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Imbalances of bipartite multitournaments

Shariefuddin Pirzada
King Fahd University
of Petroleum and Minerals,
Dhahran, Saudi Arabia
email: sdpirzada@yahoo.co.in

Abdulaziz M. Al-Assaf
King Fahd University
of Petroleum and Minerals,
Dhahran, Saudi Arabia
email: alassaf@kfupm.edu.sa

Koko K. Kayibi
King Fahd University
of Petroleum and Minerals,
Dhahran, Saudi Arabia
email: kokokayibi@googlemail.com

Antal Iványi
Eötvös Loránd University
Department of Computer Algebra
Budapest, Hungary
email: tony@compalg.inf.elte.hu

Abstract. A bipartite (a, b)-tournament is a bipartite tournament in which the vertices belonging to different parts of the tournament are connected with at least a and at most b arcs. The imbalance of a vertex is defined as the difference of its outdegree and indegree. In this paper existence criteria and construction algorithms are presented for bipartite (a, b)-tournaments having prescribed imbalance sequences and prescribed imbalance sets.

1 Introduction

An actual research topic of graph theory is the characterization of different special cases (as oriented, simple, multipartite, bipartite, signed and semi-complete graphs, see e.g. [1, 13, 14, 15, 27]), and different generalizations (as hypergraphs, hypertournaments, weighted graphs, signed graphs, see e.g. [17, 24, 25]) of multigraphs having prescribed degree properties.

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The classical results, as the paper published by Landau in 1953 [12], and the paper due to Erdős and Gallai in 1960 [4] contained necessary and sufficient conditions of the existence of a graph with the prescribed parameters. Later also constructive results appeared as the Havel-Hakimi theorem [7, 8] and the construction algorithm of optimal (a, b, n)-tournaments in 2010 [10].

2 Preliminary notions and earlier results

Let $a,\ b$ and n be nonnegative integers ($b \geq a,\ n \geq 1$), $\mathcal{T}(a, b.n)$ be the set of directed multigraphs $T = (V, E)$, where $|V| = n$, and elements of each pair of different vertices $u,\ v \in V$ are connected with at least a and at most b arcs [9]. $T \in \mathcal{T}(a, b, n)$ is called (a, b, n)-tournament. $(1, 1, n)$-tournaments are the usual tournaments, and $(0, 1, n)$-tournaments are also called oriented graphs or simple directed graphs [5]. The set $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcup_{b \geq a \geq 0,\ n \geq 1} \mathcal{T}(a, b, n).$$

According to this definition $\mathcal{T}$ is the set of the finite directed loopless multigraphs.

For any vertex $x \in V$ let $d(x)^+$ and $d(x)^-$ denote the outdegree and indegree of x, respectively. Define $f(x) = d(x)^+ - d(x)^-$ as the imbalance of the vertex x. The imbalance sequence of $T \in \mathcal{T}$ is formed by listing the imbalances of the vertices in nonincreasing or nondecreasing order.

The following result due to Avery [1] and Mubayi, Will and West [15] provides a necessary and sufficient condition for a nonincreasing sequence F of integers to be the imbalance sequence of a tournament $T \in \mathcal{T}(0, 1, n)$.

Theorem 1 *A nonincreasing sequence of integers* $F = [f_1, f_2, \ldots, f_n]$ *is an imbalance sequence of a tournament* $T \in \mathcal{T}(0, 1, n)$ *if and only if*

$$\sum_{i=1}^{k} f_i \geq k(n - k),$$

for $1 \leq k < n$ *with equality when* $k = n$.

Arranging the sequence F in nondecreasing order, we have the following equivalent assertion.

Corollary 2 *A nondecreasing sequence of integers* $F = [f_1, f_2, \ldots, f_n]$ *is the imbalance sequence of a* $(0, 1, n)$*-tournament if and only if*

$$\sum_{i=1}^{k} f_i \leq k(k-n)$$

for $1 \leq k < n$*, with equality when* $k = n$*.*

The following theorem gives a characterization of imbalance sequences of $(0, b, n)$-tournaments [22].

Theorem 3 *A nonincreasing sequence* $F = [f_1, f_2, \ldots, f_n]$ *of integers is the imbalance sequence of a* $(0, b, n)$*-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq bk(n-k),$$

for $1 \leq k \leq n$ *with equality when* $k = n$*.*

In [22] also a construction algorithm of a $(0, b, n)$-tournament can be found. Some other results on imbalances of $(0, b, n)$-tournaments and their special cases can be found in [16, 19, 23, 27, 28].

K. B. Reid in 1978 [26] introduced the concept of the score set of $(1, 1, n)$-tournaments as the set of different scores (outdegrees) of the given tournament. At the same time he formulated the conjecture that for any set of nonnegative integers S there exists a tournament T having S as its score set, and proved the conjecture for sets containing 1, 2, or 3 elements. Hager in 1986 [6] proved the conjecture for $|S| = 4$ and $|S| = 5$ and Yao in 1989 [29] published an existence proof of the whole conjecture. Constructive proof of the therem is not known.

There are some known results on the imbalance sets of some $(0, 1, n)$-tournaments (see e.g. [18, 20, 21, 22]).

3 Imbalances in $(0, \infty, p, q)$-tournaments

Let $a,\ b,\ p$ and q be nonnegative integers $(b \geq a,\ p \geq 1,\ q \geq 1)$, $\mathcal{B}(a, b, p, q)$ be the set of directed bipartite multigraphs $B = (U \cup V, E)$, where $|U| = p$ and $|V| = q$, and the elements of each pair of vertices $u \in U$ and $v \in V$ are connected with at least a and at most b arcs. Then $B \in \mathcal{B}(a, b, p, q)$ is called an (a, b, p, q)-tournament. $B \in \mathcal{B}(0, 1, p, q)$ is an oriented bipartite graph and a $(1, 1, p, q)$-tournament is a bipartite tournament.

The set $\mathcal{B}$ is defined by

$$\mathcal{B} = \bigcup_{b \geq a \geq 0,\ p \geq 1,\ q \geq 1} \mathcal{T}(a, b, p, q).$$

According to this definition $\mathcal{B}$ is the set of the finite directed bipartite multigraphs.

For any vertex $x \in U \cup V$ of $T \in \mathcal{T}(a, b, p, q)$ let d_x^+ and d_x^- denote the outdegree and indegree of x, respectively. Define $f_(x) = d(x)^+ - d(x)^-$ and $g(x) = d(x)^+ - d(x)^-$ as the imbalances of the vertex x. Then the nonincreasing or nondecreasing equences $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ are the imbalance sequences of the (a, b, p, q)-tournament $T = (U \cup V, E)$.

3.1 Basic properties of imbalance sequences

If in an (a, b, p, q)-tournament $B(U \cup V, E)$ there are x arcs directed from vertex $u \in U$ to $v \in V$ and y arcs directed from v to u, with $a \leq x \leq b$, $a \leq y \leq b$ and $a \leq x + y \leq b$, then it is denoted by $u(x - y)v$. We also call $u(x - y)v$ as a double. A tetra in an (a, b, p, q)-tournament is an induced $(0, 1, 2, 2)$-tournament. Define tetras of the form $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ and $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ to be of α-type, and all other tetras to be of β-type. An (a, b, p, q)-tournament is said to be of α-type or β-type according as all of its tetras are of α-type or β-type respectively. We note that an α-type tetra $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ or $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ can be respectively transformed to the β-type tetra $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-0)u_1$ or $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-1)u_1$ and vice-versa with imbalances of the vertices $u_1, u_2 \in U$ and $v_1, v_2 \in V$ remaining unchanged (see Figure 1). Further we note that a double of the form $u(x-x)v$ can be transformed to the double of the form $u(0-0)v$ making number of arcs less by $2x$ while imbalances remain unchanged.

The above facts lead us to the following assertion.

Lemma 4 *Among all (a, b, p, q)-tournaments with given imbalance sequences, those with the fewest arcs are of β-type.*

Proof. Let $B = B(U \cup V, E)$ be an (a, b, p, q)-tournament with imbalance sequences F and G. If B is not of β-type, it contains an oriented tetra of α-type. Thus for $u_1, u_2 \in U$ and $v_1, v_2 \in V$, we have $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$, or $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ as an oriented tetra of

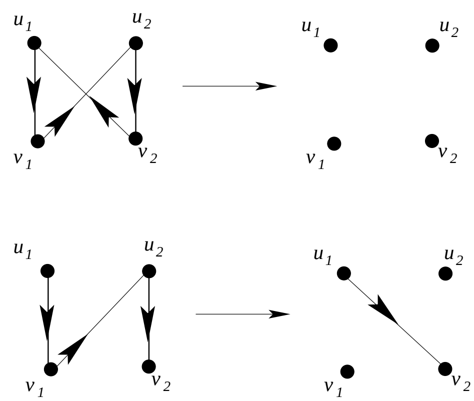

Figure 1: Transformation of an α-type tetra to a β-type tetra

α-type in B. Clearly $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ can be changed to $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-0)u_1$ with the same imbalance sequences and four arcs fewer, and $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ can be changed to $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-1)u_1$ with same imbalance sequences and two arcs fewer. Hence in both cases we obtain a realization $B'(U\cup V, E)$ of F and G with fewer arcs. In case there is a double of the form $u(x-x)v$, it can be transformed to the double of the form $u(0-0)v$ making number of arcs lesser by $2x$. ■

A transmitter is a vertex whose indegree is zero. We have the following assertion about the transmitters in a β-type $(0, b, p, q)$-tournament.

Lemma 5 *In a β-type $(0, b, p, q)$-tournament with ???creasing imbalance sequences $F = [f_1, f_2, \dots, f_p]$ and $G = [g_1, g_2, \dots, g_q]$, either a vertex with imbalance a_p, or a vertex with imbalance b_q, or both may act as transmitters.*

Proof. Let $U = \{u_1, u_2, \dots, u_p\}$ and $V = \{v_1, v_2, \dots, v_q\}$ be the parts of a $(0, b, p, q)$-tournament $B(U\cup V, E)$, so that $g(u_p) = f_p$ and $g(v_q) = g_q$. Assume that neither u_p nor v_q is a transmitter. Then there exist some vertices $u_i \in U$ and $v_j \in V$ such that $u_i(1-0)v_q$ and $v_j(1-0)u_p$. Since $g(u_p) \geq g(u_i)$ and $g(v_q) \geq g(v_j)$, there exist vertices $u_r \in U$ and $v_s \in V$ such that $u_p(1-0)v_s$ and $v_q(1-0)u_r$ (see Figure 2(a)). We have the following possibilities.

Case (i). $v_s(1-0)u_r$ and $u_r(0-0)v_j$. Here $v_j(1-0)u_p(1-0)v_s(1-0)u_r(0-0)v_j$ is a tetra of α-type, a contradiction (see Figure 2(b)).

Case (ii). $v_s(1-0)u_r$ and $u_r(1-0)v_j$. Here $v_j(1-0)u_p(1-0)v_s(1-0)u_r(1-0)v_j$ is a tetra of α-type, a contradiction (see Figure 2(c)).

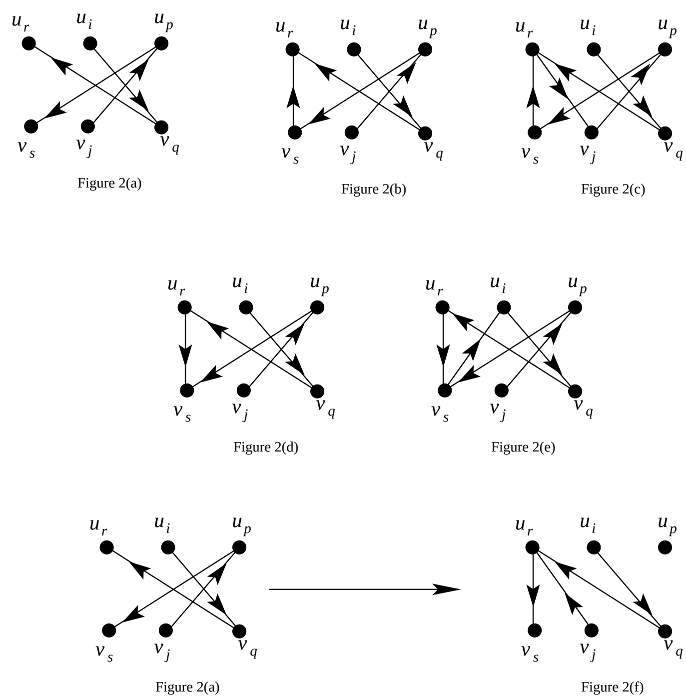

Figure 2: Illustration of the different cases in the proof of Lemma 5

Case (iii). $u_r(1-0)v_s$ and $v_s(0-0)u_i$. In this case $u_i(1-0)v_q(1-0)u_r(1-0)v_s(0-0)u_i$ is a tetra of α-type, again a contradiction (Figure 2(d)).

Case (iv). $u_r(1-0)v_s$ and $v_s(1-0)u_i$. Clearly $u_i(1-0)v_q(1-0)u_r(1-0)v_s(1-0)u_i$ is a tetra of α-type, again a contradiction (Figure 2(e)).

Case (v). If $u_r(1-0)v_s$ and $u_i(1-0)v_s$, then $b(u_i) > b(u_p)$, which is a contradiction. Similarly if $v_s(1-0)u_r$ and $v_j(1-0)u_r$, then $b(v_j) > b(v_q)$, again a contradiction.

Case (vi). Finally if $u_r(0-0)v_s$, $u_r(0-0)v_j$ and $u_i(0-0)v_s$, then there is a tetra $v_j(1-0)u_p(1-0)v_s(0-0)u_r(0-0)v_j$ and this can be transformed to the tetra $v_j(0-0)u_p(0-0)v_s(0-1)u_r(0-1)v_j$ and the imbalances remain unchanged (see Figure 2(f)). This means there is an α-type tetra $u_i(1-0)v_q(1-0)u_r(1-0)v_s(0-)u_i$, a contradiction. ■

4 Extremal reconstruction of imbalance sequences

Since each arc of a tournament adds +1 and −1 to the sum of the imbalances of the vertices, therefore if $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ are imbalance sequences of a $(0, \infty, p, q)$-tournament, then the total sum of the imbalances equals to zero.

Therefore a natural necessary condition of the realizability of sequences of integer numbers $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ as imbalance sequences of $B \in \mathcal{B}$ is

$$\sum_{i=1}^{p} f_i + \sum_{j=1}^{q} g_j = 0. \qquad (1)$$

for all elements of $B \in \mathcal{B}$.

Let F_{max} and G_{max} be defined as follows:

$$F_{max} = \max_{1 \le i \le p} |f_i|$$

and

$$G_{max} = \max_{1 \le j \le p} |g_j|$$

Easy to design an algorithm which constructs an (a, z, p, q)-tournament having F and G as its imbalance sequences, where

$$z = \max\left(\frac{F_{max}}{q}, \frac{G_{max}}{p}\right).$$

The following program constructes a tournament with prescribed imbalances. This and the following programs are given using the pseudocode in [3].

Inputs. p and q: the numbers of the elements in the prescribed imbalance sequences;

$F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$: given nonincreasing sequences of integers.

Output. $\mathrm{M}_{(p+q)\times(p+q)}$: arc matrix of a tournament $B \in \mathcal{T}(0, \infty, p, q)$.

Working variables. i and j: cycle variables;

S: actual sum of the imbalances;

L and R: the actual value of the left and right side of (5).

Easy-Bipartite(b, p, q, F, G)

But we are interested not only in the construction of some tournament having the

prescribed imbalances, but of tournament having the least possible b allowing the realization of F and G.

4.1 Lower bound for b

The following result is an existence criterion for determining whether some prescribed sequences are realizable as imbalance sequences of a tournament $T \in \mathcal{T}$. This is analogous to a result on degree sequences by Erdős and Gallai [4] and a result on bipartite tournaments due to Beineke and Moon [2].

Theorem 6 *Two nonincreasing sequences $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ of integers are the imbalance sequences of some $(0, b, p, q)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j \geq b\left[k(q-l) + l(p-k)\right], \tag{2}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$, with equality when $k = p$ and $l = q$.

Proof. The necessity follows from the fact that a directed bipartite subgraph of a $(0, b, p, q)$-tournament induced by k vertices from the first part and l vertices from the second part has a sum of imbalances at most $bk(q-l) + bl(p-k)$.

For sufficiency, assume that $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ are the sequences of integers in nonincreasing order satisfying conditions (1) but are not the imbalance sequences of any $(0, b, p, q)$-tournament. Let these sequences be chosen in such a way that p is the smallest possible and q is the smallest possible among the tournaments with the smallest p, and a_1 is the least with that choice of p and q. We consider the following two cases.

Case (i). ????????

Case (ii). Suppose equality in (1) holds for some $k \leq p$ and $l < q$, so that

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j = b\left[k(q-l) + l(p-k)\right].$$

By the minimality of p and q, $F = [f_1, f_2, \ldots, f_k]$ and $G = [g_1, g_2, \ldots, g_l]$ are the imbalance sequences of some $(0, b, p, q)$-tournament $B_1(U_1 \cup V_1, E_1)$. Let $F_2 = [f_{k+1}, f_{k+2}, \ldots, f_p]$ and $G_2 = [g_{l+1}, g_{l+2}, \ldots, g_q]$.

Now,

$$\begin{aligned}
\sum_{i=1}^{r} f_{k+i} + \sum_{j=1}^{l} g_{l+j} &= \sum_{i=1}^{k+r} f_i + \sum_{j=1}^{l+s} b_j - \left(\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j\right) \\
&\geq b[2(k+r)(l+s) - (k+r)q - (l+s)p] - b(2kl + kq + lp) \\
&= b(2kl + 2ks + 2lr + 2rs - kq - qr - lp - ps - 2kl + kq + lp) \\
&= b(2rs - qr - ps + 2ks + 2lr) \\
&\geq b(2rs - qr - ps),
\end{aligned}$$

for $1 \leq r \leq p-k$ and $1 \leq s \leq q-l$, with equality when $r = p-k$ and $s = q-l$. So, by the minimality for p and q, the sequences F_2 and G_2 form the imbalance sequences of the $(0, b, p-k, q-l)$-tournament $B_2(U_2 \cup V_2, E_2)$. Now construct a $(0, b, p, q)$-tournament $B(U \cup V, E)$ as follows.

Let $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ with $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$ and the arc set containing those arcs which are between U_1 and V_1 and between U_2 and V_2. Then we obtain a $(0, b, p, q)$-tournament $B(U \cup V, E)$ with the imbalance sequences F and G, which is a contradiction.

Case (iii). Suppose that the strict inequality holds in (1) for all $k \neq p$ and $l \neq q$. That is,

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j > b\left[k(q-l) + l(p-k)\right]$$

for $1 \leq k < p$, $1 \leq l < q$.

Let $F_1 = [f_1 - 1, f_2, \ldots, f_{p-1}, f_p + 1]$ and $G_1 = [g_1, g_2, \ldots, g_q]$, so that F_1 and G_1 satisfy the conditions (1). Thus, by the minimality of f_1, the sequences F_1 and G_1 are the imbalances sequences of some $(0, b, p, q)$-tournament $B_1(U_1 \cup V_1, E_1)$. Let $f_{u_1} = f_1 - 1$ and $f_{u_p} = f_p + 1$. Since $f_{u_p} > f_{u_1} + 1$, therefore there exists a vertex $v_1 \in V_1$ such that $u_p(0-0)v_1(1-0)u_1$, or $u_p(1-0)v_1(0-0)u_1$, or $u_p(1-0)v_1(1-0)u_1$, or $u_p(0-0)v_1(0-0)u_1$, in $D_1(U_1 \cup V_1, E_1)$ and if these are changed to $u_p(0-1)v_1(0-0)u_1$, or $u_p(0-0)v_1(0-1)u_1$, or $u_p(0-0)v_1(0-0)u_1$, or $u_p(0-1)v_1(0-1)u_1$ respectively, the result is a $(0, b, p, q)$-tournament with imbalances sequences F and G, which is a contradiction proving the result. ■

Since $(0, 1, p, q)$-tournaments are special $(0, b, p, q)$-tournaments, the following corollary of Theorem 6 gives a necessary and sufficient condition for nonincreasing sequences of integers to be imbalance sequences of some $(0, 1, p, q)$-tournaments.

Corollary 7 *Two nonincreasing sequences $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ of integers are the imbalance sequences of some $(0, 1, p, q)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j \geq k(q-l) + l(p-k), \tag{3}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$ with equality when $k = p$ and $l = q$.

From the other side, for arbitrary sequences of integer numbers F and G one can find such a b, that F and G are imbalance sequences of some $(0, b, p, q)$-tournament.

4.2 Computation of b_{min} using Theorem 6

Using Theorem 6 we can compute the minimal such value of b which allows that prescribed sequences are the imbalance sequences of a $(0, b, p, q)$-tournament.

Let

$$\alpha(k, l) = \sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j = 0$$

and

$$\beta(k, l) = b\,[k(q-l) + l(p-k)]$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$.

The following theorem allows quickly to compute b_{min}.

Theorem 8 *Two nonincreasing sequences $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ of integers are the imbalance sequences of some $(0, b, p, q)$-tournament B if and only if $b \geq b_{min}$, where*

$$b_{min} = \min_{1 \leq k \leq m, 1 \leq l \leq n} \{b \mid \alpha(k, l) \leq \beta(k, l)\}. \tag{4}$$

Inputs. p and q: the numbers of the elements in the prescribed imbalance sequences;
b: maximum number of permitted arcs between two vertices $u \in U$ and $v \in V$;
$F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$: given nonincreasing sequences of integers.
Output. b_{min}: the minimal number of allowed arcs between two vertices belonging to different parts of B.
Working variables. i and j: cycle variables;
S: actual sum of the imbalances;
$L = \alpha(k, l)$: the actual value of the left side of (5)
$R = \alpha(k, l)$: the actual value of the right side of (5).

MINIMAL-$b(b, p, q, F, G)$

```
S ← 0
b_min ← 1
for i ← 1 to p
    S ← S + f_i
    L ← S
    for j ← 1 to q
        L ← S + g_j
        R ← b_min[k(q − l) + l(p − k)]
        if L < R and ⌈L/R⌉
           then b_min ← ⌈L/R⌉
return b_min
```

This simple algorithm computes b_{min} in $\Theta(pq)$ time.

4.3 Lower bound for b in a generalized problem

let the positive integers p and q, the nonincreasing sequences of integers $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$, further the $(p+q) \times (p+q)$ sized matrix $\mathcal{M} = [m_{ij}]$ be given.

At first we investigate the condition of the existence of a $(0, b, p, q)$-tournament a $(p+q) \times (p+q)$ sized arc matrix $N = n_{ij}$ with $n_{ij} \geq m_{ij}$ for $1 \leq i, j \leq (p+q)$ resulting the prescribed imbalance sequences.

The first observation is natural: the elements of $\mathcal{M}$ have to satisfy the following inequaqualities:

$$m_{ij} \begin{cases} = 0, & \text{if} \quad 1 \le i, j \le p, \\ = 0, & \text{if} \quad p+1 \le i, j \le q, \\ \ge 0, & \text{otherwise.} \end{cases} \tag{5}$$

We can interpret this problem so, that some results of the tornament are prescribed, e.g. in the form of a set S, where

$$S = \{(u_1 v_1)^{e_{11}}, (u_1 v_2)^{e_{12}}, \ldots, (u_p v_q)^{e_{pq}}\},$$

and if all exponents e_{ij} are equal to zero, then we have the original problem.

Another natural necessary condition is 1.

The following program demonstrates that (1) and (5) together form a sufficient condition for the existence of a correspondig tournament.

Inputs. p and q: the numbers of the elements in the prescribed imbalance sequences;

b: maximum number of permitted arcs between two vertices $u \in U$ and $v \in V$;

$F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$: given nonincreasing sequences of integers.

Output. b_{min}: the minimal number of allowed arcs between two vertices belonging to different parts of B.

Working variables. i and j: cycle variables;

S: actual sum of the imbalances;

$L = \alpha(k, l)$: the actual value of the left side of (5)

$R = \alpha(k, l)$: the actual value of the right side of (5).

Easy-Modified-Bipartite-$(b, p, q, F, G, \mathcal{M})$

```
for i ← 1 to p
    for j ← 1 to q
        m_ij ← m_ij + m_ji
???
???
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???
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???
???
???
return M′
```

This algorithm computes runs in worst case in $\Theta(??)$ time.

The problem becomes harder if we try to get a $(0, b, p, q)$-tournament with minimal b. Here we meet a natural bound

$$b_{min} \geq \max_{1 \leq i,j \leq p+q} (m_{ij} + m_{ji}).$$

Using the method described in [9, 10, 11] we get the following extension of Theorem 6.

Theorem 9 *Let p and q be positive integers, $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1, g_2, \ldots, g_q]$ nonincreasing sequences of integers, then F and G are the imbalance sequences of some $(0, b, p, q)$-tournament B if and only if $b \geq b_{min}$, where*

$$b_{min} = \min_{1 \leq k \leq m, 1 \leq l \leq n} \{b \mid \alpha(k, l) \leq \beta(k, l)\}. \tag{6}$$

Proof. Let ■

4.4 Computation of b_{min} using Theorem ??

4.5 Construction of a $(0, b_{min}, p, q)$-tournament with prescribed imbalances

The next result provides a useful recursive test to decide whether given sequences of integers are the imbalance sequences of some $(0, b, p, q)$-tournament.

Theorem 10 *Let p and q be positive integers, $F = [f_1, f_2, \ldots, f_p]$ and $G = [g_1,$
$g_2, \ldots, g_q]$ nonincreasing sequences of integers ????*

Proof. Let ■

4.6 Algorithm and program for constructing a $(0, b_{min}, p, q)$-tournament with prescribed imbalance sequences

The successive application of Theorem 9 provides an algorithm for determining whether the two sequences of integers in nonincreasing order are the imbalance sequences, and for constructing a corresponding tournament.

The following program realizes the construction described in Theorem 9.

Input. p and q: the numbers of the elements in the prescribed imbalance sequences;

b_{min}: the minimal number of allowed arcs between the vertices, determined by algorithm Minimal-b.

$(f_1, f_2, \ldots, f_p)$ and $(g_1, g_2, \ldots, g_q)$: given nonincreasing sequences of positive integers;

Output. $\mathcal{M}_{(p+q)\times(p+q))}$: the arc matrix of the reconstructed tournament.

Working variables. i, j: cycle variables.

Bipartite-Sequences-with-Results(b, p, q, F, G)

The running time of this algorithm is $\Theta(pq)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has mn elements, this algorithm is asymptotically optimal.

4.7 Examples

The first three examples show the result of the computation of b_{min} in the following cases:

a) $p = 4$, $q = 5$, $F = [-3, 1, 2, 2]$, and $G = [-3, -1, 0, 1, 1]$.

b) $p = 4$, $q = 3$, $F = [-2, -2, 3, 4]$, and $G = [-5, -1, 3]$.

c) $p = 6$, $q = 3$, $F = [(-12)^2, (4)^4)]$, and $G = [9^2, -10]$.

The next three examples show the constructed $(0, b_{min}, p, q)$-tournaments.

d) $A_2 = [-3, 1, 2]$, $B_2 = [-2, 0, 0, 1, 1]$, $u_4(1-0)v_1$, $u_4(1-0)v_2$,
$A_3 = [-3, 1]$, $B_3 = [-1, 1, 0, 1, 1]$, $u_3(1-0)v_1$, $u_3(1-0)v_2$,

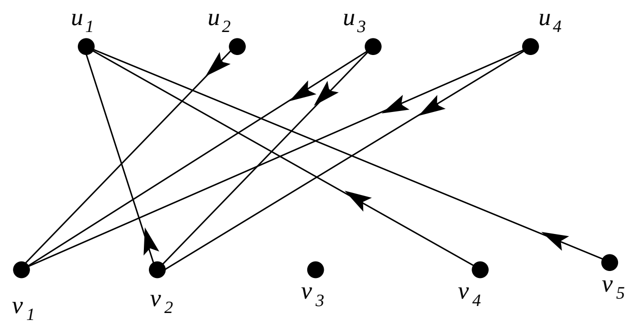

Figure 3: Result of the construction of the imbalance sequences F and G

or $A_3 = [-3, 1]$, $B_3 = [-1, 0, 1, 1, 1]$,
$A_4 = [-3]$, $B_4 = [0, 0, 1, 1, 1]$, $u_2(1-0)v_1$,
$A_5 = [-2]$, $B_5 = [0, 0, 1, 1]$, $v_5(1-0)u_1$,
$A_6 = [-1]$, $B_6 = [0, 0, 1]$, $v_4(1-0)u_1$,
$A_7 = [0]$, $B_7 = [0, 0]$, $v_2(1-0)u_1$.

Obviously, an oriented bipartite graph D with parts $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$ in which $u_4(1-0)v_1$, $u_4(1-0)v_2$, $u_3(1-0)v_1$, $u_3(1-0)v_2$, $u_2(1-0)v_1$, $v_5(1-0)u_1$, $v_4(1-0)u_1$, $v_2(1-0)u_1$ are arcs has imbalance sequences $[-3, 1, 2, 2]$ and $[-3, -1, 0, 1, 1]$ (see Figure 4).

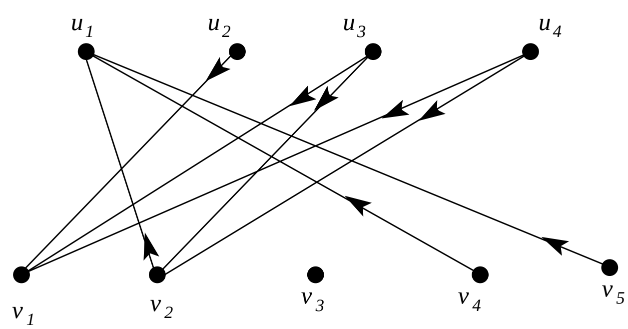

Figure 4: Result of the construction of the imbalance sequences F and G

e) $F_2 = [-2, -2, 3]$, $G_2 = [-3, 0, 4]$, $u_4(2-0)v_1$, $u_4(1-0)v_2$, $u_4(1-0)v_3$,
$A_3 = [0, -1, 4]$, $B_3 = [-3, 0]$, $v_3(2-0)u_1$, $v_3(1-0)u_2$, $v_3(1-0)u_3$,
$A_4 = [0, -1]$, $B_4 = [-1, 2]$, $u_3(2-0)v_1$, $u_3(2-0)v_2$,
$A_5 = [1, 0]$, $B_5 = [-1]$, $v_2(1-0)u_1$, $v_2(1-0)u_2$,

or $A_5 = [0,1]$, $B_5 = [-1]$,
$A_6 = [0]$, $B_6 = [0]$, $u_1(1-0)v_1$.

A $(0,b,p,q)$-tournament B with parts $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3\}$ in which $u_4(2-0)v_1,\ u_4(1-0)v_2,\ u_4(1-0)v_3,\ v_3(2-0)u_1,\ v_3(1-0)u_2,\ v_3(1-0)u_3,\ u_3(2-0)v_1,\ u_3(2-0)v_2,\ v_2(1-0)u_1,\ v_2(1-0)u_2$ are arcs, has imbalance sequences $[-2,-2,3,4]$ and $[-5,-1,3]$ (see Figure 5).

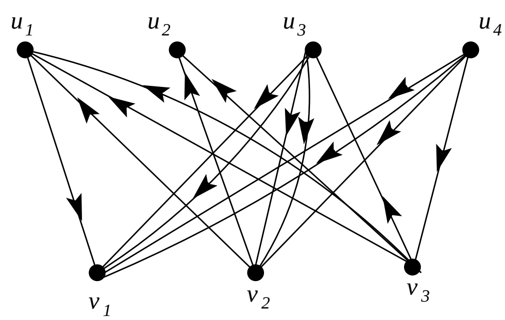

Figure 5: Illustration of the reduction for $b_{min} = 2$

f) $F_2 =, G_2 =$

5 Imbalance sets in $(0,b,n)$-tournaments

K. B. Reid in 1978 [26] introduced the concept of the score set of tournaments as the set of different scores (outdegrees) of a tournament. At the same time he formulated the conjecture that for any set of nonnegative integers S there exists a tournament T having S as its score set. At the same time he proved the conjecture for sets containing 1, 2, or 3 elements. Hager in 1986 [6] proved the conjecture for $|S| = 4$ and $|S| = 5$ and Yao [29] published a proof of the conjecture.

Let define the imbalance sets of a $(0,b,p,q)$-tournament $B = (U \cup V, E)$ having imbalance sequences $J = [j_1, j_2, \cdots, j_p]$ and $K = [k_1, k_2, \cdots, k_q]$ as the union of the sets of different imbalances of the values in J and K.

First we show the existence of a $(0,1,p,q)$-tournament, then the existence of a special $(0,1,p,q)$-tournament, having prescribed imbalance sets.

5.1 Existence of a $(0, 1, p, q)$-tournament with prescribed imbalances

The following assertion shows the existence of a $(0, 1, p, q)$-tournament having prescribed imbalance sets in the case when the first set contains nonnegative, and the second set contains only nonpositive elements.

Theorem 11 *Let p and q be positive integers, $J = [j_1, j_2, \ldots, j_p]$ and $K = [-k_1, -k_2, \ldots, -k_q]$, where $j_1, j_2, \ldots, j_p, k_1, k_2, \ldots, k_q$ are nonnegative integers with $j_1 < j_2 < \cdots < j_p$ and $k_1 < k_2 < \cdots < k_q$. Then there exists a connected $(0, 1, p, q)$-tournament with imbalance set $J' \cup K'$.*

Proof.

Case 1. $j_1 \cdot k_1 > 0$.Construct a $(0, 1, p, q)$-tournament $B(U \cup V, E)$ as follows. Let $U = U_1 \cup U_2 \cup \cdots \cup U_p$, $V = V_1 \cup V_2 \cup \cdots \cup V_p$ with $U_i \cap U_j = \emptyset$ $(i \neq j)$, $V_i \cap V_j = \emptyset$ $(i \neq j)$, $|U_i| = b_i$ for all i, $1 \leq i \leq p$ and $|V_j| = a_j$ for all j, $1 \leq j \leq p$. Let there be an arc from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq p$, so that we obtain the $(0, 1, p, q)$-tournament $B(U \cup V, E)$ with the given imbalance sets of vertices as follows.

For $1 \leq i, j \leq p$, $f_{u_i} = |V_i| - 0 = f_i$, for all $u_i \in U_i$ and $g_{v_j} = 0 - |U_j| = -g_j$, for all $v_i \in V_i$.

Therefore, the imbalance set of $B(U \cup V, E)$ is $F \cup G$.

The oriented bipartite graph constructed above is not connected. In order to see the existence of oriented bipartite graph, whose underlying graph is connected, we proceed as follows.

Taking $U_i = \{u_1, u_2, \ldots, u_{b_i}\}$ and $V_j = \{v_1, v_2, \ldots, v_{a_j}\}$, and let there be an arc from each vertex of U_i to every vertex of V_j except the arcs between u_{g_i} and v_{f_j}, that is $u_{b_i}(0 - 0)g_{a_j}$, $1 \leq i \leq p$ and $1 \leq j \leq p$. We take $u_{g_1}(0 - 0)g_{f_2}$, $u_{g_2}(0-0)v_{f_3}$, and so on $u_{g_{(n-1)}}(0-0)v_{f_n}$, $u_{g_n}(0-0)v_{f_1}$. The underlying graph of this $(0, 1, p, p)$-tournament is connected.

Case 2. $j_1 \cdot k_1 = 0$. If $j_1 = 0$, then we use the construction proposed in the first case not taking into account the vertex u_1 and its imbalance $j_1 = 0$. At the end the construction process we add two arcs $u_1(1 - 1)v_1$ resulting a tournament in which the imbalance of u_1 is the prescribed 0 and the imbalance of v_1 is the prescribed k_1. ■

5.2 Existence of a $(0, b, p, q)$-tournament with prescribed imbalance sets

Finally, we prove the existence of a $(0, b, p, q)$-tournament with prescribed sets of positive integers as its imbalance set.

Let $(f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q)$ denote the greatest common divisor of $f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q$.

Theorem 12 *Let $b \geq 1$ a positive integer, $F = \{f_1, f_2, \ldots, f_p\}$ and $Q = \{-g_1, -g_2, \ldots, -g_q\}$, where $f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q$ are positive integers with $f_1 < f_2 < \cdots < f_p$, $g_1 < g_2 < \cdots < g_q$ and $(f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q) = t \leq b_{min}$. Then there exists a connected $(0, b, p, q)$-tournament with imbalance set $P \cup Q$.*

Proof. Since $(f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q) = t$, where $1 \leq t \leq b$, there exist positive integers $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q$ with $x_1 < x_2 < \cdots < x_p$, $y_1 < y_2 < \cdots < y_q$ such that $f_i = tx_i$ for $1 \leq i \leq p$ and $g_j = ty_j$ for $1 \leq j \leq q$.

Construct a $(0, b, p, q)$-tournament $B(U \cup V, E)$ as follows. Let $U = U_1 \cup U_2 \cup \cdots \cup U_p \cup U^1 \cup U^2 \cup \cdots \cup U^p$, $V = V_1 \cup V_2 \cup \cdots \cup V_p \cup V^1 \cup V^2 \cup \cdots \cup V^p$ with $U_i \cap U_j = \emptyset$, $U_i \cap U^j = \emptyset$, $U^i \cap U^j = \emptyset$, $V_i \cap V_j = \emptyset)$, $V_i \cap V^j = \emptyset$, $V^i \cap V^j = \emptyset$, $i \neq j$, $|U_i| = x_i$ for all i, $1 \leq i \leq p$ and $|U^i| = g_i$ for all i, $1 \leq i \leq p$, $|V_i| = x_i$ for all i, $1 \leq i \leq p$ and $|V^i| = g_i$ for all i, $1 \leq i \leq q$. Let there be t arcs directed from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq p$ and let there be t arcs directed from every vertex of U^i to each vertex of V^i for all i, $1 \leq i \leq q$, so that we obtain the $(0, b, p, q)$-tournament $B(U \cup V, E)$ with the imbalances of vertices as follows.

For $1 \leq i \leq p$,

$$f_{u_i} = t|V_i| - 0 = tx_i = f_i, \text{ for all } u_i \in U_i,$$

$$g_{v_i} = 0 - t|U_i| = -ty_1 = -g_1, \text{for all } v_i \in V_i,$$

for $1 \leq i \leq q$,

$$f_{u^i} = t|V^i| - 0 = tf_1 = g_1, \text{ for all } u^i \in U^i,$$

$$g_{v^i} = 0 - t|U^i| = -ty_i = -g_i, \text{ for all } v^i \in V^i.$$

Therefore the imbalance set of $B(U \cup V, E)$ is $P \cup Q$.

The $(0, b, p, q)$-tournament constructed above is not connected. In order to see the existence of a $(0, b, p, q)$-tournament, whose underlying graph is connected, we proceed as follows.

Let $U_i = \{u_1, u_2, \ldots, u_{g_i}\}$ and $V_j = \{v_1, v_2, \ldots, v_{f_j}\}$, and let there be an arc from each vertex of U_i to every vertex of V_j except the arcs between u_{g_i} and v_{f_j}, that is $u_{g_i}(0-0)v_{f_j}$, $1 \le i \le q$ and $1 \le j \le q$. We take $u_{g_1}(0-0)v_{f_2}$, $u_{b_2}(0-0)v_{a_3}$, and so on $u_{b_{(n-1)}}(0-0)v_{a_n}$, $u_{b_n}(0-0)v_{a_1}$. The underlying graph of this $(0, b, p, q)$-tournament is connected. ■

5.3 Program for constructing a connected $(0, b, p, q)$-tournament with prescribed imbalance sets

The following program realizes the construction described in Theorem 12.

Input. p and q: the numbers of the elements in the prescribed imbalance sequences;

b: maximum number of permitted arcs between two vertices $u \in U$ and $v \in V$;

$F = (f_1, f_2, \ldots, f_p)$ and $G = (g_1, g_2, \ldots, g_q)$: given sequences of nonpositive integers with $0 \le f_1 < f_2 < \cdots < f_p$ and $0 \le g_1 < g_2 < \cdots < g_q$;

$t = (f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_q)$.

Output. $\mathcal{M}_{(p+q)\times(p+q)}$: the arc matrix of the reconstructed tournament (m_{ij} gives the number of arcs directed from the vertex u_i to the vertex v_j).

Working variables. i, j: cycle variables;

BIPARTITE-SETS(b, p, q, F, G)

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The running time of this algorithm in worst case is $\Theta(???)$.

6 Open problems

We list several unsolved problems connected with the topic investigated in the given paper.

??????????????????????

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