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SCORE SETS OF MULTITOURNAMENTS

k-tournaments (multitournaments) are directed graphs in which each pair of vertices is connected with exactly k directed edges. The increasingly ordered sequence of outdegrees of the vertices in such a graph is called the score sequence of the given graph and the set of the different outdegrees is called the score set of the given graph.

The classical result of this topic belongs to Landau [1]:

Theorem 1 *The sequence* $\mathbf{s} = \langle s_1, \ldots, s_n \rangle$ *with* $s_1 \le s_2 \le \cdots \le s_n$ *is a score sequence of a 1-tournament* T, *if and only if*

$$\sum_{i=1}^{j} s_i \ge \frac{j(j-1)}{2} \quad (j \in [1:n]) \tag{1}$$

and

$$\sum_{i=1}^{n} s_i = \frac{n(n-1)}{2} \; . \tag{2}$$

A similar assertation holds for k-tournaments: then the right side of the formulas is multiplied by k.

Reid formulated the following conjecture in [2].

Conjecture 2 *Any set of nonnegative numbers is the score set of some 1-tournament* T.

Using Landau's theorem this conjecture can be formulated in the following arithmetic form too.

Conjecture 3 *If* $0 \leq r_1 < r_2 < \cdots < r_m$*, then there exist such positive integers* $x_1,\ x_2,\ \ldots,\ x_m$*, that*

$$\sum_{i=1}^{j} x_i r_i \geq \frac{(\Sigma_{i=1}^{j} x_i)(\Sigma_{i=1}^{j} x_i - 1)}{2},\ j \in [1:m]$$

and

$$\sum_{i=1}^{m} x_i r_i = \frac{(\Sigma_{i=1}^{m} x_i)(\Sigma_{i=1}^{m} x_i - 1)}{2}.$$

In this case we say that the sequence $\mathbf{s} = \langle s_1, \ldots, s_n \rangle = \langle r_1^{x_1}, \ldots, r_m^{x_m} \rangle$ realizes the sequence $\mathbf{r} = \langle r_1, \ldots, r_m \rangle$ or $\mathbf{s}$ is a solution for $\mathbf{r}$.

Reid gave a constructive proof of his conjecture for sets containing one, two or three elements [2].

Later Hager published a constructive proof for sets with four and five elements [3] and Yao [4] published the outline of a nonconstructive proof of the general case.

A score set is called k-unique, if there exists exactly 1 score sequence generating the given set. In the talk we investigate the following questions:

1. characterization of the unique score sets of 1-tournaments;
2. extension of the Reid's conjecture to 2-tournaments;
3. extensions of Landau's theorem

1. 1-UNIQUE SCORE SETS

At first we formulate a useful necessary condition.

Lemma 4 *If the sequence* $\mathbf{s}$ *is a solution for* $\mathbf{r}$, *then in the case* $m = 1$ *we have*

$$n = 2r_1 + 1$$

and in the case $m \geq 2$ *we have*

$$2r_1 + 2 \leq n \leq 2r_m$$

and

$$n \geq r_m + 1.$$

This lemma implies the exact answer for the case $m = 1$.

Collorary 5 *If* $\mathbf{r} = \langle r_1 \rangle$, *then exactly the sequence* $\mathbf{s} = \langle r_1^{2r_1+1} \rangle$ *is a solution for* $\mathbf{r}$.

Proof: Lemma implies that only this solution is acceptable. One can check that it satisfies the required inequality and equality. Q.e.d.

Now we present a useful method of the investigation of the uniqueness. Let $\mathbf{r} = \langle a, a+d \rangle$. Then according to the Reid-equality we get

$$2ax + 2(a + d)y = n(n - 1)$$

implying

$$y = \frac{n(n - 2a - 1)}{2d}\,. \tag{3}$$

But here only the values $n = 2a+1+i$ ($i \in [1, 2d-1]$) are permitted where

$$i \geq d + 1 - a. \tag{4}$$

By substitution $a = (q-1)d$ from (3) we get

$$y = \frac{(2qd - 2d + 2r + 1 + i)i}{2d} . \tag{5}$$

Here y must be an integer, so transform this formula into

$$y = i(q-d) + \frac{i(2r+1+i)}{2d} . \tag{6}$$

Theorem 6 *If* $0 \leq a < b$, *then there exist positive integers* x *and* y *satisfying*

$$ax \geq \frac{x(x-1)}{2}$$

and

$$ax + by = \frac{(x+y)(x+y-1)}{2}.$$

In the following cases there is only one solution:

- *$a = 0$;*
- *$d = 1$;*
- *$d = 2$.*

In the following case there are at least two solutions:

- *d is odd and $3 \le d \le a$.*

Proof: a) Existence of a solution (due to Waldrop). Let $d = b - a$ and $i = 2d - 2r - 1$. Then $n = 2(b - r)$, $y = q(2d - 2r - 1)$, $x = q(2r + 1)$ satisfy all requirements.

b) Uniqueness. If $a = 0$, then $d = b$, $q = 1$ and y is integer only if $i = 2b - 1$. So we get the unique $\langle 0^1, b^{2b-1} \rangle$ solution.

If $d = 1$, then only $i = 1$ is permitted, implying the unique solution $\langle a^b, b^b \rangle$.

If $d = 2$ or d is odd, then we also can analyse formula (6). Q.e.d.

This theorem left open the case when the difference d is odd and the investigated set is sparse and also the case when the difference is an even number greater then 2.

2. 2-UNIQUE SCORE SETS

Now we present a new form of Reid-problem for 2-tournaments.

For a fixed sequence $\mathbf{R}[m] = \langle R_1, \ldots, R_m \rangle$ with $R_1 < \cdots < R_m$ of positive integers, we shall denote by $\mathbf{G}(\mathbf{R}[m])$ the set $\mathbf{G}$ of sequences $\mathbf{g} = \langle g_1, \ldots, g_m \rangle$ such that

$$\sum_{i=1}^{k} R_i g_i \ \geq \ \left(\sum_{i=1}^{k} g_i\right)^2, \quad k \in [1 : m-1]$$

and

$$\sum_{i=1}^{m} R_i g_i \ = \ \left(\sum_{i=1}^{m} g_i\right)^2.$$

Here we also say that $\mathbf{g}$ is a solution for $\mathbf{R}$.

We wish to give necessary and sufficient conditions for $\mathbf{R}[m]$ to have a solution that is a nonempty $\mathbf{G}(\mathbf{R}[m])$.)

Theorem 7 *For the sequence* $\mathbf{R}[1] = \langle R_1 \rangle$, *we have* $\mathbf{G}(\mathbf{R}[1]) = \langle R_1 \rangle$.

Proof: If $\mathbf{R}[1] = \langle R_1 \rangle$, then it is obvious that the solution of $R_1 g_1 = g_1^2$ is given in the form $g_1 = R_1$. Hence we have $\mathbf{G}(\mathbf{R}[1]) = \langle R_1 \rangle$.

Theorem 8 *Let* $\mathbf{R}[2] = \langle R_1, R_2 \rangle$ *be a sequence of positive integers with* $d = R_2 - R_1 > 0$. *Then* $\mathbf{G}(\mathbf{R}[2]) \neq \emptyset$ *if and only if either* $d \not| (R_1, R_2)$ *or* $d | (R_1, R_2)$ *and there is a prime* p *such that* $p^2 | d$.

Proof: According to the definition of $\mathbf{G}(\mathbf{R}[m])$, we need only find positive integers g_1, g_2 such that $R_1 \geq g_1$ and $R_1 g_1 + R_2 g_2 = (g_1 + g_2)^2$.

Let $q,\ r$ be integers for which $R_2 = qd+r$, where $0 \leq r < d$. If $d \not| (R_1, R_2)$, then $r \neq 0$ and let $g_1 = rq$ and $g_2 = R_2 - r(q+1)$. Hence we have

$$g_1 = rq = r\frac{R_2 - r}{R_2 - R_1} = r + r\frac{R_1 - r}{R_2 - R_1}$$

$$< r + (R_1 - r) = R_1,$$

$$g_2 = R_2 - r(q+1) =$$

$$R_2 - (R_2 - r)\frac{r}{R_2 - R_1} - r$$

$$> R_2 - (R_2 - r) - r = 0$$

and

$$R_1 g_1 + R_2 g_2 = R_1 rq + R_2^2 - R_2 r(q+1)$$

$$= R_2^2 + r(R_1 q - R_2 q + R_2) - 2R_2 r =$$

$$= (R_2 - r)^2 = (g_1 + g_2)^2.$$

Now assume that $d|(R_1, R_2)$ and there is a prime p such that $p^2|d$. In this case $r = 0$

and we choose $g_1,\ g_2$ as follows:

$$g_1 := \frac{R_2}{p} - \frac{d}{p^2} \ \text{ and } \ g_2 := g_1(p-1).$$

It is obvious that

$$g_1 > 0,\ g_2 > 0\ ,\ g_1 \le R_1$$

and

$$R_1 g_1 + R_2 g_2 = g_1(R_1 + (p-1)R_2)$$
$$= g_1(pR_2 - d) =$$
$$= g_1 p^2 (\frac{R_2}{p} - \frac{d}{p^2}) = (g_1 p)^2 = (g_1 + g_2)^2.$$

Finally, assume that $d = 1$ or $d|(R_1, R_2)$ and d is the product of distinct primes. If there are positive integers g_1, g_2 such that $R_1 \ge g_1$ and $R_1 g_1 + R_2 g_2 = (g_1 + g_2)^2$, then we have $d|g_1 + g_2$ and

$$\frac{1}{d}(g_1 + g_2)^2 - \frac{R_1}{d}(g_1 + g_2) = g_2 > 0,$$
$$\frac{1}{d}(g_1 + g_2)^2 - \frac{R_2}{d}(g_1 + g_2) = -g_1 < 0,$$

consequently

$$\frac{R_2}{d} = \frac{R_1}{d} + 1 > \frac{g_1 + g_2}{d} > \frac{R_1}{d}.$$

This is impossible. Q.e.d.

Theorem 9 *Let* $\mathbf{R}[2] =< R_1,\ R_2 >$ *be the sequence of positive integers with conditions* $R_1 < R_2$, $(R_1, R_2) = 1$, $2R_1 > R_2$ *and* $d := R_2 - R_1$ *has* s *dictinct prime factors. Then*

$$|\mathbf{G}(\mathbf{R}[2])| = 2^s - 1.$$

Proof: . Since $d = R_2 - R_1 < R_1$ and $(R_1, R_2) = 1$, the congruence $x^2 \equiv R_2 x \pmod{d}$ has $2^s - 1$ solutions in positive integers less than d. For each solution x we set $g_1 = \frac{x(R_2 - x)}{d}$ and $g_2 = (d - x)\frac{R_2 - x}{d}$. One can check that g_1, g_2 satisfy conditions $R_1 \geq g_1$ and $R_1 g_1 + R_2 g_2 = (g_1 + g_2)^2$. Q.e.d.

3. EXTENDED PROBLEMS

(a, b)-tournaments

Here each pair of vertices is connected at least a and at most with b (arbitrarily) directed edges. E.g. Hungarian football championships are (2,3)-tournaments.

Tournaments with incomplete data

Here we suppose that only some pairs of vertices are connected with edges (and we either know or do not now which pairs are connected – that is tournament maybe is not finished). János Csirik proposed this problem.

Collective tournaments

Here each match has $m \geq 2$ participants that is we investigate matroids (András Frank proposed this problem).

References

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