List coloring of Latin and Sudoku graphs

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Abstract

In 2001 Galvin [44] proved a generalization of Hall theorem [49]. We improve his result in two directions, giving a weaker precondition and also a stronger consequence. We show some basic properties of the Latin and Sudoku graphs considering the solution of the Latin and Sudoku puzzles as a list coloring problem of graphs [5, 11, 14, 20, 33, 39, 57, 58, 60, 62, 84].

1 Introduction

List coloring of graphs was proposed by Vizing in 1976 [87]. Substantial progress was made by Erdős, Rubin and Taylor in 1979 [31].

Today graph coloring is a popular research topic of combinatorics and computer science. Concrete applications appear in connection with VLSI design [30, 61], networks [27, 38, 66, 73], resource allocation [45], and determination of partially given Latin [34] and Sudoku [17, 18, 19, 37, 40, 63, 64, 65] squares.

Let $m \geq 1$ and $n \geq 2$ be integers and $G = (V, E)$ be a finite simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Assume further that an infinite set $\mathcal{C} = \{c_1, c_2, \ldots\}$ of so called *colors,* and a collection $\mathcal{L} = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ of lists of colors are given, where $L(v_i) \subseteq \mathcal{C}$ for $1 \leq i \leq n$. Here each list $L(v_i)$ is considered as the set of allowed colors for vertex v_i. $\mathcal{L}$ is called *list arrangement.*

For the simplicity and without the loss the generality we suppose that $\mathcal{C}$ is the set of positive integers, and so each list is a finite set of positive integers. If the sizes of the lists have a join upper bound, that is if $|L(v_i)| \leq k$, then the collection $\mathcal{L}$ is called a *k-assignment* on G. Since the components of G are colorable independently, for the simplicity we suppose that G is connected.

A sequence of colors (positive integers) $S = (c_{i_1}, c_{i_2}, \ldots, c_{i_n}) = (i_1, i_2, \ldots, i_n)$ is called *a proper vertex coloring* of G with a given collection of lists $\mathcal{L}$, if for all $1 \leq j < k \leq n$

1. $i_j \in L(v_j)$;
2. if $v_j v_k \in E$, then $i_j \neq i_k$.

Graph G with a given $\mathcal{L}$ is $\mathcal{L}$-*colorable* (list colorable), if there exists a proper *vertex coloring function* $\varphi : V \to \mathcal{C}$ satisfying for all for all $1 \leq j < k \leq n$

1. $\varphi(v_j) \in L(v_j)$;
2. if $v_j v_k \in E$, then $\varphi(v_j) \neq \varphi(v_k)$.

Figure 1 shows a graph G_1 with some list assignment A_1.

Graph G_1 with list assignment A_1 has two proper colorings S_1 and S_2, which are represented in Figure 2.

Figure 3 shows graph G_2 with list assignment A_2, but this graph has no proper list coloring.

Let f a function from V to the positive integers, that is

$$f : V \to \{1, 2, \ldots\},$$

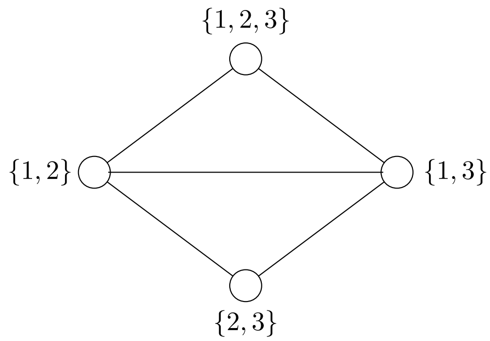

Figure 1: Graph G_1 with list assignment A_1.

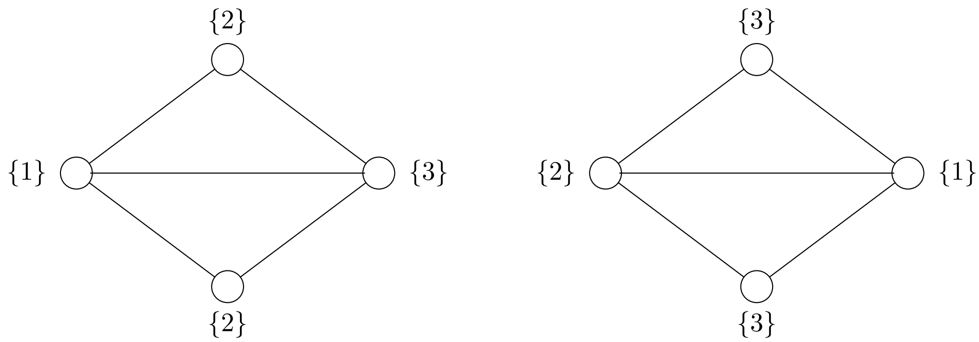

Figure 2: Proper colorings S_1 and S_2 of graph G_1 with list assignment A_1.

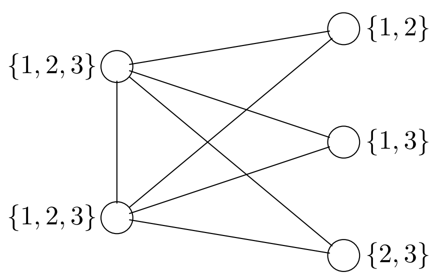

Figure 3: Graph G_2 with list assignment A_2.

then a graph G is called *f-choosable* [31] if there is a proper $\mathcal{L}$-coloring of G whenever $\mathcal{L}$ satisfies

$$|\mathcal{L}| \geq f(v_i) \text{ for all } v_i \in V.$$

For example the graph G_1 represented in Figure 1 is not f-choosable, if f is defined according to Figure 4. In Figure 5 such list assignment A_3 is represented which shows that the graph G_1 is not f-choosable.

The most important graph parameter in list coloring is the *choice number* of a graph G denoted by $\chi_\ell(G)$ and defined as the smallest integer k such that G is $\mathcal{L}$-colorable for every k-assignment $\mathcal{L}$.

The choice number of the graphs represented in Figures 1 and 3 is 3. The $\chi(G)$ chromatic number of these graphs is also 3. It is known that $\chi(G) \leq \chi_\ell(G)$ for all G [31, 87].

The exact characterization of the graphs for which the chromatic number is equal with the choice number is an open problem. The graph G_4 represented in 6 represented with a list assignment A_4 shows $\chi_\ell(G_4) = 3$, while $\chi(G_4) = 2$.

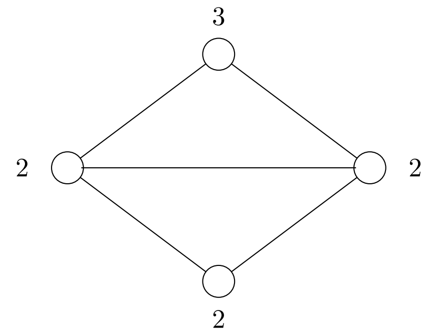

Figure 4: Function f_1 for graph G_1.

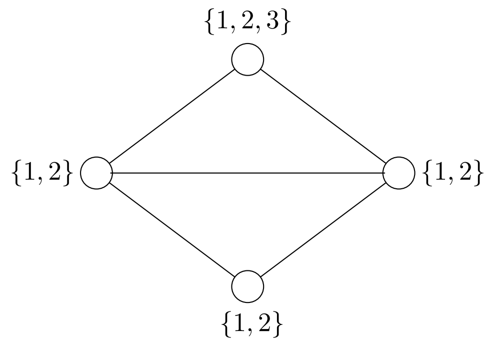

Figure 5: List assignment A_3 showing that graph G_1 is not f_1-choosable.

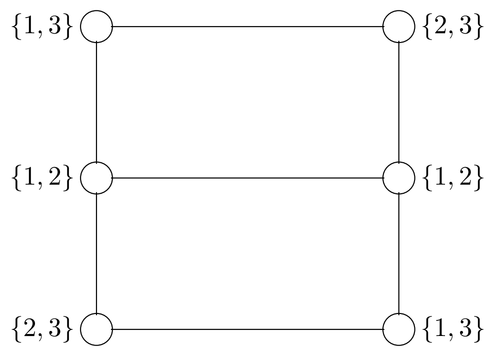

Figure 6: Graph G_4 with list assignment A_4.

2 Extension of Hall theorem

The Hall theorem [49] results the following necessary and sufficient condition of the $\mathcal{L}$-colorability of K_n (the complete graph on n vertices): K_n is $\mathcal{L}$-colorable if and only if the union of any k $(1 \leq k \leq n)$ lists contain at least k elements.

The classical Hall theorem has several new proofs [7, 8, 12, 71, 72, 76, 82] and many extensions [1, 2, 3, 4, 13, 25, 24, 26, 31, 35, 41, 42, 47, 48, 50, 52, 54, 55, 56, 69, 70, 74, 89].

In 1998 Fred Galvin [43] proposed the following problem as *Problem 1071* in *The American Mathematical Monthly.*

Let G be a finite, undirected, simple graph with vertex set V. Let $C = \{C_x : x \in V\}$ be a family of sets indexed by the vertices of G. For $X \subseteq V$, let $C_X = \cup_{x \in X} C_x$. A set $X \subseteq V$ is C-colorable if one can assign to each vertex $x \in X$ a "color" $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are adjacent in G.

a) Prove that if $|C_X| \geq |X|$ whenever X induces a connected subgraph of G, then V is C-colorable.

b) Prove that if every proper subset of V is C-colorable and $|C_V| \geq |V|$, then V is C-colorable.

c) For every connected graph G, find a family $C = \{C_x : x \in V\}$ showing that the condition $|C_V| \geq |V|$ in part **b)** cannot be weakened to $|C_V| \geq |V| - 1$.

Part **a)** was solved by Stephen C. Locke (Florida Atlantic University, Boca Raton, FL), part **(b)** by Sung Soo Kim (Hanyang University, Ansan, Kyunggi, Korea), and part **c)** by David Callan (University of Wisconsin, Madison, WI) in 2001 [44, 67].

These assertions on list coloring of graphs can be strengthened in two directions:

- there is a weaker sufficient condition than in part **a)**;
- the condition in part **a)** implies a stronger consequence for which the given condition is necessary and sufficient.

Let $\Gamma(x) = \{y : x \text{ and } y \text{ are adjacent in } G\}$ and $\deg(x) = |\Gamma(x)|$. A set $X \subseteq V$ is D-colorable if one can assign to each vertex $x \in X$ a color $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are connected with a path in G.

d) Prove that if $\deg(v) > |C_v|$ for each $v \in V$, then G is C-colorable.

e) Prove that G is D-colorable iff $|C_X| \geq |X|$ whenever X induces a connected subgraph of G.

Solution to part **d)**

Let $V = (v_1, v_2, \ldots, v_{|V|})$. Then we can assign colors to vertices e.g. in the increasing order of their indices, since a preceding vertex can exclude at most the colors of $\Gamma(v_i)$ adjacent vertices, but the degree of v_i is greater than the number of its adjacent vertices.

Solution to part **e)**

If G is D-colorable and X induces a connected subgraph of G, then the elements of X are connected with a path in G, therefore $|C_X| \geq |X|$ holds due to the definition of D-colorability.

If $|C_X| \geq |X|$ whenever X induces a connected subgraph S_X of G, then S_X is D-colorable due to Hall's theorem and D-colorability of the connected subgraphs of G implies the D-colorability of G.

We remark that if G is connected, then part **e)** is equivalent with Hall's theorem [49].

3 List coloring of Latin and Sudoku graphs

For a positive integer $n \geq 2$ the n-order Latin graph $\lambda_n = (V_n, E_n)$ corresponds to an n-order Latin square [28, 29] and is the finite simple graph with $V_n = \{v_{ij} : 1 \leq i,\ j \leq n\}$, and $E_n = \{(v_{ab}, v_{cd}) : a = c \text{ and } b \neq d, \text{ or } b = d \text{ and } a \neq c\}$.

Figure 9 shows the $n^2 = 16$ vertices of λ_4. λ_4 has $n \times n \times (n-1) = 48$ edges:

$$E_4 = \{v_{11}v_{12}, v_{11}v_{13}, v_{11}v_{14}, v_{12}v_{13}, v_{12}v_{14}, v_{13}v_{14}, \ldots, v_{43}v_{44}, v_{11}v_{21}, \ldots, v_{34}v_{44}\}.$$

v_{11}	v_{12}	v_{13}	v_{14}
v_{21}	v_{22}	v_{23}	v_{24}
v_{31}	v_{32}	v_{33}	v_{34}
v_{41}	v_{42}	v_{43}	v_{44}

Figure 7: The vertices of λ_4.

For a positive integer $m \geq 2$ the m-order Sudoku graph $\sigma_m = (V_m, E_m)$ [51, 78, 90] corresponds to an m-order Sudoku square [17, 18, 19, 37, 40, 63, 64, 65], which is an m^2-order Latin square divided into $m \times m$ disjoint subsquares of size $m \times m$ called *blocks*. So σ_m is the finite simple graph with $V_m = \{v_{ij} : 1 \leq i,\ j \leq m^2\}$, and $E_m = \{(v_{ab}, v_{cd} : a = c$ and $b \neq d$, or $b = d$ and $a \neq c$ or $\lfloor (a+1)/m \rfloor = \lfloor (c+1)/m \rfloor$, and $\lfloor (b+1)/m \rfloor = \lfloor (d+1)/m \rfloor$ and $(a, b) \neq (c, d)$.

Figure 8 shows the $9^2 = 81$ vertices of σ_3. σ_3 has $m^2 \times m^2 \times (m^2 - 1) = 648$ edges in the rows and columns, plus $m^2 \times m^2 \times (m-1)^2/2 = 162$ edges in the blocks, so $|E_3| = 810$ as follows:

$$E_3 = \{v_{11}v_{12}, \ldots v_{18}v_{19}, \ldots, v_{91}v_{92}, \ldots, v_{98}v_{99},$$

$$v_{11}v_{21}, \ldots, v_{81}v_{91}, \ldots, v_{19}v_{29}, \ldots, v_{89}v_{99},$$

$$v_{11}v_{22}, \ldots, v_{23}v_{32}, \ldots, v_{77}v_{88}, \ldots, v_{89}v_{98}\}.$$

v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}
v_{31}	v_{32}	v_{33}	v_{34}	v_{35}	v_{36}	v_{37}	v_{38}	v_{39}
v_{41}	v_{42}	v_{43}	v_{44}	v_{45}	v_{46}	v_{47}	v_{48}	v_{49}
v_{51}	v_{52}	v_{53}	v_{54}	v_{55}	v_{56}	v_{57}	v_{58}	v_{59}
v_{61}	v_{62}	v_{63}	v_{64}	v_{65}	v_{66}	v_{67}	v_{68}	v_{69}
v_{71}	v_{72}	v_{73}	v_{74}	v_{75}	v_{76}	v_{77}	v_{78}	v_{79}
v_{81}	v_{82}	v_{83}	v_{84}	v_{85}	v_{86}	v_{87}	v_{88}	v_{89}
v_{91}	v_{92}	v_{93}	v_{94}	v_{95}	v_{96}	v_{97}	v_{98}	v_{99}

Figure 8: The vertices of σ_3,

The following algorithm [28, 29]—defined using the pseudocode of [22]—produces a list coloring of an n-order Latin square λ_n for the case $L(v_{ij}) = \{1, 2, \ldots, n\}$ for all v_{ij}. The input parameter is n, and the output is a matrix $M_\lambda = [m_{ij}]_{n \times n}$ of the colors of the vertices of the graph λ_n.

```
LATIN(n, M_λ)
for i ← 1 to n
    do for j ← 1 to n
        do M_λ[i, j] ← i + j − 1 mod n
return M_λ
```

The order of the running time of this algorithm corresponds to the number of elements of the matrix M_λ, it is $\Theta(n^2)$.

If the input is $n = 4$, then the output of LATIN is the coloring represented in Figure 9.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Figure 9: An $\mathcal{L}$-coloring of λ_4 produced by algorithm LATIN.

It is known [79] that λ_4 has 576 different $\mathcal{L}$-colorings, if all vertex v_{ij} have the same color list $L(v_{ij}) = (1, 2, 3, 4)$, and λ_9 has $5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \sim 5.5 \cdot 10^{27}$ different colorings [79, 88], if all color lists are $L_(v_{ij}) = (1, 2, 3, 4, 5, 6, 7, 8, 9)$. These numbers give the number of 4-order, resp. 9-order Latin squares.

The following algorithm [83] produces a list coloring of the m^2-order Sudoku graph σ_m for the case $L(v_{ij}) = \{1, 2, \ldots, m^2\}$ for all v_{ij}. Its basic idea is similar to the idea of algorithm Latin. The input parameter is m, and the output is the matrix $M_\sigma = [m_{ij}]_{n\times n}$.

```
Sudoku(m, M_σ)
for i ← 1 to m^2
    do for j ← 1 to m^2
        do M_σ[i, j] ← i + j − 1 + ⌊(i − 1)/m⌋ + m⌊(i − 1)/m⌋ mod m^2
return M_σ
```

The running time of this algorithm also corresponds to the number of elements of the matrix M_σ : it is $\Theta(m^4)$.

If the input is $m = 3$, then the output of Sudoku is the coloring represented in Figure 10.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

Figure 10: The $\mathcal{L}$-coloring of σ_3, produced by algorithm Sudoku.

It is known [80] that σ_2 has 288 different $\mathcal{L}$-colorings, if all lists in $\mathcal{L}$ are equal to $\{1, 2, 3, 4\}$. It is also known [36, 77, 80, 85] that σ_3 has $6\,670\,903\,752\,021\,072\,936\,960 \sim 6.7 \cdot 10^{21}$ different $\mathcal{L}$-colorings, if all lists in $\mathcal{L}$ are equal with $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. These numbers give the number of 4-order, resp. 9-order Sudoku squares.

In the typical Latin and Sudoku puzzles the color of some subset of vertices is fixed (these vertex-color pairs are the input data of the puzzle). The aim is to complete the coloring, that is determine the number of possible complete colorings and to determine the proper colorings.

The first complexity result is due to Colbourn who proved in 1984 [21], that the problem of decision whether the coloring of a partially colored Latin graph can be completed is NP-complete.

Using Colburn's result Yato and Seta proved in 2003 [83], that the Sudoku ASP problem is NP-complete: if we know a proper vertex coloring of a partially colored Sudoku graph, then it is NP-complete problem to decide whether the given graph has another coloring or not.

In 1956 Behrens [9] proposed a more general problem, than the coloring of the Sudoku graph. He divided the set of the n^2 vertices of the n-order Latin graph λ_n into n disjunct subsets, and introduced the additional requirement that the elements of each subset have to be colored by different colors.

In 2007 Cameron asked [16] the complexity of the completion of the coloring of such generalized graphs, and Vaughan proved in 2009 [86] the NP-completeness.

Of course there are special graphs which are colorable by polynomial algorithms. E.g. the books written by Bach, Berthier, Erickson, Gordon, Inkala and Stuart [6, 10, 32, 46, 59, 81] and papers written by Crook [23] and Provan [75] contain numerous polynomial algorithms solving special Sudoku puzzles having unique solution.

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