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BALANCED RECONSTRUCTION OF MULTIGRAPHS

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1. PRESCRIBED OUT-DEGREES

Let a, b and n be integers with $b \geq a \geq 0$ and $n \geq 2$. An (a, b, n)-tournament is defined as a loopless directed multigraph on n vertices, in which every pair of vertices is cennected with at least a and at most b arcs.

Theorem 1 (Landau, 1953) *A sequence* $(r_1, r_2, \ldots, r_n)$ *satisfying* $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ *is the out-degree sequence of some* $(1, 1, n)$-*tournament* T *if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{n}{2}, \quad 1 \leq k \leq n, \quad (1)$$

with equality when $k = n$.

Theorem 2 (Landau, 1953) *If i is a positive integer, then the sequence $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ satisfying $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ is the out-degree sequence of some (j, j, n)-tournament T if and only if*

$$\sum_{i=1}^{k} r_i \geq j\binom{n}{2}, \quad 1 \leq k \leq n,$$

with equality when $k = n$.

These theorems allow to check the realisability of **r** in linear time, but are not constructive. The following theorem requires more time to decide the existence, but allows the reconstruction in quadratic time.

Theorem 3 (Havel, 1955; Hakimi, 1962) *A sequence* $(r_1, r_2, \ldots, r_n)$ *satisfying* $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ *is the out-degree sequence of some* $(1,1,n)$*-tournament* T *if and only if the increasingly sorted version of the sequence* $(r_1, \ldots, r_m,\ \ r_{m+1} - 1,\ r_{m+2} - 1,\ \ r_{n-1} - 1)$ *is the out-degree sequence of some* $(1,1,n)$*-tournament, where* $m = r_n$.

Theorem 4 (Iványi, 2009) *Let* a, b, k, n, r_1, r_2, $\ldots$, r_n *be nonnegative integers* $(a \leq b, 0 < b, r_1 \leq r_2 \leq \ldots \leq r_n)$. *Further let* $L_0 = 0$, *and if* $1 \le k \le n$, *then let*

$$L_k = \max\left(L_{k-1},\ b\binom{n}{2} - \sum_{i=1}^{k} r_i\right).$$

The sequence $(r_1, r_2, \ldots, r_n)$ *is the out-degree sequence of an* (a, b, n)*-tournament* T, *if and only if*

$$a\binom{k}{2} \le \sum_{i=1}^{k} r_i \le$$

$$\le b\binom{n}{2} - L_k - (n-k)r_i \ (1 \le k \le n).$$

2. PRESCRIBED IN- AND OUT-DEGREES

Let $n \geq 2$ be a positive integer and let $\mathcal{T}_n(\mathbf{r}, \mathbf{c})$ be the set of directed multigraphs on $n+1$ vertices having prescribed out-degrees $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ and prescribed in-degrees $\mathbf{c} = (c_1, c_2, \ldots, c_n)$.

Let the element m_{ij} of the matrix $M_{(n+1)\times(n+1)}$ denote the number of arcs directed from the vertex V_i to vertex V_j. We present an algorithm MiniMax constructing a directed multigraph $D \in \mathcal{T}_n(\mathbf{r}, \mathbf{c})$ having the following properties:

- a) D has the prescribed in-degrees and out-degrees;

- b) D has the minimal value of $r_{n+1} + c_{n+1}$;
- c) D has the minimal value of $\max(m_{ij} + m_{ji})$;
- d) D has the maximal value of $\min(m_{ij} + m_{ji})$.

This algorithm is based on the algorithms due to Landau [7], Havel [4], Hakimi [2, 3], Ryser [10], and Iványi [6]. The algorithm generalizes the results due to Landau [7], Moon [9], Havel [4], Hakimi [2, 3], Meierlink and Volkman [8], Iványi [5, 6], Erdős, Miklós and Toroczkai [1], Ryser [10].

Let

$$\sum_{i=1}^{n} r_i = R, \quad \text{and} \quad \sum_{i=1}^{n} c_i = C.$$

The sequences **r** and **c** are called **balanced**, if $R = C$. It is a natural observation, that a corresponding D exists only for balanced **r** and **c**.

Hakimi gave in 1965 a necessary and sufficient condition of the existence of a $(0, \infty, n)$-tournament having a prescribed out-degree sequence **r** and in-degree sequence **c**.

Let's consider examples.

T_i/T_j	SL	RO	HU	Sum
SL	–	6	0	**6**
RO	4	–	0	**4**
HU	0	0	–	0
Sum	**4**	**6**	0	–

Table 1: A simple example

Now $r_1+r_2 = R = 10$ and $c_1+c_2 = C = 10$, $R = C$ and we have a unique solution without the third team.

T_i/T_j	SL	RO	HU	Sum
SL	–	4	2	**6**
RO	4	–	0	**4**
HU	0	0	–	**0**
Sum	**4**	**4**	**2**	–

Table 2: An example with $C < R$

Now $r_1+r_2 = R = 10$, but $c_1+c_2 = C = 8$, $R \neq C$ and we have NO solution without the third team. Choosing $r_3 = 0$ and $c_3 = 2$ we reach $R = C = 10$ and get a unique solution.

T_i/T_j	SL	RO	HU	Sum
SL	–	3	3	**6**
RO	4	–	0	**4**
HU	2	0	–	**2**
Sum	**6**	**3**	**1+2**	–

Table 3: Example with $C < R$, where r_1 is too large

Now $R = 10 > 9 = C$. Choosing $c_3 = 1$ and $r_3 = 0$ we get a balanced situation, but $r_1 > c_2 + c_3$. Choosing $c_3 = 3$ helps in some sense: now $r_1 = c_2 + c_3$, but $c_1 + c_2 + c_3 = 12$ became too large. $r_3 = 2$ results a unique solution.

These examples show, that

$$r_i \leq C - c_i \ (i = 1,\ 2,\ \ldots,\ n)$$

is also a necessary condition of the existence of a tournament $T \in \mathcal{T}_n(\mathbf{r}, \mathbf{c})$.

Hakimi in 1965 [3] proved the following theorem.

Theorem 5 *Let* $n \geq 2$ *and the sequences* $\boldsymbol{r}$ *and* $\boldsymbol{c}$ *of nonnegative integers have the property* $0 < r_1 + c_1 \leq r_2 + c_2 \leq \cdots \leq r_n + c_n$. *Then there exists a* $T \in \mathbf{r}, \mathbf{c}$ *toutnament if and only if the pair* $(\boldsymbol{r}, \boldsymbol{c})$ *balanced and*

$$\sum_{i=1}^{n} (r_i + c_i) \geq r_n + c_n.$$

If this necessary condition does not hold, then the following algorithm produces balanced (**r'**,**c'**) sequences.

Sequence-Augment$(n, \mathbf{r}, \mathbf{c}, \mathbf{r}', \mathbf{c}')$

```
R ← 0
C ← 0
for i ← 1 to n
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$\quad\quad R \leftarrow R + r_i$
$\quad\quad C \leftarrow C + c_i$
$r_{n+1} \leftarrow 0$
$c_{n+1} \leftarrow 0$
if $R > C$
$\quad$ **then** $c_{n+1} \leftarrow R - C$
$\quad\quad\quad\quad C \leftarrow C + c_{n+1}$
if $C > R$
$\quad$ **then** $r_{n+1} \leftarrow C - R$
$\quad\quad\quad\quad R \leftarrow R + r_{n+1}$
$r \leftarrow 0$
for $i \leftarrow 1$ **to** n
$\quad\quad$ **if** $r_i > C - c_i$
$\quad\quad\quad$ **then** $r \leftarrow \max(r, r_i - (C - c_i))$
$r_{n+1} \leftarrow r_{n+1} + r$
$c_{n+1} \leftarrow c_{n+1} + r$
$c \leftarrow 0$

for $i \leftarrow 1$ **to** n
  if $c_i > R - r_i$
    then $c \leftarrow \max(c, c_i - (R - r_i))$
$c_{n+1} \leftarrow c_{n+1} + c$
$r_{n+1} \leftarrow r_{n+1} + c$

Then a developed version of the algorithm MiniMax [6] results a multitournament D having properties a), b), c) and d).

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