geq 2$);
$D = (d_1, d_2, \ldots, d_n)$: a nondecreasing sequence of integers satisfying (7).

Output. $\mathcal{M} = [1 \,.\,.\, n, 1 \,.\,.\, n]$: the point matrix of the reconstructed tournament.

Local working variables. i, j: cycle variables.

Global working variables. $p = (p_0, p_1, \ldots, p_n)$: provisional score sequence;
$P = (P_0, P_1, \ldots, P_n)$: the partial sums of the provisional scores;
$\mathcal{M}[1 \,.\,.\, n, 1 \,.\,.\, n]$: matrix of the provisional points.

Mini-Max(n, D)

01 MinF-MaxG(n, D) ▷ Initialization
02 $p_0 \leftarrow 0$

```
P0 ← 0
for i ← 1 to n
    do for j ← 1 to i − 1
            do M[i, j] ← b
        for j ← i to n
            do M[i, j] ← 0
    pi ← di
if n ≥ 3                                    ▷ Score slicing for n ≥ 3 players
  then for k ← n downto 3
          do Score-Slicing(k)
if n = 2                                    ▷ Score slicing for 2 players
  then m1,2 ← p1
       m2,1 ← p2
return M
```

5.2 Definition of the score slicing algorithm

The key part of the reconstruction is the following algorithm Score-Slicing [38].

During the reconstruction process we have to take into account the following bounds:

$$a \le m_{i,j} + m_{j,i} \le b \quad (1 \le i < j \le n); \tag{9}$$

modified scores have to satisfy (7); (10)

$$m_{i,j} \le p_i \ (1 \le i,\ j \le n, i \ne j); \tag{11}$$

the monotonicity $p_1 \le p_2 \le \ldots \le p_k$ has to be saved $(1 \le k \le n)$ (12)

$$m_{ii} = 0 \quad (1 \le i \le n). \tag{13}$$

Input. k: the number of the actually investigated players $(k > 2)$;
$p_k = (p_0, p_1, p_2, \ldots, p_k)$ $(k = 3,\ 4,\ \cdots,\ n)$: prefix of the provisional score sequence p;
$\mathcal{M}[1 \mathinner{.\,.} n, 1 \mathinner{.\,.} n]$: matrix of provisional points;

Output. Local working variables. $A = (A_1, A_2, \ldots, A_n)$ the number of the additional points;
M: missing points: the difference of the number of actual points and the number of maximal possible points of P_k;
d: difference of the maximal decreasable score and the following largest score;
y: number of sliced points per player;

f: frequency of the number of maximal values among the scores $p_1,\ p_2,\dots,p_{k-1}$;
$i,\ j$: cycle variables;
m: maximal amount of sliceable points;
$P = (P_0, P_1, \dots, P_n)$: the sums of the provisional scores;
x: the maximal index i with $i < k$ and $m_{i,k} < b$.

Global working variables: n: the number of players ($n \geq 2$);
$B = (B_0, B_1, B_2, \dots, B_n)$: the sequence of the binomial coefficients;
a: minimal number of points divided after each match;
b: maximal number of points divided after each match.

```
Score-Slicing(k)
for i ← 1 to k − 1                          ▷ Initialization
    do P_i ← P_{i−1} + p_i
       A_i ← P_i − aB_i
M ← (k − 1)b − p_k
while M > 0 and A_{k−1} > 0                 ▷ There are missing and additional points too
      do x ← k − 1
         while r_{x,k} = b
               do x ← x − 1
      f ← 1
      while p_{x−f+1} = p_{x−f}
            do f = f + 1
      d ← p_{x−f+1} − p_{x−f}
      m ← min(b, d, ⌈A_x/b⌉, ⌈M/b⌉)
      for i ← f downto 1
          do y ← min(b − r_{x+1−i,k}, m, M, A_{x+1−i}, p_{x+1−i})
             r_{x+1−i,k} ← r_{x+1−i,k} + y
             p_{x+1−i} ← p_{x+1−i} − y
             r_{k,x+1−i} ← b − r_{x+1−i,k}
             M ← M − y
          for j ← i downto 1
              A_{x+1−i} ← A_{x+1−i} − y
while M > 0                                 ▷ No missing points
      i ← k − 1
      y ← max(m_{ki} + m_{ik} − a, m_{ki}, M)
      r_{ki} ← r_{ki} − y
      M ← M − y
      i ← i − 1
return π_k, M
```

Let's consider an example. Figure 2 shows the point table of a $(2, 10, 6)$-tournament T.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	1	5	1	1	1	09
P_2	1	—	4	2	0	2	09
P_3	3	3	—	5	4	4	19
P_4	8	2	5	—	2	3	20
P_5	9	9	5	7	—	2	32
P_6	8	7	5	6	8	—	34

Figure 2: The point table of a $(2, 10, 6)$-tournament T.

The score sequence of T is D = (9,9,19,20,32,34). In [38] the algorithm Score-Slicing resulted the point table represented in Figure 3.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	1	1	6	1	0	9
P_2	1	—	1	6	1	0	9
P_3	1	1	—	6	8	3	19
P_4	3	3	3	—	8	3	20
P_5	9	9	2	2	—	10	32
P_6	10	10	7	7	0	—	34

Figure 3: The point table of T reconstructed by Score-Slicing.

The algorithm Mini-Max starts with the computation of f. MinF-MaxG called in line 01 begins with initialization, including provisional setting of the elements of $\mathcal{M}$ so, that $m_{ij} = b$, if $i > j$, and $m_{ij} = 0$ otherwise. Then MinF-MaxG sets the lower bound $l = \max(9, 7) = 9$ of f in line 07 and tests it in line 10 Interval-Test. The test shows that $l = 9$ is large enough so Mini-Max sets $b = 9$ in line 12 and jumps to line 23 and begins to compute g. Interval-Test called in line 25 shows that $a = 9$ is too large, therefore MinF-MaxG continues with the test of $a = 5$ in line 30. The result is positive, therefore comes the test of $a = 7$, then the test of $a = 8$. Now $u = l + 1$ in line 35, so $a = 8$ is fixed, and the control returns to line 02 of Mini-Max.

Lines 02–09 contain initialization, and Mini-Max begins the reconstruction of a $(8, 9, 6)$-tournament in line 10. The basic idea is that Mini-Max successively determines the won and lost points of P_6, P_5, P_4 and P_3 by repeated calls of SCORE-SLICING in line 12, and finally it computes directly the result of the match between P_2 and P_1.

At first MINI-MAX computes the results of P_6 calling calling SCORE-SLICING with parameter $k = 6$. The number of additional points of the first five players is $A_5 = 89 - 8 \cdot 10 = 9$ according to line 03, the number of missing points of P_6 is $M = 5 \cdot 9 - 34 = 11$ according to line 04. Then SCORE-SLICING determines the number of maximal numbers among the provisional scores p_1, p_2, ..., p_5 ($f = 1$ according to lines 09–14) and computes the difference between p_5 and p_4 ($d = 12$ according to line 12). In line 13 we get, that $m = 9$ points are sliceable, and P_5 gets these points in the match with P_6 in line 16, so the number of missing points of P_6 decreases to $M = 11 - 9 = 2$ (line 19) and the number of additional point decreases to $A = 9 - 9 = 0$. Therefore the computation continues in lines 22–27 and m_{64} and m_{63} will be decreased by 1 resulting $m_{64} = 8$ and $m_{63} = 8$ as the seventh line and seventh column of Figure 4 show. The returned score sequence is $p = (9, 9, 19, 20, 23)$.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	4	4	0	0	0	9
P_2	4	—	4	1	0	0	9
P_3	4	4	—	7	4	0	19
P_4	7	7	1	—	5	0	20
P_5	8	8	4	3	—	9	32
P_6	9	9	8	8	0	—	34

Figure 4: The point table of T reconstructed by MINI-MAX.

Second time MINI-MAX calls SCORE-SLICING with parameter $k = 5$, and get $A_4 = 9$ and $M = 13$. At first A_4 gets 1 point, then A_3 and A_4 get both 4 points, reducing M to 4 and A_4 to 0. The computation continues in line 22 and results the further decrease of m_{54}, m_{53}, m_{52}, and m_{51} by 1, resulting $m_{54} = 3$, $m_{53} = 4$, $m_{52} = 8$, and $m_{51} = 8$ as the sixth row of Figure 4 shows.

Third time MINI-MAX calls SCORE-SLICING with parameter $k = 4$, and get $A_3 = 11$ and $M = 11$. At first P_3 gets 6 points, then P_3 further 1 point, and P_2 and P_1 also both get 1 point, resulting $m_{34} = 7$, $m_{43} = 2$, $m_{42} = 8$, $m_{24} = 1$, $m_{14} = 1$ and $m_{14} = 8$, further $A_3 = 0$ and $M = 2$. The computation continues in lines 22–27 and results a decrease of m_{43} by 1 point resulting $m_{43} = 1$, $m_{42=8}$, and $m_{41} = 8$, as the fifth row and fifth column of Figure 4 show. The returned score sequence is $p = (9, 9, 15)$.

Fourth time Mini-Max calls Score-Slicing with parameter $k = 3$, and gets $A_2 = 10$ and $M = 9$. At first P_2 gets 6 points, then ... The returned point vector is $p = (4, 4)$.

Finally Mini-Max sets $m_{12} = 4$ and $m_{21} = 4$ in lines 14–15 and returns the point matrix represented in Figure 4.

The comparison of Figures 3 and 4 shows a large difference between the simple reconstruction of Score-Slicing and the minimax reconstruction of Mini-Max: while in the first case the maximal value of $m_{ij} + m_{ji}$ is 10 and the minimal value is 2, in the second case the maximum equals to 9 and the minimum equals to 8, that is the result is more balanced (the given D does not allow to build a perfectly balanced (k, k, n)-tournament).

5.3 Analysis of the minimax reconstruction algorithm

The main result of this paper is the following assertion.

Theorem 2 *If $n \geq 2$ is a positive integer and $D = (d_1, d_2, \ldots, d_n)$ is a nondecreasing sequence of nonnegative integers, then there exist positive integers f and g, and a (g, f, n)-tournament T with point matrix $\mathcal{M}$ such, that*

$$f = \min(m_{ij} + m_{ji}) \leq b, \tag{14}$$

$$g = \max m_{ij} + m_{ji} \geq a \tag{15}$$

for any (a, b, n)-tournament, and algorithm Linear-MinF-MaxG *computes f and g in $\Theta(n)$ time, and algorithm* Mini-Max *generates a suitable T in $O(d_n n^2)$ time.*

Proof. The correctness of the algorithms Score-Slicing, MinF-MaxG implies the correctness of Mini-Max.

Lines 1–46 of Mini-Max require $O(\log(d_n/n))$ uses of MinG-MaxF, and one search needs $O(n)$ steps for the testing, so the computation of f and g can be executed in $O(n \log(d_n/n))$ times.

The reconstruction part (lines 47–55) uses algorithm Score-Slicing, which runs in $O(bn^3)$ time [38]. Mini-Max calls Score-Slicing $n - 2$ times with $f \leq 2\lceil d_n/n \rceil$, so $n^3 d_n/n = d_n n^2$ finishes the proof. ∎

The property of the tournament reconstruction problem that the extremal values of f and g can be determined independently and so there exists a tournament T having both extremal features is called linking property. This concept was introduced by Ford and Fulkerson in 1962 [15] and later extended by A. Frank in [20].

6 Summary

A nondecreasing sequence of nonnegative integers $D = (d_1, d_2, \ldots, d_n)$ is a score sequence of a $(1, 1, 1)$-tournament, iff the sum of the elements of D equals to B_n and the sum of the first i $(i = 1, 2, \ldots, n-1)$ elements of D is at least B_i [51].

D is a score sequence of a (k, k, n)-tournament, iff the sum of the elements of D equals to kB_n, and the sum of the first i elements of D is at least kB_i [43, 56].

D is a score sequence of an (a, b, n)-tournament, iff (7) holds [38].

In all 3 cases the decision whether D is digraphical requires only linear time.

In this paper the results of [38] are extended proving that for any D there exists an optimal minimax realization T, that is a tournament having D as its out-degree sequence and maximal G and minimal F in the set of all realization of D.

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