On Realizability of a Set of Integers as Degrees of the Vertices of a Linear Graph. I
Author(s): S. L. Hakimi
Published by: Society for Industrial and Applied Mathematics
Stable URL: http://www.jstor.org/stable/2098746
Accessed: 14/05/2009 04:19

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=siam.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We work with the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact support@jstor.org.
ON REALIZABILITY OF A SET OF INTEGERS AS DEGREES OF THE VERTICES OF A LINEAR GRAPH. I*

S. L. HAKIMI†

Abstract. This paper is mainly concerned with the realizability of a set of \( n \) integers as the degrees of vertices of an \( n \)-vertex linear graph. Other related problems, such as when a set of integers is realizable as a connected graph, connected graph without "parallel" elements, separable graph, and nonseparable graph, are considered. The relationship between this problem and the problem of isomers in the organic chemistry is described. A similar problem in weighted graphs is also studied.

1. Introduction. Let \( G \) be a linear graph with \( n \) vertices (nodes) [1, 2, 3]. Let the degree of a vertex in \( G \) be the number of elements (arcs) connected to (incident at) that vertex. Suppose we are given a set of positive integers \( d_1, d_2, \ldots, d_n \), \( d_i \leq d_{i+1}, 1 \leq i \leq n - 1 \). How can we tell whether or not there exists a graph \( G \) whose vertices \( v_1, v_2, \ldots, v_n \) have degrees \( d_1, d_2, \ldots, d_n \)? If such a graph exists, we say that the set of integers is realizable, or the graph realizes the set of integers. Other questions, such as whether or not the set of integers is realizable as a connected graph, separable graph, and nonseparable graph\(^1\) are considered.

Suppose a positive number is attached to every element of a linear graph. Such a graph, represented here by \( G_w \), is called a weighted graph. Let \( w_i \) represent the sum of the weights of the elements incident at vertex \( v_i \) of \( G_w \). Then, given a set of positive numbers \( w_1, w_2, \ldots, w_n \), the necessary and sufficient condition for this set of numbers to be realizable as the "degrees" of the vertices of a weighted graph are found.

We shall make use of many concepts in linear graph theory. Definitions of some of the terms used in this paper are found in the Appendix.

2. The basic theorem.

Theorem 1. The necessary and sufficient conditions for positive integers\(^2\) \( d_1, d_2, \ldots, d_n \) to be realizable (as the degrees of the vertices of a linear graph) are:

(i) \[ \sum_{i=1}^{n} d_i = 2e, \text{ where } e \text{ is an integer}, \]

(ii) \[ \sum_{i=1}^{n-1} d_i \geq d_n. \]

* Received by the editors November 17, 1961 and in revised form January 22, 1962. This work was supported by the U. S. Army Research Office (Durham).
† Department of Electrical Engineering, Northwestern University, Evanston, Illinois. Formerly at the University of Illinois, Urbana, Illinois.

\(^1\) Since the realization, in general, is not unique, it is possible that a set of integers is realizable as a separable and as a non-separable graph simultaneously. The question of uniqueness is a complex one and is considered in the companion paper.

\(^2\) Throughout this paper every given set of integers is assumed to contain only positive integers and to be in a nondecreasing order.

496
REALIZABILITY OF A SET OF INTEGERS

Proof. The necessity of (i) is proved by noting that \( \sum_{i=1}^{n} d_i = 2e \) where \( e \) is the number of elements in a graph \( G \). To prove the necessity of (ii), assume otherwise, that is, there exists a graph \( G \) in which \( \sum_{i=1}^{n-1} d_i < d_n \). Let the vertex \( v_n \) in \( G \) correspond to the integer \( d_n \). Then, the inequality \( \sum_{i=1}^{n-1} d_i < d_n \) implies that there exists in \( G \) at least one element incident at \( v_n \) which is not incident at any other vertex. This is impossible, hence \( \sum_{i=1}^{n-1} d_i \geq d_n \).

The sufficiency is proved by induction. If \( n = 2 \), then (ii) implies that \( d_1 = d_2 \). Such a pair of integers is realized as a graph with two vertices between which \( d_1 = d_2 \) elements are connected. For \( n = 3 \), we have positive integers \( d_1, d_2, d_3 \) which satisfy (i) and (ii), and we must find a three-vertex graph realizing these integers.

Consider a graph with three vertices. Let \( n_{ij} = n_{ji} \) represent the number of elements connected between vertices \( v_i \) and \( v_j \), \( (i, j = 1, 2, 3, i \neq j) \). Then, we have

\[
\begin{align*}
    d_1 &= n_{12} + n_{13}, \\
    d_2 &= n_{23} + n_{21} = n_{32} + n_{12}, \\
    d_3 &= n_{32} + n_{41} = n_{23} + n_{13}.
\end{align*}
\]

Solving the above linear equations for \( n_{12}, n_{13}, \) and \( n_{23}, \) we obtain

\[
\begin{align*}
    n_{12} &= \frac{(d_1 + d_2 + d_3) - 2d_3}{2}, \\
    n_{13} &= \frac{(d_1 + d_2 + d_3) - 2d_2}{2}, \\
    n_{23} &= \frac{(d_1 + d_2 + d_3) - 2d_1}{2}.
\end{align*}
\]

The existence of the desired graph is established, if we show that the numbers \( n_{12}, n_{13}, \) and \( n_{23} \) are all nonnegative integers. Condition (ii) implies that the sum of any \( n - 1 \) integers is no less than the remaining one, therefore, \( n_{12}, n_{13}, \) and \( n_{23} \) are all nonnegative. Condition (i) implies that \( \sum_{i=1}^{n} d_i - 2d_j, (1 \leq j \leq n) \) is an even number, hence \( n_{12}, n_{13} \) and \( n_{23} \) are integers.

To complete the induction, we assume that the assertion is true for \( n < k, (k > 3) \), we will then show that it is also true for \( n = k \). Let \( d_1, d_2, \ldots, d_k \) be \( k \) positive integers satisfying (i) and (ii). Consider the set of nonnegative integers \( d_2, d_3, \ldots, d_{k-1}, d_k - d_1 \). For the moment let us assume \( d_k - d_1 \neq 0 \). We would like to establish that this new set of integers satisfies (i) and (ii). The sum of these integers is equal to \( \sum_{i=1}^{k} d_i - 2d_1 \) which is an even number, for \( \sum_{i=1}^{k} d_i \) is an even number,
hence (i) is satisfied. It remains to be shown that this new set of integers also satisfies (ii). However, we have \( \sum_{i=1}^{k-1} d_i \geq d_k \) which implies \( \sum_{i=2}^{k-1} d_i \geq d_k - d_1 \). If \( d_k - d_1 \) is the largest integer in the set of integers \( d_2, d_3, \ldots, d_{k-1}, d_k - d_1 \), (ii) is satisfied. If not, then \( d_{k-1} \) is the largest, and we must show that \( d_2 + d_3 + \cdots + d_{k-2} + (d_k - d_1) \geq d_{k-1} \). The above inequality is satisfied for \( k > 3 \), because \( d_k \geq d_{k-1} \) and \( d_2 \geq d_1 \).

According to the hypothesis any set of \( n < k \) integers satisfying (i) and (ii) is realizable, therefore, there exists a graph \( G_1 \) which realizes integers \( d_2, d_3, \ldots, d_{k-1}, d_k - d_1 \), as the degrees of its vertices. To the graph \( G_1 \) we add a vertex called \( v_1 \) and we connect \( d_1 \) elements between vertex \( v_1 \) and the vertex in \( G_1 \) corresponding to the integer \( d_k - d_1 \). The graph thus constructed realizes the original set of integers. If \( d_k - d_1 = 0 \), then the new set of integers will be \( d_2, d_3, \ldots, d_{k-1} \), which also satisfies (i) and (ii), and hence, is realizable as a graph \( G_1 \). However, we must now add two vertices to \( G_1 \), and connect \( d_1 = d_k \) elements between these vertices. This ends the proof of the theorem. As a direct consequence of the proof of this theorem, we can state the following corollary.

**Corollary.** For any set of realizable three integers \( d_1, d_2, d_3 \), there exists only one graph whose vertices have degrees \( d_1, d_2, d_3 \).

Suppose we are given the set of integers 1, 1, 1, 1. This set of integers satisfies conditions (i) and (ii). However, the graph realizing these integers is not connected. A connected graph is a graph in which there exists a path between every pair of vertices. In the next section we shall give necessary and sufficient conditions for a set of integers to be realizable as a connected graph.

### 3. Realizability as a connected graph and \( d \)-invariant transformation.

Suppose a graph \( G \) realizes integers \( d_1, d_2, \ldots, d_n \). Consider a pair of elements \( e(v_i, v_j) \) and \( e(v_k, v_0) \) in \( G \) which are connected between vertices \( v_i \) and \( v_j \), \( v_k \) and \( v_0 \) respectively. Assume that vertices \( v_i, v_j, v_k \), and \( v_0 \) are all distinct. Suppose we transform graph \( G \) into a graph \( G_1 \) by removing elements \( e(v_i, v_j) \) and \( e(v_k, v_0) \) from \( G \) and either adding (1) elements \( e(v_i, v_k) \) and \( e(v_j, v_0) \) between vertices \( v_i \) and \( v_k \), \( v_j \) and \( v_0 \), or adding (2) elements \( e(v_i, v_0) \) and \( e(v_j, v_k) \) between vertices \( v_i \) and \( v_0 \), \( v_j \) and \( v_k \). We should notice that in this transformation of graphs the degrees of the vertices remain unchanged. Such a transformation is called an elementary \( d \)-invariant transformation. The class of all graphs realizing the same set of integers is called a class of \( d \)-invariant graphs. It will be shown (in the companion paper) that the class of all invariant graphs can be found by a finite number of elementary \( d \)-invariant transformations.

Consider a graph \( G \). The subgraphs \( g_1, g_2, \ldots, g_r, \) \( (r > 1) \), of \( G \) are called the maximally connected subgraphs\(^3\) (or components) of \( G \) if (1)

\[^3\] It must be noted that if \( d_1 = d_k \), then \( d_1 = d_2 = \cdots = d_k \).
every element of $G$ is in only one of the subgraphs $g_1, g_2, \ldots, g_r$, i.e.,
the "union" $g_1 \cup g_2 \cup \cdots \cup g_r = G$, and the "intersection" $g_i \cap g_j = 0,$
i \neq j; and (2) there exist no paths from a vertex in $g_i$ to a vertex in $g_j$,
i \neq j. A circuitless graph is a graph which contains no circuits.

**Lemma 1.** Let $g_1, g_2, \ldots, g_r, (r > 1)$, be the maximally connected sub-
graphs of $G$ not all of which are circuitless. Then, there exists a graph $G_1$ which
is $d$-invariant from $G$, with $r - 1$ maximally connected subgraphs.

**Proof.** Let $g_i$ be a maximally connected subgraph of $G$ containing a cur-
nic. Let $e_{i_1}$ in $g_i$ be an element which is a circuit in $g_i$. Take an arbitrary
element $e_{j_1}$ in $g_j$, $i \neq j$. By an elementary $d$-invariant transformation in-
volving $e_{i_1}$ and $e_{j_1}$, we obtain a graph $G_1$ in which subgraph $g_i$ and $g_j$ are
combined into a connected subgraph, which reduces the number of maxim-
ally connected subgraphs by one.

It should be noted that we can keep on reducing the number of maximally
connected subgraphs as long as there are circuit elements in any of the maxi-
mally connected subgraphs.

**Theorem 2.** The necessary and sufficient conditions for a set of integers
d_1, d_2, \ldots, d_n to be realizable as a connected graph are:

(i) the set $d_1, d_2, \cdots, d_n$ is realizable, i.e., it satisfies the conditions of
Theorem 1.

(ii) $\sum_{i=1}^{n} d_i \geq 2(n - 10)$.

**Proof.** The necessity of condition (i) is already established. To establish
the necessity of (ii), consider a connected graph $G$. In a connected graph
of $n$ vertices the number of elements $e$ is at least equal to $n - 1$ [3]. We
know $\sum_{i=1}^{n} d_i = 2e$; therefore, $\sum_{i=1}^{n} d_i \geq 2(n - 1)$.

To prove the sufficiency, we assume a given set of integers $d_1, d_2, \cdots, d_n$
satisfy (i) and (ii), and then prove that among the class of $d$-invariant
graphs realizing $d_1, d_2, \cdots, d_n$, there exists one which is connected.
Since the set $d_1, d_2, \cdots, d_n$ satisfy (i), there exists a graph $G$ realizing
d_1, d_2, \cdots, d_n. If $G$ is connected, there is no problem; therefore, let us
assume $G$ is not connected. Let the maximally connected parts of $G$ be
$g_1, g_2, \cdots, g_r, (r > 1)$. Suppose $g_1, g_2, \cdots, g_r$ are all circuitless sub-
graphs, for if there is a subgraph $g_i$ which contains a circuit, as suggested
by Lemma 1, we can combine this subgraph with another subgraph $g_j$, hence
reducing the number of maximally connected parts. We can continue
this operation until all subgraphs are circuitless. If $g_1, g_2, \cdots, g_r$ are all
circuless, then the number of elements in $G$ is equal to $n - r$ [3]. This
implies that $\sum_{i=1}^{n} d_i = 2(n - r)$, which contradicts (ii), hence the theorem.

The following corollaries can be easily established as a consequence of
the previous theorem:

**Corollary 1.** The set of integers $d_1, d_2, \cdots, d_n$ is realizable as a circuitless
but connected graph if and only if,

(i) $\sum_{i=1}^{n} d_i = 2(n - 1)$. 

**Corollary 2.** The set of integers \( d_1, d_2, \cdots, d_n \) is realizable as a connected graph if,

(i) the set \( d_1, d_2, \cdots, d_n \) is realizable,

(ii) \( d_4 \neq 1 \).

4. Realizability as separable and nonseparable graphs. A graph \( G \) is nonseparable if every subgraph \( g \) of \( G \) has at least two vertices in common with its complement. All other graphs are separable [2, 3]. If \( G \) is a connected but separable graph, then there exists a subgraph \( g \) in \( G \) which has only one vertex \( v_c \) in common with its complement; \( v_c \) is called a cut vertex of \( G \) [3].

**Theorem 3.** The necessary and sufficient conditions for a set of integers \( d_1, d_2, \cdots, d_n \) to be realizable as a nonseparable graph are:

(i) \( \sum_{i=1}^{n} d_i = 2e \), where \( e \) is an integer,

(ii) \( d_i \neq 1 \) for all \( i \),

(iii) \( \sum_{i=1}^{n-1} d_i \geq d_n + 2(n - 2) \).

**Proof.** Necessity of (i) is already established. The necessity of (ii) is self-explanatory. To prove that condition (iii) is also necessary, suppose there exists a graph \( G \) in which

\[
\sum_{i=1}^{n-1} d_i < d_n + 2(n - 2).
\]

We will show that \( G \) is separable and the cut vertex is vertex \( v_n \) corresponding to the integer \( d_n \). To prove this we will show that deleting all elements incident at \( v_n \) from \( G \) results in a graph \( G_1 \) which is not connected. Let the degrees of the vertices in \( G_1 \) be \( d_1', d_2', \cdots, d_{n-1}' \). We know that

\[
\sum_{i=1}^{n-1} d_i' < 2(n - 2).
\]

According to Theorem 2, graph \( G_1 \) cannot be connected. This ends the proof of the necessity of these conditions.

To prove sufficiency, we assume we are given a set of integers \( d_1, d_2, \cdots, d_n \), which satisfies (i), (ii), and (iii). We construct a new set of nonnegative integers \( d_1', d_2', \cdots, d_n' \), where \( d_i' = d_i - 2 \), \( 1 \leq i \leq n \). We now show that the set of integers \( d_1', d_2', \cdots, d_n' \) is realizable. To do this, we show that these integers satisfy conditions of Theorem 1. The first condition of Theorem 1 is satisfied, for

\[
\sum_{i=1}^{n} d_i' = \sum_{i=1}^{n} d_i - 2n
\]

which is an even number, due to condition (i) of the hypothesis. Now, we must show that

\[
\sum_{i=1}^{n-1} d_i' \geq d_n'.
\]

From (iii), we have

\[
\sum_{i=1}^{n-1} d_i \geq d_n + 2(n - 2),
\]
which may be written as
\[ \sum_{i=1}^{n-1} d_i - 2(n - 1) \geq d_n - 2, \]
or finally,
\[ \sum_{i=1}^{n-1} (d_n - 2) \geq (d_n - 2) \]
which is what was desired. Let a realization of the set of integers
\( d'_1, d'_2, \ldots, d'_n \) be a graph \( G' \). It is possible that some of these integers are zero, in which case we allow the corresponding vertices in \( G' \) to be isolated vertices (points) in \( G' \). Suppose we superimpose upon \( G' \) a collection of \( n \) elements forming a Hamiltonian circuit (a circuit going through all vertices of \( G \)). It is clear that \( G \) realizes the original set of integers. Since \( G \) contains a Hamiltonian circuit, \( G \) is also nonseparable.

Since a set of integers may be realizable as a nonseparable graph as well as a separable graph, the following question remains to be answered: Given a set of integers, how do we determine whether or not there exists a separable but connected graph realizing these integers as the degrees of its vertices?

**Theorem 4.** A necessary and sufficient condition for a set of integers which is realizable as a connected graph (satisfies conditions of Theorem 2) to be realizable as a separable but connected graph is:

(i) If \( n = 3 \), \( d_1 + d_2 = d_3 \),

(ii) If \( n = 4 \), there must exist among the set of integers a \( d_i \) and a \( d_j \) such that \( d_i \neq d_j \),

(iii) If \( n > 4 \), there must exist among the set an integer \( d_i \neq 2 \).

**Proof.** If \( n = 3 \), the necessity and sufficiency of the condition \( d_1 + d_2 = d_3 \) is self-evident. The necessity of (ii) is established by showing that there exists no four-vertex separable graph which is connected and in which all vertices are of the same degree. Suppose a separable connected graph \( G \) is realizing the set of integers \( d_1 = d_2 = d_3 = d_4 \). Suppose \( v_1 \) is a cut vertex in \( G \). It is clear that one of the separable parts of \( G \) must contain two vertices one of which is \( v_1 \). This implies that there exist positive integers \( d_{11} \) and \( d_{12} \) such that \( d_{11} + d_{12} = d_1 \) and integers \( d_{11} \) and \( d_{12} \) is a realizable two-vertex graph. However, integers \( d_{11} \) and \( d_{12} \) are realizable if, and only if, \( d_{11} = d_{12} \). This requires \( d_{12} = 0 \) which is a contradiction. The necessity of (iii) is proved, by merely observing that a connected graph in which all vertices are of degree two is a circuit [3], which is a nonseparable graph.

Let us consider a set of integers \( d_1, d_2, \ldots, d_n (n \geq 4) \) that satisfies conditions (ii) and (iii). We want to show we can find a separable graph realizing these integers. We shall consider two cases. Case 1: \( d_1 = d_2 = d_3 = \cdots = d_n, n > 4 \). Suppose \( d_1 \) is an even number. Since \( d_1 \neq 2 \), then \( d_1 = 2 + d_{12} \). Consider the following two sets of integers: \( 2, d_2, d_3 \) and \( d_{12}, d_4, d_5, \ldots, d_n \). Both of these sets of integers satisfy the condi-
tions of Theorem 2; hence they are realizable as connected graphs \( g_1 \) and \( g_2 \). Superimposing the vertices of the lowest degree in \( g_1 \) and \( g_2 \), we obtain a separable graph which realizes the original set of integers. If \( d_1 \) is an odd number, then \( n \) must be an even number. Furthermore, \( d_1 \neq 1 \), for otherwise the conditions of Theorem 2 are not satisfied. Let \( d_1 = 2 + d_{12} \). Consider the two sets of integers \( 2, d_2, d_3 \) and \( d_{12}, d_4, d_5, d_6, \ldots, d_n \). These two sets of integers satisfy the conditions of Theorem 2, hence they are realizable as connected graphs \( g_1 \) and \( g_2 \). As before, superimposing the vertices of lowest degree in \( g_1 \) and \( g_2 \), we obtain a separable and connected graph \( G \) which realizes the given set of integers. Case 2: \( d_1 \neq d_n \). In this case, we shall consider the following two sets of integers, \( d_1 \), \( d_1 \) and \( d_2, d_3, \ldots, d_n - d_1 \). The first set is obviously realizable. The proof of the realizability of the set of integers \( d_2, d_3, \ldots, d_{n-1}, d_n - d_1 \) is already given in the proof of Theorem 1. However, we must show that \( d_2, d_3, \ldots, d_{n-1}, d_n - d_1 \) is realizable as a connected graph. To do this, we must show that \( d_2 + d_3 + \ldots + d_{n-1} + (d_n - d_1) \geq 2(n - 2) \). The left side of the above inequality is equal to \( \sum_{i=1}^{n} d_i - 2d_1 \). If \( d_1 = 1 \), then

\[
\sum_{i=1}^{n} d_i - 2d_1 \geq 2(n - 1) - 2 = 2(n - 2).
\]

If \( d_1 > 1 \), then

\[
\sum_{i=1}^{n} d_i - 2d_1 \geq nd_1 - 2d_1 = (n - 2)d_1 \geq 2(n - 2).
\]

This ends the proof of the theorem, for the two sets of integers are realizable as connected graphs \( g_1 \) and \( g_2 \) and superimposing a vertex of \( g_1 \) upon the vertex of \( g_2 \) corresponding to the integer \( d_n - d_1 \), we obtain the desired graph.

5. **Realizability as a graph without parallel elements.** A graph \( G \) has parallel elements if there exist at least two elements in \( G \) which are connected between the same vertices. The question we would like to answer is: Under what circumstances may a set of integers be realized as a graph in which there are no parallel elements? We shall make use of the following reduction process.

Let a set of integers \( d_1, d_2, \ldots, d_n \) be called reducible if \( d_n \leq n - 1 \). The set of integers \( d_{i_1}, d_{i_2}, \ldots, d_{i_{n-1}} \) are obtained from the original set as follows:

\[
d_{i_j} = \begin{cases} 
  d_{n-j} - 1 & (1 \leq j \leq d_n), \\
  d_{n-j} & (j > d_n).
\end{cases}
\]

Let \( d_1', d_2', \ldots, d_p' \) (\( p \leq n - 1 \)) be positive integers in the set \( d_{i_1}, d_{i_2}, \ldots, d_{i_{n-1}} \) arranged in a nondecreasing order. The set of integers \( d_1', d_2', \ldots, d_p' \) are called a reduced set of integers and the process just described is called a reduction cycle. Starting with \( d_1', d_2', \ldots, d_p' \), we
can go through another reduction cycle and obtain another set of integers, if \( d_p' \leq p - 1 \).

**Theorem 5.** A set of realizable integers, \( d_1, d_2, \cdots, d_n \) is realizable as a graph without parallel elements if, and only if, the set is reducible and every set of integers obtained from this set by successive reduction cycles is reducible.

**Proof.** Let \( G \) be a graph with no parallel elements. Let the degrees of the vertices of \( G \) be \( d_1, d_2, \cdots, d_n \). There are at most \( n - 1 \) elements incident at vertex \( v_n \), hence \( d_n \leq n - 1 \). Let \( G_1 \) be a graph obtained from \( G \) by removing from \( G \) the vertex \( v_n \) and all elements incident at \( v_n \). Let the degrees of the vertices in \( G_1 \) be \( d_1^{(1)}, d_2^{(1)}, \cdots, d_p^{(1)} \), \( d_i^{(1)} \leq d_{i+1}^{(1)} \) for \( 1 \leq i \leq p' - 1 \). Since \( G_1 \) has no parallel elements, \( d_i^{(1)} \leq p' - 1 \). Let \( d_1', d_2', \cdots, d_p' \) be the set of integers obtained from integers \( d_1, d_2, \cdots, d_n \) by a single reduction cycle. It can easily be seen that \( d_p' \leq d_p^{(1)} \) and \( p' \leq p \), which implies that \( d_p' \leq p - 1 \). The continuation of this process also establishes the necessity of the reducibility of the sets of integers obtained in the successive reduction cycles.

To prove this condition is also sufficient, we shall outline a procedure that results in a graph without parallel elements. Let us pick \( n \) points (vertices) \( v_1, v_2, \cdots, v_n \). We now connect an element between each of the following pairs of vertices \( v_n - v_{n-1}, v_n - v_{n-2}, \cdots, v_n - v_{n-d_n} \). Since \( d_n \leq n - 1 \), this can always be done. There are now exactly \( d_n \) elements incident at \( v_n \), none of which are in parallel. To construct the rest of the graph, we note that the degrees of the vertices of the remaining graph must be the reduced set of integers, \( d_1', d_2', \cdots, d_p' \). Since \( d_p' \leq p - 1 \), we can successively continue on similar cycles and finally obtain a graph which has no parallel elements. However, there remains a question that must be answered. That is, we must show that if \( d_1, d_2, \cdots, d_n \) is realizable so is the reduced set of integers \( d_1', d_2', \cdots, d_p' \). However, since \( \sum_{i=1}^{n} d_i' = \sum_{i=1}^{n} d_i - 2d_n \), the sum of the reduced set of integers is even. Also, since \( d_p' \leq p - 1 \), we have \( \sum_{i=1}^{p-1} d_i' \geq d_p' \). This shows that if the original set of integers is realizable and successively reducible, then all of the reduced sets of integers are also realizable. This ends the proof of the theorem.

6. **Realizability of a set of positive numbers as the degrees of a weighted graph.** The concept of a weighted graph is found to be extremely useful in the applications of graph theory to electrical networks and other physical systems. It is, therefore, natural to extend the result discussed here to the case of weighted graphs.

**Theorem 6.** Given a set of numbers \( w_1, w_2, \cdots, w_n \), \( (w_i \leq w_{i+1}) \) a necessary and sufficient condition for this set to be realizable as weighted degrees of the vertices of a linear graph is

\[
(i) \quad \sum_{i=1}^{n-1} w_i \geq w_n.
\]
Proof. Proof of the necessity is entirely the same as Theorem 1, hence will not be repeated here. As in the case of Theorem 1, we shall prove the sufficiency by induction. For \( n = 2 \), \( w_1 \) must be equal to \( w_2 \). An element of weight \( w_1 \) connected between a pair of vertices would realize such a pair of numbers. For \( n = 3 \), we define numbers \( n_{ij} = n_{ji} \ (i, j = 1, 2, 3, i \neq j) \) which is to be the weight of the element connected between vertices \( v_i \) and \( v_j \). The unique realization for this case is determined by calculating the values of these weights according to the following equations:

\[
\begin{align*}
n_{12} &= \frac{w_1 + w_2 - w_3}{2}, \\
n_{13} &= \frac{w_1 - w_2 + w_3}{2}, \\
n_{23} &= -\frac{w_1 + w_2 + w_3}{2}.
\end{align*}
\]

Now let us assume the theorem is true for \( n < k \); we prove it for \( n = k \). Consider a set of numbers \( w_1, w_2, \ldots, w_k \). We construct the set of numbers \( w_2, w_3, \ldots, w_{k-1}, w_k - w_1 \) and prove as in the case of Theorem 1, that this set satisfies the condition (i) of the theorem, and hence is realizable. Let the weighted graph realizing \( w_2, w_3, \ldots, w_{k-1}, w_k - w_1 \), be \( G_w' \). To this graph we add a vertex \( v_1 \), and between \( v_1 \) and the vertex in \( G_w' \) corresponding to the number \( w_k - w_1 \) we connect an element whose weight is \( w_1 \). Thus, we obtain a graph \( G_w \) realizing the given set of numbers as the degrees of the vertices of a weighted graph.

7. Applications and conclusion. Consider the chemical compound \( \text{N}_2\text{O}_3 \). The five structural representations of this compound are shown in Fig. 1a. The corresponding graphs of these structures are shown in Fig. 1b. These five structures are called the isomers of \( \text{N}_2\text{O}_3 \). Apparently, these isomers are also distinguishable from their physical properties.

The common characteristic of the graphs in Fig. 1b is that they are all \( d \)-invariant connected graphs realizing the set of integers 2, 2, 2, 3, 3. Therefore, it seems that Theorem 2 enables us to tell whether or not a hypothetical chemical compound could exist. An important, and as yet unsolved problem is to find all isomers of a given chemical compound. This problem essentially boils down to the problem of generating all \( d \)-invariant graphs corresponding to a set of integers, for which there is no efficient solution available.

Many theoretical questions remain unanswered, some of which are: Which realization of a set of integers is a graph with a maximum (or mini-
num) number of trees, circuits, or cut sets\(^4\). Which realization has a maximum (or minimum) number of parallel elements, separable parts, maximally connected parts, or noncircuit elements?

An important question that is answered in the companion paper is: What are the necessary and sufficient conditions for a set of integers to be realizable as a unique and connected graph (i.e., all graphs realizing the set of integers are one-isomorphic)?

**APPENDIX**

A linear graph is a finite collection of two types of entities, *elements* (arcs) and *vertices* (nodes). Each element \(e(v_i, v_j)\) is connected between a pair of distinct vertices \(v_i\) and \(v_j\). If a graph \(g\) is included in \(G\), then \(g\) is said to be a *subgraph* of \(G\). The *complement* of a subgraph \(g\) is a subgraph \(\bar{g}\) which contains all of the elements in \(G\) which are not in \(g\), together with every vertex associated with these elements. A *path* in a graph \(G\) is a set of elements (a subgraph of \(G\)) that can be ordered in the form \(e(v_i, v_j), e(v_j, v_k), \ldots, e(v_r, v_l), e(v_l, v_s)\), where the vertices \(v_i, v_j, v_k, \ldots, v_r, v_l, v_s\) are distinct. A circuit in a graph \(G\) (loop) is a set of elements (a subgraph of \(G\)) that can be ordered in the form \(e(v_i, v_j), e(v_j, v_k), \ldots, e(v_r, v_l), e(v_l, v_i)\) where the vertices \(v_i, v_j, v_k, \ldots, v_r, v_l\) are distinct.

**Acknowledgment.** It is a pleasure for the author to acknowledge with

\(^4\) The number of cut sets and unions of disjoint cut sets in a connected \(n\)-vertex graph is \(2^{n-1} - 1\); however, the number of cut sets, in general, is a function of the structure of the graph.
gratitude the enlightening discussions he had with Professor W. Mayeda of the University of Illinois who also suggested the original problem.

REFERENCES