Perfect hypercubes

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Abstract

An $(n, d, \mathbf{a}, \mathbf{b})$-perfect array is a d-dimensional $b_1 \times b_2 \times \cdots \times b_d$ sized n-ary periodic array containing all possible $a_1 \times a_2 \times \cdots \times a_d$ sized n-ary array exactly once as subarray. If $a_1 = a_2 = \cdots = a_d$ and $b_1 = b_2 = \cdots = b_d$, then the notation (n, d, a, b) and term double cube are used. If $d \geq 4$, then the double cube is called double hypercube. We prove the existence of (N, d, a, b)-perfect double cubes for arbitrary $d \geq 1$, $a \geq 2$ and $n \geq 2$, where $N = kn$ with a suitable $k \geq 1$. Further we illustrate the main theorem constructing 4 and 5-dimensional hypercubes.

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1 Introduction

Cyclic sequences in which every possible subsequence of fixed length occurs exactly once have been studied for more than a hundred years. The problem

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later was extended to two- and more-dimensional arrays.

Let $\mathbb{Z}$ be the set of integers. For $u,\ v \in \mathbb{Z}$ we denote the set $\{j \in \mathbb{Z} \mid u \leq j \leq v\}$ by $[u..v]$ and the set $\{j \in \mathbb{Z} \mid j \geq u\}$ by $[u..\infty]$. Let $d \in [1..\infty]$, $b_i,\ c_i,\ j_i \in [1..\infty]$ $(i \in [1..d])$ and $a_i \in [2..\infty]$ $(i \in [1..d])$. Let $\mathbf{a} = \langle a_1, a_2, \ldots, a_d\rangle$, $\mathbf{b} = \langle b_1, b_2, \ldots, b_d\rangle$, $\mathbf{c} = \langle c_1, c_2, \ldots, c_d\rangle$, $\mathbf{j} = \langle j_1, j_2, \ldots, j_d\rangle$.

A d-dimensional n-ary array A is a mapping $A : [1..\infty]^d \rightarrow [0, n-1]$. If there exist a vector $\mathbf{b}$ and an array M such that

$$\forall \mathbf{j} \in [1..\infty]^d : A[\mathbf{j}] = M[(j_1 \bmod b_1) + 1, (j_2 \bmod b_2) + 1, \ldots, (j_d \bmod b_d) + 1],$$

then A is a **b-periodic** array and M is a period of A. The possible **a**-sized **subarrays** of A are the **a**-periodic n-ary arrays.

Although our arrays are infinite we say that a **b**-periodic array is **b**-sized.

A d-dimensional **b**-periodic n-ary array A is called $(n, d, \mathbf{a}, \mathbf{b})$**-perfect**, if all possible n-ary arrays of size **a** appear in A exactly once as a subarray. Here n is the alphabet size, d gives the number of dimensions of the "window" (or "pattern") and the perfect array M, the vector **a** characterizes the size of the window, and the vector **b** is the size of the perfect array M.

If $b_1 = b_2 = \cdots = b_d$, then the $(n, d, \mathbf{a}, \mathbf{b})$-perfect array A is called **symmetric**. If A is symmetric and $a_1 = a_2 = \cdots = a_d$, then A is called **doubly symmetric.** If A is doubly symmetric and $d = 1$, then A is called a **double sequence** (here the size b_1 means $1 \times b_1$); $d = 2$, then A is called a **double square**; $d = 3$, then A is called a **double cube**; $d \geq 4$, then A is called a **double hypercube**.

According to this definition, all perfect sequences are doubly symmetric. In the case of symmetric arrays we use the notion $(n, d, \mathbf{a}, b)$ and in the case of doubly symmetric arrays we use (n, d, a, b) instead of $(n, d, \mathbf{a}, \mathbf{b})$.

One-dimensional perfect arrays are often called de Bruijn or Good sequences. Two-dimensional perfect arrays are called also perfect maps or de Bruijn tori.

Even de Bruijn sequences are useful in construction of perfect arrays when the size of the alphabet is an even number and the window size is 2×2. If n is an even integer then an $(n, 1, 2, n^2)$-perfect sequence $M = (m_1, m_2, \ldots, m_{n^2})$ is called **even**, if $m_i = x,\ m_{i+1} = y,\ x \neq y, m_j = y$ and $m_{j+1} = x$ imply $j - i$ is even.

The concept of perfectness can be extended to infinite arrays in various ways. In **growing arrays** the window size is fixed, the alphabet size is increasing and the prefixes grow in all d directions. In **superperfect arrays** the alphabet size is perfect and the window size is growing.

2 Necessary condition and earlier results

Since in the period M of a perfect array A each element is the head of a pattern, the volume of M equals the number of the possible patterns. Since each pattern—among others the pattern containing only zeros—can appear only once, any size of M is greater then the corresponding size of the window. So we have the following necessary condition [5,10,11]: If M is an $(n, d, \mathbf{a}, \mathbf{b})$-perfect array, then

$$|\mathbf{b}| = n^{|\mathbf{a}|} \tag{1}$$

and

$$b_i > a_i \text{ for } i \in [1..d]. \tag{2}$$

The first known result originates from Flye-Sainte [4] who in 1884 proved the existence of $(2, 1, a, 2^a)$-perfect sequences for all possible values of a and gave an explicit formula for the number of $(2, 1, a, 2^a)$-perfect sequences.

It is known [4] that in the one-dimensional case the necessary condition (1) is sufficient too. There are many construction algorithms, like the ones of Fan, Fan, Ma and Siu [3], Martin [13] or any algorithm for the construction of directed Euler cycles . The most popular algorithm is probably due to H. M. Martin. It has several implementations having different time complexity [1,9,10].

N. Vörös [15] in 1984, Cummings and Wiedemann in 1986 [2] and Iványi in 1987 [6,10] proposed algorithms for $n \geq 3$ to construct superperfect sequences. It is known [6,10] that it does not exist superperfect sequence for $n = 2$. Iványi and Tóth [11] and later Hurlbert and Isaak [5] provided a constructive proof of the existence of even sequences. The conditions (1) and (2) are sufficient for the existence of $(2, 2, \mathbf{a}, \mathbf{b})$-perfect arrays [3]. Paterson in [14] supplied further sufficient conditions.

Hurlbert and Isaak [5] gave a construction for one- and two-dimensional growing arrays. Iványi [8] constructed $(n, d, \mathbf{a}, \mathbf{b})$-perfect arrays for arbitrary $n \geq 2$, $d \geq 1$ and $\mathbf{a}$, but his arrays are rarely symmetric.

Different construction algorithms and other results concerning one- and two-dimensional perfect arrays can be found in the fourth volume of *The Art of Computer Programming* [12], e.g. a (2,1,5,32)-perfect array, a (2,2,2,4)-perfect array and a (4,2,2,16)-perfect array.

3 Construction of hypercubes

In the talk—using the algorithms Cellular, Optimal-Martin, Even, Mesh, Shift, Colour [7,8,9,10,11,12]—we prove the following theorem and

column/row	1	2	3	4
1	0	0	0	1
2	0	0	1	0
3	1	0	1	1
4	0	1	1	1

Table 1
A (2,2,4,4)-square

illustrate it by the construction of two hypercubes.

Theorem 3.1 *If $n \geq 2$, $d \geq 1$, $a \geq 2$, and $b \geq 2$ satisfy 1 and*

a) $d \mid a^d$ and $(un)^{a^d/d} \geq n^{a^d} - a^{d-1}$, then there exists a $(un, d, a, (un)^{a^d/d})$-perfect array;

b) $(vn)^{a^d} \geq n^{a^d - a^{d-1}}$, then there exists a $\left((vn)^d, d, a, (vn)^{a^d}\right)$-perfect array,where u and v are suitable positive integers.

At first we compute the minimal k, then construct an $(n, d, a, \mathbf{b})$-perfect array, where

$$\mathbf{b} = \langle n^a, n^{a^2-a}, n^{a^3-a^2}, \ldots, n^{a^n - a^{n-1}} \rangle.$$

Finally we fill an empty hypercube with such prisms and using Colour we get the requuired hypercube.

3.1 Construction of a 4-dimensional double hypercube

In 4 dimensions the smallest b's satisfying (1) are $b = 16$ and $b = 81$. But we do not know algorithm which can construct $(2, 4, 2, 16)$-perfect or $(3, 4, 2, 81)$-perfect hypercube. The third chance is the $(4, 4, 2, 256)$-perfect hypercube. Let $n = 2$ and $a = 2$. Cellular calculates $N = 2$, then calls Optmial-Martin receiving the cellular $(2, 1, 2, 4)$-perfect sequence 00|11. Then Cellular calls Mesh which constructs the cellular $(2, 2, 2, 4)$-perfect square shown in Table 1a.

Now Shift calls Optimal-Martin with $n = 1$ and $a = 1$ to get the shift sizes for the layers of the $(2, 3, 2, \mathbf{b})$-perfect output P of Cellular, where $\mathbf{b} = \langle 4, 4, 16 \rangle$. Shift uses P as zeroth layer and the jth layer is generated by cyclic shifting of the previous layer downwards by w_i (div 4) and right by w_i (mod 4), where $\mathbf{w} = \langle 0\ 15\ 14\ 13\ 1211\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \rangle$. The layers of the $(2, 3, 2, \langle 4\ 4\ 16 \rangle)$-perfect array are shown in Table 2.

Layer 0	Layer 1	Layer 2	Layer 3	Layer 4	Layer 5	Layer 6	Layer 7
0 0 0 1	0 0 0 1	0 1 0 0	1 1 0 1	1 1 0 1	0 1 0 0	1 1 0 1	0 0 1 0
0 0 1 0	0 0 1 0	0 1 1 1	1 0 1 1	0 1 0 0	1 0 0 0	1 0 1 1	0 1 0 0
1 0 1 1	1 0 1 1	1 1 1 0	1 0 0 0	1 0 0 0	1 1 1 0	1 0 0 0	0 1 1 1
0 1 1 1	0 1 1 1	0 0 1 0	0 0 0 1	1 1 1 0	1 1 0 1	0 0 0 1	1 1 1 0
Layer 8	Layer 9	Layer 10	Layer 11	Layer 12	Layer 13	Layer 14	Layer 15
1 0 1 1	0 0 0 1	1 1 1 0	1 1 0 1	1 0 0 0	0 1 0 0	1 0 0 0	0 0 1 0
0 1 1 1	0 0 1 0	0 0 1 0	1 0 1 1	1 1 1 0	1 0 0 0	0 0 0 1	0 1 0 0
0 0 0 1	1 0 1 1	0 1 0 0	1 0 0 0	1 1 0 1	1 1 1 0	1 1 0 1	0 1 1 1
0 0 1 0	0 1 1 1	0 1 1 1	0 0 0 1	0 1 0 0	1 1 0 1	1 0 1 1	1 1 1 0

Table 2
16 layers of the $(2, 3, 2, 16)$-perfect output of SHIFT

Up to this point the construction is the same as in [9], but now $d = 4$, therefore we use SHIFT again to get a $(2, 4, 2, 256)$-perfect prism, then we fill an empty $256 \times 256 \times 256 \times 256$ cube with $4 \times 4 \times 16 \times 256$-sized prisms and finally colouring results the required 4-dimensional hypercube.

3.2 Construction of a 5-dimensional double hypercube

If $d = 5$, then $a = 2$, $n = 2^5$ and $b = 2^{32}$ satisfy (1), and 3 is the smallest value of v, corresponding to part b) of Theorem 3.1. Therefore we start with a $(2^5, 5, 2, 2^{32})$-perfect prism and finish with a $(2^{15}, 5, 2, 2^{96})$-perfect hypercube.

The remaining used auxiliary arrays for both construction and the constructed $256 \times 256 \times 256 \times 256$ sized $(4, 4, 2, 256)$-perfect double hypercube can be found in [10] and its supplements.

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