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ON THE EXISTENCE OF GRAPHS WITH PRESCRIBED DEGREES AND CONNECTIVITY*

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Abstract. Given a set $\delta = \{d_0, d_1, \cdots, d_{n-1}\}$ of positive integers and a positive integer p, the necessary and sufficient conditions are presented for existence of a p-connected n-vertex graph (which allows parallel edges but no self-loops) whose vertex degrees are $d_0, d_1, \cdots, d_{n-1}$. Some related problems are discussed.

1. Introduction. A *graph*, denoted by $G(V, E)$ or simply G, consists of a set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ of *vertices* and a set $E = \{e_0, e_1, \cdots, e_{m-1}\}$ of *edges*. With each edge $e_k \in E$, there is associated a pair of distinct vertices v_i and $v_j \in V$. Such an edge may be represented by the unordered pair (v_i, v_j). If $e_k = (v_i, v_j)$, then e_k is said to be *incident* at v_i and v_j, and v_i and v_j are called *adjacent* vertices in G. If two edges, say e_k and $e_l \in E$, have the same unordered pair representation, say (v_i, v_j), then they are called *parallel* edges. A graph may or may not have parallel edges. Following Harary [1], we call a graph with parallel edges a *multigraph* but reserve the word "*unigraph*" for those graphs which do not have parallel edges. In a multigraph, it is implicitly assumed that parallel edges are distinguishable from one another although they have the same unordered pair representations.

A graph $g(V', E')$ is called a *subgraph* of $G(V, E)$ if $E' \subseteq E$ and $V' \subseteq V$. Let $g = g(V', E')$ be a subgraph of G; the *degree of vertex* $v_i \in V'$ with respect to g, denoted by $d_i(g)$, is the number of edges of g incident at v_i. For each pair of distinct vertices v_r and $v_s \in V$, *a path between* v_r *and* v_s *in* G, denoted by $p(v_r, v_s)$, is a sequence of edges $(v_r, v_{k1})(v_{k1}, v_{k2}) \cdots (v_{kt}, v_s)$ of G where all vertices in the sequence are distinct. In the above path, the set of vertices $\{v_{k1}, v_{k2}, \cdots, v_{kt}\}$ is the set of *nonterminal vertices* of $p(v_r, v_s)$. Two paths $p_1(v_r, v_s)$ and $p_2(v_r, v_s)$ in G are said to be *nonintersecting* if their sets of nonterminal vertices are disjoint. Let us define $\rho_{rs}(G)$ to be equal to the maximum number of nonintersecting paths between v_r and v_s in G with the stipulation that if v_r and v_s are adjacent in G, then at most one of these paths is edge (v_r, v_s). The *connectivity* of G, denoted by $\rho(G)$, is

$$\rho(G) = \min_{\substack{v_r \text{ and } v_s \in V \\ v_r \neq v_s}} \rho_{rs}(G). \tag{1}$$

Let $G(V, E)$ be given. Let $X \subset V$. By $G - X$, we mean the subgraph $g(V', E')$ of G, where $V' = V - X$ and $E' = \{(v_i, v_j) \in E | v_i \text{ and } v_j \in V'\}$. We define $\rho(G - X)$ to be equal to $\min \rho_{rs}(G - X)$, where the minimum is taken over all v_r and $v_s \in V'$, $v_r \neq v_s$. Let $X_0 \subset V$ be a subset of V with the smallest cardinality such that $\rho(G - X_0) = 0$. Then by Menger's theorem [1], $\rho(G) = |X_0|$, where $|X_0|$ is the cardinality of X_0. A graph G is *p-connected*, if $\rho(G) \geqq p$.

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Let $\delta = \{d_0, d_1, \cdots, d_{n-1}\}$ be a set of nonnegative integers and assume $d_0 \leqq d_1 \leqq \cdots \leqq d_{n-1}$. A graph $G(V, E)$, with $V = \{v_0, v_1, \cdots, v_{n-1}\}$, *realizes* δ if $d_i(G) = d_i$ for $i = 0, 1, 2, \cdots, n-1$, and such a set δ is said to be *realizable* and G is a *realization* of δ. It is known that [2], [3] δ is realizable if and only if $S(\delta) = \sum_{i=0}^{n-1} d_i$ is even and $S(\delta) \geqq 2d_{n-1}$.

This paper presents a complete solution to the following problem: Given a realizable set δ and a positive integer p, what are the necessary and sufficient conditions for existence of a p-connected realization of δ? A simple procedure for construction of the desired graphs is described. Also it is easy to find a realization of δ with maximum connectivity.

Suppose we are given a set δ which is realizable as a unigraph [1], [2], [4], [5] and a positive integer p. We may be asked: is there a p-connected unigraph realization of δ? This problem is more difficult. Results for $p \leqq 3$ are available [1], [6]. We close the paper by discussing this problem and some practical variations of it.

2. The main result. Two special cases of the problem for $p = 1$ and $p = 2$ were solved previously. Since the general theorem (Theorem 3) does not cover the special cases, we begin this section by stating these known results.

THEOREM 1 (Senior [3] and Hakimi [2]). *Let $\delta = \{d_0, d_1, \cdots, d_{n-1}\}$ be a given realizable set of integers with $d_i \leqq d_{i+1}$ for $i = 0, 1, \cdots, n-2$. Then, δ is realizable as a 1-connected graph if and only if $d_0 \geqq 1$ and $S(\delta) = \sum_{i=0}^{n-1} d_i \geqq 2(n-1)$.*

THEOREM 2 (Hakimi [2]). *Let $\delta = \{d_0, d_1, \cdots, d_{n-1}\}$ be a given realizable set of integers with $d_i \leqq d_{i+1}$ for $i = 0, 1, \cdots, n-2$. Then, δ is realizable as a 2-connected graph if and only if $n > 2$, $d_0 \geqq 2$, and $S(\delta) \geqq 2\,d_{n-1} + 2(n-2)$.*

The main result of this paper is stated as follows.

THEOREM 3. *Let $\delta = \{d_0, d_1, \cdots, d_{n-1}\}$ be a given realizable set of integers with $d_i \leqq d_{i+1}$ for $i = 0, 1, \cdots, n-2$, and $p \geqq 3$. Then, δ is realizable as a p-connected graph if and only if $p \leqq n-1$, $d_0 \geqq p$, and $S(\delta) \geqq 2d_{n-1} + (n-1)(p-1)$.*

Proof. The *necessity* is proved as follows. Suppose $G(V, E)$ is a p-connected realization of δ. By the definition of connectivity, p cannot exceed $n-1$. Also $d_0(G) = d_0 \geqq p$, because otherwise, $\rho_{0i}(G) < p$, and G would not be p-connected. By Menger's theorem, $\rho(G - \{v_{n-1}\}) \geqq p-1$. But the number of edges in $G - \{v_{n-1}\}$ is equal to $(1/2)(\sum_{i=0}^{n-1} d_i) - d_{n-1}$ which must be at least $(1/2)(n-1) \cdot (p-1)$. This implies the necessity of $S(\delta) \geqq 2d_{n-1} + (n-1)(p-1)$.

The *sufficiency* is proved in two parts. Part (A): $S(\delta) - 2d_{n-1} \geqq (n-2)p$. Part (B): $(n-1)(p-1) \leqq S(\delta) - 2d_{n-1} < (n-2)p$. Since $p \leqq n-1$, $(n-1) \cdot (p-1) \leqq (n-2)p$. Therefore, the above two parts are valid and together exhaust all possibilities. By $[x]$, we mean the integral part of real number x.

Part (A): $S(\delta) - 2d_{n-1} \geqq (n-2)p$. We first construct a unigraph $G_1(V, E_1)$ in three steps as follows.

Step (a). If $i - j \equiv 1, 2, \cdots, [p/2] \bmod n$ and $0 \leqq i, j \leqq n-1$, then

$$(v_i, v_j) \in E_1.$$

Step (b). If p is odd and $j \equiv i + [(n+1)/2] \bmod n$ and $i = 0, 1, \cdots, [n/2] - 1$, then

$$(v_i, v_j) \in E_1.$$

Step (c). If p is odd and n is odd, then

$$(v_{[n/2]}, v_{n-1}) \in E_1.$$

It can be shown (Harary [7] and Hakimi [8]) that the graph $G_1 = G_1(V, E_1)$ as constructed by the above procedure is p-connected. Furthermore, it can be seen that

$$d_i(G_1) = \begin{cases} p + 1, & \text{if } p \text{ and } n \text{ are odd and } i = n - 1, \\ p, & \text{otherwise.} \end{cases} \tag{2}$$

Consider the set of integers $\delta^* = \{d_0^*, d_1^*, \cdots, d_{n-1}^*\}$, where $d_i^* = d_i - d_i(G_1)$ for $i = 0, 1, \cdots, n - 1$. We would like to show that δ^* is realizable as a graph $G_2(V, E_2)$. To do this, we first note that $d_i^* \geqq 0$ for $i = 0, 1, \cdots, n - 1$, because if $d_i = p$ for $i = 0, 1, \cdots, n - 1$, then both p and n cannot be odd. Also we have $d_i^* \leqq d_{i+1}^*$ for $i = 0, 1, \cdots, n - 2$, except when n and p are both odd. If both n and p are odd, then it is possible that $d_{n-2}^* > d_{n-1}^*$, in which case $d_{n-2}^* = d_{n-1}^* + 1$. Since $S(\delta^*) = \sum_{i=0}^{n-1} (d_i - d_i(G_1)) = S(\delta) - 2|E_1|$, where $|E_1|$ is the number of edges in G_1, $S(\delta^*)$ is an even number. Finally, to prove δ^* is realizable, we must show that $S(\delta^*) \geqq 2 \max(d_{n-2}^*, d_{n-1}^*)$. We note that $\max(d_{n-2}^*, d_{n-1}^*) \leqq d_{n-1} - p$. This is because, if p and n are not both odd, then $\max(d_{n-2}^*, d_{n-1}^*) = d_{n-1}^* = d_{n-1} - p$; on the other hand, if p and n are both odd, then $\max(d_{n-2}^*, d_{n-1}^*) \leqq d_{n-1}^* + 1 = d_{n-1} - (p + 1) + 1 = d_{n-1} - p$. Therefore, we need only to show that $S(\delta^*) \geqq 2(d_{n-1} - p)$. Since $S(\delta^*) = S(\delta) - (n - 1)p - d_{n-1}(G_1)$, we must show that

$$S(\delta) - (n - 1)p - d_{n-1}(G_1) \geqq 2(d_{n-1} - p), \tag{3}$$

or

$$S(\delta) - 2d_{n-1} \geqq (n - 2)p + d_{n-1}(G_1) - p. \tag{4}$$

If both p and n are not odd, then $d_{n-1}(G_1) = p$, hence (4) is implied by the hypothesis of Part (A). If both p and n are odd, then $d_{n-1}(G_1) = p + 1$ and (4) becomes

$$S(\delta) - 2d_{n-1} \geqq (n - 2)p + 1. \tag{5}$$

But since $S(\delta) - 2d_{n-1}$ is even and no less than $(n - 2)p$ which is odd, (5) holds. This ends the proof of the existence of $G_2(V, E_2)$ which realizes δ^*. The proof of Part (A) is complete by noting that the graph $G(V, E_1 \cup E_2)$, obtained by superimposing G_1 and G_2, is a p-connected realization of δ.

Part (B). $(n - 1)(p - 1) \leqq S(\delta) - 2d_{n-1} < (n - 2)p$. Let us pick integer r such that $S(\delta) - 2d_{n-1} = pn - p - r$. Comparing this equality with the hypothesis of Part (B), we see that $p < r \leqq n - 1$. Let us define the set $\delta^* = \{d_0^*, d_1^*, \cdots, d_{n-1}^*\}$ with $d_i^* = d_i - p$ for $i = 0, 1, \cdots, n - 2$ and $d_{n-1}^* = d_{n-1} - r$. By definition of r, we have

$$\sum_{i=0}^{n-1} d_i - 2d_{n-1} = (n - 1)p - r \tag{6}$$

which may be rewritten as

$$\sum_{i=0}^{n-2} (d_i - p) = d_{n-1} - r. \tag{7}$$

However (7), by definition of δ^*, implies

$$\sum_{i=0}^{n-2} d_i^* = d_{n-1}^*. \tag{8}$$

Equation (8) and the fact that $d_i \geqq p$ for $i = 0, 1, \cdots, n-1$ immediately imply that δ^* is realizable as a graph, say $G_1(V, E_1)$. If the set of n integers $\delta(p, r) = \{p, p, \cdots, p, r\}$ is realizable as a p-connected graph $G_2(V, E(p, r))$, then $G(V, E_1 \cup E(p, r))$ would be a p-connected realization of δ. Thus all that is left to prove is to show that $\delta(p, r)$ is realizable as a p-connected graph. We note here that $\delta(p, r)$ is realizable, because $S(\delta(p, r)) = (n-1)p + r = S(\delta) - S(\delta^*)$ which is even, and since $p \geqq 3$ and $r \leqq n-1$, $S(\delta(p, r)) \geqq 2r$. The realizability of $\delta(p, r)$ as a p-connected graph is established in the next section.

3. Basic lemma. The main result of this section is to prove the following lemma which would complete the proof of Theorem 3.

LEMMA. *Let $\delta(p, r) = \{p, p, \cdots, p, r\}$ be a realizable set of n integers with $p \geqq 3$ and $p < r \leqq n-1$. Then, $\delta(p, r)$ is realizable as a p-connected unigraph.*

It should be noted that the above lemma holds even if $r = p$ for $p \geqq 2$, but this is a known result [7], [8] and the method of construction, for this case, is given in Part (A) of the proof of Theorem 3. Before we give a proof of the lemma, as stated, we describe an algorithm for the construction of the desired unigraph, denoted by $G(V, E(p, r))$. Then, we prove two "assertions" about the algorithm which will pave the way for a proof of the lemma.

For the remainder of this section, we represent the set of vertices $V = \{v_0, v_1, \cdots, v_{n-1}\}$ of G by the set of integers $\{0, 1, 2, \cdots, n-1\}$, that is, vertex v_i is represented by integer i, $0 \leqq i \leqq n-1$. By $\bar{j} \equiv a$, we mean j modulo $(n-1) = a$. The algorithm is divided into three cases. These cases together cover all possibilities. We assume $r < n-1$; the case when $r = n-1$ will be covered separately. For each case, the algorithm is completed in three steps.

Case 1. p is even, then r is even.

Step (a). If $0 \leqq i, j \leqq n-2$ and $\overline{i-j} \equiv 1, 2, \cdots, [(p-1)/2]$, then $(i, j) \in E(p, r)$.

Step (b). If $j = i + [(n-2)/2]$ and $i = 0, 1, \cdots, [r/2] - 1$, then $(i, n-1)$ and $(j, n-1) \in E(p, r)$.

Step (c). If $j \equiv \overline{i + [(n-2)/2]}$ and $i = [r/2], [r/2] + 1, \cdots, n-2$, then $(i, j) \in E(p, r)$.

Case 2. p is odd, $n-1$ is even, then r is even.

Step (a). The same as Step (a) in Case 1.

Step (b). If $j = i + [(n-1)/2]$ and $i = 0, 1, \cdots, [r/2] - 1$, then $(i, n-1)$ and $(j, n-1) \in E(p, r)$.

Step (c). If $j = i + [(n-1)/2]$ and $i = [r/2], [r/2] + 1, \cdots, [(n-1)/2] - 1$, then $(i, j) \in E(p, r)$.

Case 3. p is odd and $n-1$ is odd, then r is odd.

Step (a). The same as Step (a) in Case 1.

Step (b). The same as Step (b) in Case 2.

Step (c). If $j = i + [(n-1)/2]$ and $i = [r/2], [r/2] + 1, \cdots, [(n-1)/2] - 1$, then $(i, j) \in E(p, r)$, and $(n-2, n-1) \in E(p, r)$.

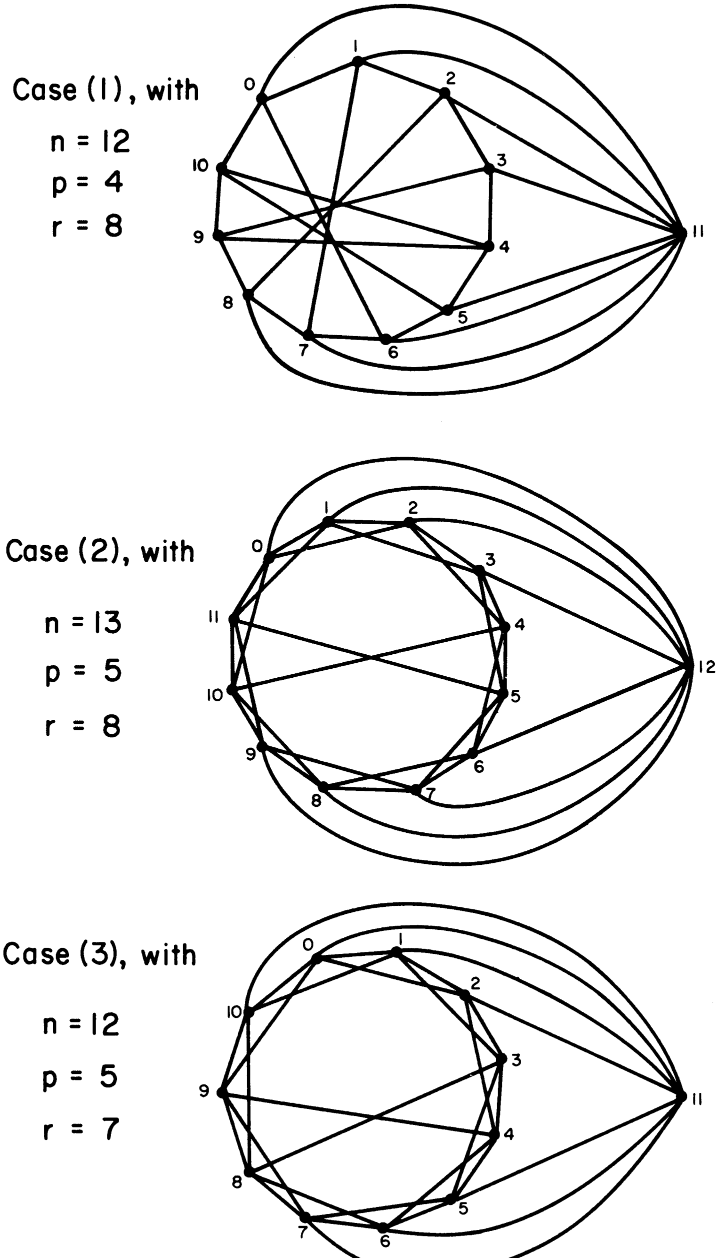

FIG. 1

Figure 1 illustrates examples of graphs constructed by this algorithm for each of the above three cases. About this algorithm, we state and prove the following two elementary assertions.

ASSERTION 1. *Let $\delta(p, r) = \{p, p, \cdots, p, r\}$ be a realizable set of n integers with $3 \leqq p < r < n - 1$. Then the graph $G(V, E(p, r))$, constructed by the algorithm, realizes $\delta(p, r)$, i.e., $d_i(G) = p$ for $i = 0, 1, \cdots, n - 2$ and $d_{n-1}(G) = r$.*

Proof. We first consider Case 1. After completion of Step (a), we have a graph $G'(V, E'(p, r))$ with $d_i(G') = p - 2$ for $i = 0, 1, \cdots, n - 2$ and $d_{n-1}(G') = 0$. By the end of Step (b), we obtain G'' with

$$d_i(G'') = \begin{cases} p - 1 & \text{for } i = 0, 1, \cdots, [r/2] - 1, \\ p - 2 & \text{for } i = [r/2], [r/2] + 1, \cdots, [(n - 2)/2] - 1, \\ p - 1 & \text{for } i = [(n - 2)/2], [(n - 2)/2] + 1, \cdots, [(n - 2)/2] + [r/2] - 1, \\ p - 2 & \text{for } i = [(n - 2)/2] + [r/2], [(n - 2)/2] + [r/2] + 1, \cdots, n - 2, \\ r & \text{for } i = n - 1. \end{cases}$$

This is because $[r/2] - 1 < [(n - 2)/2]$ and $[(n - 2)/2] + [r/2] - 1 < n - 2$. To see that by the end of Step (c) we obtain the desired graph, we proceed as follows. We note that if in Step (b), instead of placing $(i, n - 1)$ and $(j, n - 1) \in E(p, r)$, we would place $(i, j) \in E(p, r)$, then we could combine Steps (b) and (c) into a single step which would require $(i, j) \in E(p, r)$ if $\overline{(i - j)} \equiv [(n - 2)/2]$. Such a modification of the algorithm would result in construction of a graph G^* with $d_i(G^*) = p$ for $i = 0, 1, \cdots, n - 2$ and $d_{n-1}(G^*) = 0$. However, this modification merely affects the degree of vertex $n - 1$, which was already equal to r at the end of Step (b). This ends the proof for Case 1.

Now, let us consider Case 2. By the end of Step (a), we have a graph G', with $d_i(G') = p - 1$ for $i = 0, 1, \cdots, n - 2$ and $d_{n-1}(G') = 0$. By the end of Step (b), we obtain G'' with

$$d_i(G'') = \begin{cases} p & \text{for } i = 0, 1, \cdots, [r/2] - 1, \\ p - 1 & \text{for } i = [r/2], [r/2] + 1, \cdots, [(n - 1)/2] - 1, \\ p & \text{for } i = [(n - 1)/2], [(n - 1)/2] + 1, \cdots, [(n - 1)/2] + [r/2] - 1, \\ p - 1 & \text{for } i = [(n - 1)/2] + [r/2], [(n - 1)/2] + [r/2] + 1, \cdots, n - 2, \\ r & \text{for } i = n - 1. \end{cases}$$

At the end of Step (c), we have the desired graph, because $[(n - 1)/2] - 1 + [(n - 1)/2] = n - 2$.

To prove the assertion for Case 3, we note that by the end of Step (b), we have a graph G'' with $d_i(G'')$ being the same as in (10), except $d_{n-1}(G'') = r - 1$. Since $n - 1$ is odd, $[(n - 1)/2] - 1 + [(n - 1)/2] = n - 3$ and, as a result, at the end of the first part of Step (c) all degrees are correct except the degrees of vertices $n - 2$ and $n - 1$. These deficiencies are corrected by placing $(n - 2, n - 1) \in E(p, r)$. This completes the proof of Assertion 1.

ASSERTION 2. *Let $\delta(p, r) = \{p, p, \cdots, p, r\}$ be a realizable set of n integers with $3 \leqq p < r < n - 1$. Let $G(V, E(p, r))$ be the graph obtained using the algorithm* *which by Assertion* 1 *realizes* $\delta(p, r)$. *Let* $X \subset V$ *with* $|X| \leqq p$ be such that $\rho(G - X) = 0$. *Then there exist integers* x, s, $t \in [0, n - 2]$ *with* s *and* $t \neq 0$, *which define the disjoint sets of vertices*

$$V_a = \{\overline{x+1}, \overline{x+2}, \cdots, \overline{x+s}\}, \quad V_b = \{\overline{x+s+1}, \overline{x+s+2}, \cdots, \overline{x+s+\pi}\},$$

$$V_c = \{\overline{x+s+\pi+1}, \overline{x+s+\pi+2}, \cdots, \overline{x+s+\pi+t}\},$$

$$V_d = \{\overline{x+s+\pi+t+1}, \overline{x+s+\pi+t+2}, \quad \cdots, \overline{x+s+\pi+t+\pi}\}$$

such that $V_b \cup V_d \subseteq X$, *where* $\pi = [(p-1)/2]$ *and* $s + t + 2\pi = n - 1$.

Proof. $\rho(G - X) = 0$ implies that there exist vertices i and $j \in V - X$ such that $\rho_{ij}(G - X) = 0$. Since $|X| \leqq p$ and $r > p$, one of the "components" of $G - X$ cannot be the vertex $n - 1$, hence i and j may be picked such that neither is $n - 1$. Without loss of generality, assume $0 \leqq i < j \leqq n - 2$. We first claim that if there does not exist an integer $y, i \leqq y < j$, such that $V_\beta = \{y + 1, y + 2, \cdots, y + \pi\} \subset X$, then there exists a path between i and j in $G - X$ which passes through only vertices in $B = \{k | i \leqq k \leqq j\} - X$. We show the existence of this path as follows. Let z be the largest integer in B which is reachable from i using only vertices in B. If $z = j$, there is nothing left to prove. If $0 < j - z \leqq \pi$, then edge (j, z) is in $G - X$ and the existence of the desired path is established. If $j - z > \pi$, then consider the set of vertices $V'_\beta = \{z + 1, z + 2, \cdots, z + \pi\}$. By assumption, $V'_\beta \not\subset X$; thus for some $c, z + c \in V'_\beta$ and $z + c \notin X$. This implies that edge $(z, z + c)$ exists in $G - X$ which contradicts the fact that z was the largest integer in B reachable from i. Thus, we conclude that there exists an integer $y, i \leqq y < j$, such that $\{y + 1, y + 2, \cdots, y + \pi\} \subset X$. Similarly, one can establish the existence of a path between i and j in $G - X$ passing through only vertices in $D = \{k | j \leqq k \leqq n - 2$, or $0 \leqq k \leqq i\} - X$ unless there exists an integer $q \in D$ such that $V_\delta = \{\overline{q+1}, \overline{q+2}, \cdots, \overline{q+\pi}\} \subset X$. We note that V_β and V_δ are disjoint. If we pick x, s and $t \in [0, n-2]$ such that $x \equiv \overline{q+\pi}$, $y \equiv \overline{x+s}$, and $q \equiv \overline{x+s+\pi+t}$, then the assertion is established. Also since $\overline{q+\pi} \equiv x$, we have $\overline{x+s+\pi+t+\pi} \equiv x$, or $s + 2\pi + t = n - 1$.

Proof of the lemma. Let $\delta(p, r) = \{p, p, \cdots, p, r\}$ be a realizable set of n integers with $3 \leqq p < r \leqq n - 1$. We would like to prove that $\delta(p, r)$ is realizable as a p-connected graph.

We first assume $r = n - 1$. Let $G_1(V_1, E_1)$ with $V_1 = \{0, 1, \cdots, n - 2\}$ be a $(p - 1)$-connected graph which realizes the set of $n - 1$ integers $\{p - 1, p - 1, \cdots, p - 1\}$. (This can be done using the construction procedure given in Part (A) of the sufficiency proof of Theorem 3. It should be noted that since $(n - 1)(p - 1) + 2(n - 1) = S(\delta(p, r))$ is even, both $n - 1$ and $p - 1$ cannot be odd. As a result the degrees of all vertices in the graph obtained by that construction procedure will be equal to $p - 1$.) We construct the desired graph $G(V, E(p, r))$ from G_1 as follows: $V = V_1 \cup \{n - 1\}$ and $E(p, r) = E_1 \cup \{(i, n - 1) | i \in V_1\}$. It is clear that $G(V, E(p, r)) = G$ realizes $\delta(p, r)$. To prove G is p-connected, we note that if i and $j \in V_1$, then there are $p - 1$ disjoint paths in G_1 between i and j. In addition to these

$p - 1$ paths, there is in G the path $(i, n - 1)(n - 1, j)$ which gives a total of p disjoint paths between i and j. Therefore, $\rho_{ij}(G) \geqq p$ for all i and $j, i \neq j$, and $0 \leqq i, j \leqq n - 2$. Since vertex $n - 1$ is adjacent to all other vertices in G, $\rho_{ij}(G) \geqq p$ for all i and $j, i \neq j$, and $0 \leqq i, j \leqq n - 1$. This ends the proof of the special case, $r = n - 1$.

Now let us assume $p < r < n - 1$. Let $G(V, E(p, r)) = G$ be constructed by the algorithm. By Assertion 1, G realizes $\delta(p, r)$. We need to show that G is also p-connected. Suppose otherwise; then, by Menger's theorem, there exists a set $X \subset V$ with $|X| < p$ such that $\rho(G - X) = 0$. This, by Assertion 2, implies that there exist integers $x, s, t \in [0, n - 2]$ with s and $t \neq 0$, which define the disjoint sets of vertices: $V_a = \{\overline{x + 1}, \overline{x + 2}, \cdots, \overline{x + s}\}$, $V_b = \{\overline{x + s + 1}, \overline{x + s + 2}, \cdots, \overline{x + s + \pi}\}$, $V_c = \{\overline{x + s + \pi + 1}, \overline{x + s + \pi + 2}, \cdots, \overline{x + s + \pi + t}\}$, and $V_d = \{\overline{x + s + \pi + t + 1}, \overline{x + s + \pi + t + 2}, \cdots, \overline{x + s + \pi + t + \pi}\}$ with $V_b \cup V_d \subseteq X$, where $s + 2\pi + t = n - 1$ and $\pi = [(p - 1)/2]$. Since the cardinality of $V_b \cup V_d$ is equal to 2π, X can contain one vertex not in $V_b \cup V_d$ only if p is even. Thus, we shall consider two cases.

Suppose p is odd; then $V_b \cup V_d = X$. As was stated in the proof of Assertion 2, i and j may be picked such that $\rho_{ij}(G - X) = 0$ and neither i nor j is equal to $n - 1$. Then without loss of generality, assume $i \in V_a$ and $j \in V_c$. Note, since $V_b \cup V_d = X$, that all vertices in V_a (or V_c) are reachable from i (or j) in $G - X$, repectively, by construction. We first claim that if $\alpha \equiv \overline{x + [s/2] + 1} \in V_a$, then there exists an ε_2 such that (a) $\gamma \equiv \overline{x + s + \pi + \varepsilon_2} \in V_c$, and (b) $\gamma - \alpha \equiv [(n - 1)/2]$. To prove the claim, we note that $\gamma - \alpha \equiv [(s + 1)/2] + \pi + \varepsilon_2 - 1$. With the aid of the equality $s + t + 2\pi = n - 1$, one can easily show that if $\varepsilon_2 = [t/2] + 1$, then $\gamma - \alpha \equiv [(n - 1)/2]$ except when s is odd and t is even; in that case, the choice of $\varepsilon_2 = [t/2]$ would lead to the same result. In either case, $1 \leqq \varepsilon_2 \leqq t$, or $\gamma \in V_c$. A comparison of the above claim with Case 2 and Case 3 of the algorithm proves that if $\gamma > \alpha$ either $(\alpha, \gamma) \in E(p, r)$ or $(\alpha, n - 1)$ and $(n - 1, \gamma) \in E(p, r)$. In any case, we have a path from i to j in $G - X$. Thus, $\rho_{ij}(G - X) \neq 0$, which proves the lemma for odd p and $\gamma > \alpha$. If $\gamma < \alpha$, a similar proof would lead to the same contradiction.

Suppose p is even; then there is no more than one vertex u such that $u \in X$ and $u \notin V_b \cup V_d$. We need only consider the largest set X; thus assume $u \in X - (V_b \cup V_d)$. As before, $\rho(G - X) = 0$ implies that $\rho_{ij}(G - X) = 0$ for some i and j, $0 \leqq i, j \leqq n - 2$. Again, assume $i \in V_a$ and $j \in V_c$.

If $u = n - 1$, then by similar arguments as when p was odd, one can establish the existence of $\alpha \in V_a$ and $\gamma \in V_c$ such that either $\overline{\gamma + [(n - 2)/2]} \equiv \alpha$ or $\overline{\alpha + [(n - 2)/2]} \equiv \gamma$. This proves that $(\alpha, \gamma) \in E(p, r)$ and hence, there exists a path from i to j in $G - X$. This contradicts $\rho_{ij}(G - X) = 0$, which proves the lemma.

If $u \neq n - 1$, then assume, without the loss of generality, that $u \in V_a$. All vertices in $V_a - \{u\}$ are reachable from i in $G - X$ and all vertices in V_c are reachable from j in $G - X$. (If $p = 4$, then it is not obvious that all vertices in $V_a - \{u\}$ are reachable from i. However, $p = 4$ implies $|X| = 3$, or $X = \{u, \overline{x + s + 1}, \overline{x + s + t + \pi + 1}\}$. Also, one may pick any vertex in X to play the role of the vertex u. This flexibility may be used to show that for some choice of u and V_a, all vertices in $V_a - \{u\}$ are reachable from i.) Let us first assume that $\alpha \equiv \overline{x + [s/2]} \in V_a$ and $\overline{x + [s/2]} \neq u$. If we could prove that there exists $\gamma \in V_c$ such that $\overline{\gamma + [(n - 2)/2]} \equiv \alpha$, then either $(\alpha, \gamma) \in E(p, r)$ or $(\alpha, n - 1)$ and $(n - 1, \gamma) \in E(p, r)$

and hence, there exists a path from i to j in $G - X$, which, as before, leads to a contradiction. To do this we note that if $\gamma \equiv \overline{x + \pi + s + [(t + 1)/2]}$, then $\overline{\gamma + [(n - 2)/2]} \equiv \alpha$ unless both s and t are even. When both s and t are even, if we pick $\gamma \equiv \overline{x + \pi + s + [(t + 1)/2] + 1}$, then $\overline{\gamma + [(n - 2)/2]} \equiv \alpha$. In either case $\gamma \in V_c$. Now, let us suppose $\overline{x + [s/2]} \equiv u$. In this case, we choose $\alpha \equiv \overline{x + [s/2] + 1} \in V_a$. Then, we pick $\gamma \equiv \overline{x + \pi + s + [(t + 1)/2] + 1}$ which results in $\overline{\gamma + [(n - 2)/2]} \equiv \alpha$, unless both s and t are even, in which case we pick $\gamma \equiv \overline{x + \pi + s + [(t + 1)/2] + 2}$. However, in any case, $\gamma \in V_c$ only if $t > 2$. Thus, we must consider the case when $t \leqq 2$ separately. If $t \leqq 2$, then $\gamma \equiv \overline{x + s + \pi + \varepsilon} \in V_c$ for $1 \leqq \varepsilon \leqq t$. The reader can establish that

$$\alpha_1 \equiv \overline{x + s + \pi + \varepsilon + [(n - 2)/2]} \in V_a$$

and

$$\alpha_2 \equiv \overline{x + s + \pi + 1 - [(n - 2)/2]} \in V_a,$$

where $\varepsilon = t$ if $p = n - 3$; otherwise $\varepsilon = 1$. This in turn proves that $(\gamma, \alpha_i) \in E(p, r)$ and $(\gamma, n - 1)$ and $(n - 1, \alpha_j) \in E(p, r)$, where $i = 1$ and $j = 2$ or vice versa. Since either $\alpha_1 \neq u$ or $\alpha_2 \neq u$, this would prove the existence of a path from i to j in $G - X$ which leads to a contradiction. This ends the proof of the lemma.

4. Further problems. Realizability of a set $\delta = \{d_1, d_2, \cdots, d_n\}$ of nonnegative integers as the degrees of vertices of a unigraph was first considered by Havel [4] and later by Hakimi [3]. But the most interesting result on this problem is due to Erdos and Gallai [5]. Their result, as stated by Harary [1], is as follows.

Given a set of nonnegative integers $\delta = \{d_1, d_2, \cdots, d_n\}$ with $d_i \geqq d_{i+1}$ for $i = 1, 2, \cdots, n - 1$, then δ is realizable as a unigraph if and only if $S(\delta) = \sum_{i=1}^{n} d_i$ is even, and

$$\sum_{i=1}^{r} d_i \leqq r(r - 1) + \sum_{i=r+1}^{n} \min(r, d_i) \quad \text{for all } r, 1 \leqq r \leqq n - 1. \tag{9}$$

For the purpose of this discussion, (9) may more conveniently be expressed as

$$\frac{1}{2}\sum_{i=1}^{n} d_i - \left(\sum_{i=1}^{r} d_i - \frac{1}{2}r(r - 1)\right) \geqq \frac{1}{2}\sum_{i=r+1}^{n} \alpha_i \quad \text{for all } r, \quad 1 \leqq r \leqq n - 1, \tag{10}$$

where for each $i, r + 1 \leqq i \leqq n$,

$$\alpha_i = \begin{cases} 0 & \text{if } d_i - r \leqq 0, \\ d_i - r & \text{if } d_i - r > 0. \end{cases}$$

The equivalence of (9) and (10) can be established by rewriting (10) as

$$\sum_{i=1}^{r} d_i \leqq r(r - 1) + \sum_{i=r+1}^{n} (d_i - \alpha_i)$$

and noting that $d_i - \alpha_i = \min(r, d_i)$ for $i = r + 1, \cdots, n$.

The problem of realizability of such a set of integers as a p-connected unigraph has been studied by Rao and Rao [6] for $p \leqq 3$. The general solution to this problem has occupied much of this author's time. This author felt he had a complete proof of the following conjecture, but a flaw in a very long proof was found. In any case, the evidence is strong that the conjecture is valid.

Conjecture. Let $\delta = \{d_1, d_2, \cdots, d_n\}$ *be a given set of nonnegative integers with* $d_i \geqq d_{i+1}$ *for* $i = 1, 2, \cdots, n-1$ *which is realizable as a unigraph.* (a) *Then* δ *is realizable as a* d_n*-connected unigraph if and only if*

$$\frac{1}{2}\sum_{i=1}^{n} d_i - \left(\sum_{i=1}^{d_n-1} d_i - \tfrac{1}{2}(d_n - 1)(d_n - 2)\right) \geqq n - d_n. \tag{11}$$

(b) *If* (11) *is not satisfied, then* δ *is realizable as a* $(d_n - 1)$*-connected unigraph.*

The proof of the necessity of (11) is as follows. Let $G(V, E)$ be a d_n-connected unigraph realization of δ. Then, G has $(1/2)\sum_{i=1}^{n} d_i$ edges. Let

$$X = \{v_1, v_2, \cdots, v_{d_n-1}\}.$$

Then $G - X$ must be 1-connected. But the number of edges in $G - X$ is at most $(1/2)\sum_{i=1}^{n} d_i - (\sum_{i=1}^{d_n-1} d_i - (1/2)(d_n - 1)(d_n - 2))$. This number must exceed the minimum number of edges necessary to form a 1-connected graph with $n - d_n + 1$ vertices. This establishes the necessity of (11).

It is instructive to compare (10) and (11). If in (10) we set $r = d_n - 1$, then the left-hand sides of (10) and (11) are identical. But when $r = d_n - 1$, the right-hand side of (10) is only guaranteed to be no less than $(1/2)(n - d_n + 1)$. This indicates the fact that not every realizable δ is realizable as a d_n-connected graph.

It is interesting to note that there exist δ's which are realizable as p-connected graphs but only realizable as $(p - 1)$-connected unigraphs. For example, $\delta = \{9, 9, 8, 4, 4, 4, 4, 4, 4, 4, 4, 4\}$ is realizable as a 4-connected graph (Theorem 3) but not realizable as a 4-connected unigraph, because δ does not satisfy (11). A 3-connected unigraph realization of δ is shown in Fig. 2.

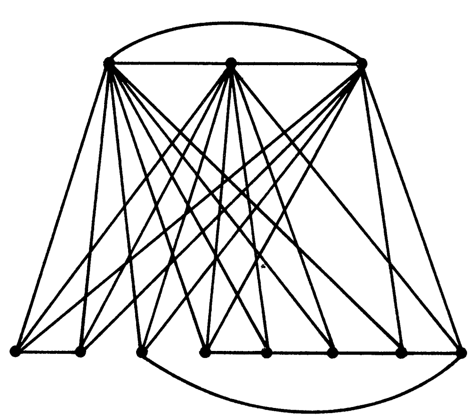

FIG. 2

Another variation of the above problem which is in fact more practical than either of the problems considered in this paper is the following.

Given a set $\delta = \{d_1, d_2, \cdots, d_n\}$ of nonnegative integers, find the necessary and sufficient conditions for the existence of a graph (or unigraph) $G(V, E)$ with

$V = \{v_1, v_2, \cdots, v_n\}$ which realizes δ such that for all i and j, $i \neq j$ and $1 \leqq i, j \leqq n$,

$$\rho_{ij}(G) = \min(d_i, d_j). \tag{12}$$

A version of this problem with $\rho_{ij}(G)$ being the "line-connectivity" [1] between v_i and v_j has been solved by Chou and Frank [9]. We have some necessary and some sufficient conditions for the existence of unigraphs realizing δ which satisfy (12), but the complete result seems to be out of reach.

Finally, the existence of unigraphs with prescribed degrees and line-connectivity was studied by Edmonds [10].

Added in proof. Since the submission of this paper, Wang and Kleitman [11] have developed a proof of the conjecture given here.

REFERENCES

[1] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.
[2] S. L. Hakimi, *On realizability of a set of integers as degrees of the vertices of a linear graph. I*, J. Soc. Indust. Appl. Math., 10 (1962), pp. 496–506.
[3] J. K. Senior, *Partitions and their representative graphs*, Amer. J. Math., 73 (1951), pp. 663–689.
[4] V. Havel, *A remark on the existence of finite graphs*, Časopis Pěst Mat., 80 (1955), pp. 477–480. (In Hungarian.)
[5] P. Erdös and T. Gallai, *Graphs with prescribed degrees of vertices*, Mat. Lapok, 11 (1960), pp. 264–274.
[6] S. B. Rao and A. R. Rao, *Existence of triconnected graphs with prescribed degrees*, Pacific J. Math., 33 (1970), pp. 203–207.
[7] F. Harary, *The maximum connectivity of a graph*, Proc. Nat. Acad. Sci., 48 (1962), pp. 1142–1146.
[8] S. L. Hakimi, *An algorithm for construction of the least vulnerable communication network on the graph with the maximum connectivity*, IEEE Trans. Circuit Theory, CT-16 (1969), pp. 229–230.
[9] W. Chou and H. Frank, *Survivable communication networks and the terminal capacity matrix*, Ibid., CT-17 (1970), pp. 192–197.
[10] J. Edmonds, *Existence of k-edge connected ordinary graphs with prescribed degrees*, J. Res. Nat. Bur. Standards Sect. B, 68 (1964), pp. 73–74.
[11] D. L. Wang and D. J. Kleitman, *On the existence of n-connected graphs with prescribed degrees* ($n \geqq 2$), Unpublished Note, Math. Dept., MIT, Cambridge, Mass.