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CONSTRUCTION OF INFINITE DE BRUIJN ARRAYS

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We construct a periodic array containing every k-ary $m\times n$ array as a subarray exactly once. Using the algorithm SUPER (which for $k\geq 3$ generates an infinite k-ary sequence whose beginning parts of length k^m, $m=1,2,\ldots$, are de Bruijn sequences) we also construct infinite $k^m\times\infty$ k-ary arrays in which each beginning part of size $k^m\times k^{mn-m}$, $n=1,2,\ldots$, as a periodic array, contains every k-ary $m\times n$ array exactly once.

Keywords. k-ary perfect maps, k-ary infinite de Bruijn matrices

1. Introduction

This paper deals with the construction of finite and infinite de Bruijn arrays (perfect maps and supercomplex arrays). Such arrays are connected with frequency allocation for multibeam satellites [2], picture coding and processing [11] and complexity of nucleotid sequences [3]. Algorithms for constructing de Bruijn sequences are described in [1, 4, 10, 14].

Definition 1.1. Let $k\geq 2$, m, n, M, N be positive integers, $X=\{0,1,\ldots,k-1\}$. A (k,m,n,M,N)*-array* (or de Bruijn array) is a periodic $M\times N$ array with elements from X and $m\leq M$, $n\leq N$, $M\times N=k^{mn}$ in which each of the different k-ary $m\times n$ arrays appears exactly once.

Definition 1.2. Let $k\geq 2$, m and M be positive integers, $X=\{0,1,\ldots,k-1\}$. A (k,m,M)*-array* (or infinite de Bruijn array) is a k-ary infinite $M\times\infty$ array with elements from X whose beginning parts of length k^{mn}/M as periodic arrays are $(k,m,n,M,k^{mn}/M)$-arrays for $n=1,2,\ldots$.

We remark that $(k,1,n,1,k^n)$-arrays are de Bruijn sequences and $(k,1,1)$-arrays are infinite de Bruijn sequences.

The following existence results are known. For any m and n there are M and

N such that a $(2, m, n, M, N)$-array exists [5, 6]. If k is odd, then for any m a $(k, m, 2, k^m, k^m)$-array exists [5]. No $(2,1,1)$-arrays exist [1, 7, 13], but for $k \geq 3$ $(k, 1, 1)$-arrays exist [1, 7, 13]. Constructions of (k, m, m, M, M)-arrays are presented in [6] for $k=2$ and in [9] for $m=2$.

2. Algorithms

The algorithm BRUIJN walks in a de Bruijn graph $B(k, n)$ defined as follows: the vertex set is X^n and the edge set is X^{n+1} in such a way that $\kappa_1\kappa_2\ldots\kappa_{n+1} \in X^{n+1}$ determines an edge going from the vertex $\kappa_1\kappa_2\cdots\kappa_n \in X^n$ to the vertex $\kappa_2\kappa_3\ldots\kappa_{n+1} \in X^n$.

If $m \geq n$, then any sequence $q = \gamma_1\gamma_2\ldots\gamma_m$ ($\gamma_i \in X$, $i = 1, \ldots, m$) determines a directed path in $B(k, n)$ beginning at the vertex $\gamma_1\gamma_2\ldots\gamma_n$, going through the vertices $\gamma_2\gamma_3\cdots\gamma_{n+1}, \ldots, \gamma_{m-n}\gamma_{m-n+1}\cdots\gamma_{m-1}$ and ending at the vertex $\gamma_{m-n+1}\gamma_{m-n+2}\cdots\gamma_m$. BRUIJN finds an Eulerian circuit p of $B(k, n)$ [12, p. 413].

Algorithm BRUIJN.

Input. The alphabet size t ($t \geq 2$) and the pattern size n ($n \geq 1$).

Output. A $(t, 1, n, 1, t^n)$-array p.

Step 1. If $n = 1$, then let $p := 01\ldots(t-1)$ and Stop.

Step 2. Let $p := \kappa_1\kappa_2\ldots\kappa_n = 00\ldots0$.

Step 3. If $p = \kappa_1\kappa_2\ldots\kappa_s$ and $s = t^n + n + 1$, then go to Step 7.

Step 4. If $p = \kappa_1\kappa_2\ldots\kappa_s$, $\kappa_s = i$ and the last vertex $V = \kappa_{s-n+1}\kappa_{s-n+2}\cdots\kappa_s$ has at least one unused outgoing edge, then let κ_{s+1} be the first suitable element in the sequence $i, i+1, \ldots, t-1, 0, 1, \ldots, i-1$ and go to Step 3.

Step 5. (Now the last vertex V in p has no unused outgoing edges.) Let us find and insert into p a suitable circuit seeking its start vertex going back in p from V and constructing it using Step 4 [12].

Step 6. Go to Step 3.

Step 7. (Now $p = \kappa_1\kappa_2\ldots\kappa_s$ and $s = t^n + n + 1$.) Let $r := t^n$, $p := \kappa_1\kappa_2\ldots\kappa_r$ and Stop.

The algorithm SUPER generates infinite de Bruijn sequences using the following characteristics of $B(k, n)$:

(a) There is a ono-to-one mapping among the Euler circuits of $B(k, n)$ and the Hamiltonian circuits of $B(k, n+1)$ [10].

(b) If $n \geq 3$ and $k \geq 1$, then any Hamiltonian circuit p of $B(k, n)$ can be continued in order to get an Eulerian circuit q of $B(k, n)$ [7].

Algorithm SUPER.

Input. The alphabet size t ($t \geq 3$).

Output. A $(t, 1, 1)$-array p.

Step 1. Let $p := 01 \cdots (t-1)$ and $n := 1$.

Step 2. Continue p in order to get an Eulerian circuit of $B(t,n)$ using Steps 3–7 of Algorithm BRUIJN.

Step 3. Let $n := n+1$ and go to Step 2.

3. Construction results

Theorem 3.1. *For any $k \geq 2$, $m \geq 1$ and $n \geq 1$ there are M and N such that a (k,m,n,M,N)-array P exists.*

Proof (Sketch). (a) If $\min(m,n)=1$, then Algorithm BRUIJN generates the required array.

(b) If $n=m=2$, then see [9].

(c) If $n \geq 3$ and $m \geq 2$, then we construct P as follows.

(c.1) If the input parameters of BRUIJN are the alphabet size k and the pattern size m, then the output as a column will be the first column of P.

(c.2) The ith, $i=2,\ldots,k^{m(n-1)}$ column of P is generated shifting cyclically downwards its $(i-1)$th column by b_{i-1} where

$$b_1 b_2 \ldots b_s, \quad s = k^{m(n-1)} - 1$$

is the output of BRUIJN for $t = k^m$.

(d) The case $n=2$, $m \geq 3$ is similar to case (c).

(e) Since in the cases (c) and (d) the height (k^m) of the constructed array is a divisor of the sum of the shift sizes and any two $m \times n$ subarrays are different (either their first columns or at least one of their corresponding shift sizes are different), the construction is correct [9]. □

Theorem 3.2. *For any odd $k \geq 3$ and $m \geq 1$ and also for any even $k \geq 2$ and $m \geq 3$ there is an M such that a (k,m,M)-array S exists.*

Proof (Sketch). (a) The output of BRUIJN as a column for alphabet size k and pattern size m gives the first column of S.

(b) The ith, $i=2,3,\ldots$ column of S is generated by cyclically downward shifting of its $(i-1)$th column by b_{i-1}, where $b_1 b_2 \ldots$ is the output of SUPER for alphabet size k^m.

(c) To prove the correctness of this construction it is enough to show that k^m divides the sum of the relative shift sizes and any two $m \times n$ subarrays are different in the $k^m \times k^{mn-m}$ beginning parts for $n=1,2,\ldots$ [9]. □

We remark that if k is even and $m=2$, then the algorithm used in the proof of Theorem 3.2 generates a k-ary $k^m \times \infty$ array whose beginning parts of length k^{mn-m} as periodic arrays are (k,m,n,k^m,k^{mn-m})-arrays for $n=1,3,4,5,\ldots$ $(n \neq 2)$.

Theorem 3.2 does not cover the case when k is even and $m \leq 2$. No $(2,1,M)$- and $(2,2,M)$-arrays exist [8]. If $s \geq 2$, then $(2s,1,1)$-arrays [1, 7] and $(2s,2,2s^2)$-arrays [8] exist.

4. An example

If $t=3$ and $m=2$, then Algorithm BRUIJN gives $p=001122021$. If $t=k^m=9$, then the 81-length beginning part of the output of SUPER is:

$$q=b_1b_2 \ldots b_{81}$$
$$=01234\ 56788\ 00224\ 46681\ 13355\ 77036\ 04714\ 82583\ 72615\ 05162$$
$$73840\ 63074\ 17528\ 53186\ 42087\ 65432\ 1.$$

In this case the sequence of the absolute shift sizes $p=c_1c_2 \cdots c_{81}$ is defined as

$$0 \leq c_j \leq 8, \quad c_j \equiv b_1 + \cdots + b_{81}, \qquad j=1,2,\ldots,81.$$

Table 1 shows the first 81 column of a (3, 2, 9)-array and under the columns the corresponding relative (q) and absolute (p) shift sizes:

Table 1

00101 11010 00010 12022 21000 22000 00220 11020 12102 00210 20120 01001 20110 10111 01000 10001 0
00022 02200 11102 11120 22021 01002 00012 12102 20220 00222 22220 02001 12021 12212 21001 22122 1
11012 02101 11101 20102 02111 20111 11201 11201 22212 11021 01201 12112 01021 21222 12111 21112 1
11100 10011 22210 20221 20102 10110 11100 20210 01001 11200 20021 10112 00102 20020 02112 00200 2
22102 12012 22210 01210 12202 01220 22010 02200 20200 22120 10212 22220 10122 00202 00222 20202 2
22211 21122 00021 21000 01210 01221 22011 21001 10110 22011 01102 21222 11210 21121 12220 11011 0
00210 20120 22221 12201 00012 12001 00121 10211 01001 00001 01000 00001 21202 11010 11002 01210 2
22020 00202 11102 02111 10221 12222 22122 00112 01021 22102 12012 20220 22001 02000 20221 03120 1
11221 21221 00022 00012 11120 20112 11202 01022 12122 11112 12111 11110 02210 02101 20110 12021 0
$q=$
01234 56788 00224 46681 13355 77036 04714 82583 72615 05162 73840 63074 17528 53186 42087 65432 1
$p=$
01361 63108 88137 28545 60384 20030 04237 68436 46340 05635 36500 60072 31687 36763 70086 38368 0

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