Számelméleti függvények segítségével értelmezett mátrixok tulajdonságai
(Matrices associated with classes of arithmetical functions)

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Introduction

- The **classical** Smith's determinant, H. J. Smith 1875

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(\mathbf{1}) \cdot \varphi(\mathbf{2}) \cdots \varphi(\mathbf{n})$$

⇝ 1875–86 H. J. Smith, E. Cesaro, P. Mansion, 1930 D. H. Lehmer

⇝ 1950–1970 E. Cohen, T. M. Apostol, B. Gyíres, Gy. Maurer

⇝ 1990–2009 S. Beslin, S. Ligh, P. Haukkanen, P. J. McCarthy, S. Hong, I. Korkee, E. Altinişik

- Let f be an arithmetical function and

$$f(n) = \sum_{d|n} g(d).$$

Then for the following generalization of Smith's determinant:

$$\det[f(i,j)]_{n\times n} = \begin{vmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)) & f((n,2)) & \cdots & f((n,n)) \end{vmatrix} = g(1)\cdots g(n).$$

- L. Carlitz 1960,

$$[f(i,j)]_{n\times n} = C \text{ diag}\Big(g(1),\ g(2),\ldots,g(n)\Big)\ C^T,$$

where $C = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & \text{if} \quad j \mid i \\ 0, & \text{if} \quad j \nmid i \end{cases},$$

▲ Examples

1. $g(n) = n$ then $f(n) = \sigma(n)$ and

$$[\sigma(i,j)]_{n\times n} = C\ \mathrm{diag}\Big(1,\ 2, \ldots, n\Big)\ C^T,$$

$$\det[\sigma(i,j)]_{n\times n} = n!.$$

2. $g(n) = q(n) = \mu^2(n)$ then

$$f(n) = \sum_{d|n} \mu^2(d) = 2^{\omega(n)} = \tau^*(n),$$

$$[\tau^*(i,j)]_{n\times n} = C\ \mathrm{diag}\Big(q(1),\ q(2), \ldots, q(n)\Big)\ C^T,$$

$$\det[\tau^*(i,j)]_{n\times n} = q(1)q(2)\cdots q(n) = \begin{cases} 1, & n \le 3 \\ 0, & n > 3 \end{cases}.$$

3. If $g(n) = \beta(n) = \sum_{i=1}^{n} (i,n)$ the Pillai function then

$$f(n) = \sum_{d|n} \beta(n) = n\tau(n),$$

$$[(i,j)\tau(i,j)]_{n\times n} = C \text{ diag}\Big(\beta(1),\ \beta(2), \ldots, \beta(n)\Big)\ C^T,$$

$$\det[(i,j)\tau(i,j)]_{n\times n} = \beta(1)\beta(2)\cdots\beta(n).$$

4. $g(n) = \dfrac{\varphi(n)}{n}$ then $f(n) = \sum_{d|n} \frac{\varphi(d)}{d} = \frac{\beta(n)}{n}$,

$$\left[\frac{\beta(i,j)}{(i,j)}\right]_{n\times n} = C \text{ diag}\left(\frac{\varphi(1)}{1},\ \frac{\varphi(2)}{2}, \ldots, \frac{\varphi(n)}{n}\right)\ C^T,$$

$$\det\left[\frac{\beta(i,j)}{(i,j)}\right]_{n\times n} = \frac{\varphi(1)\varphi(2)\cdots\varphi(n)}{n!}.$$

Definitions and problems

⇝ The GCD matrix associated with f

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)) & f((n,2)) & \cdots & f((n,n)) \end{bmatrix}$$

⇝ The LCM matrix associated with f

$$[f\Big([i,j]\Big)]_{n\times n} = \begin{bmatrix} f([1,1]) & f([1,2]) & \cdots & f([1,n]) \\ f([2,1]) & f([2,2]) & \cdots & f([2,n]) \\ \cdots & \cdots & \cdots & \cdots \\ f([n,1]) & f([n,2]) & \cdots & f([n,n]) \end{bmatrix}$$

⇝ The generalized GCD matrix associated with f

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f(1,(1,1)) & f(1,(1,2)) & \cdots & f(1,(1,n)) \\ f(2,(2,1)) & f(2(2,2)) & \cdots & f(2,(2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f(n,(n,1)) & f(n,(n,2)) & \cdots & f(n,(n,n)) \end{bmatrix}$$

$\rightsquigarrow$ The **Hadamard product** $C = A \circ B = [c_{ij}]_{n\times n}$ of two matrices $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ is simply their elementwise product,

$$c_{ij} = a_{ij}b_{ij}, \quad i, j \in \{1, 2, \ldots, b\}.$$

- Hadamard product for GCD matrices

A. Ocal, 2003; A. Nalli, 2006

Examples

$$\det\Big[[\tau(i,j)]_{n\times n} \circ [(i,j)]_{n\times n}\Big]_{n\times n} = \beta(1)\beta(2)\cdots\beta(n),$$

$$\det\left[[\beta(i,j)]_{n\times n} \circ \left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{\varphi(1)\varphi(2)\cdots\varphi(n)}{n!}.$$

⇝ The modified GCD matrix

$$[(i+1, j+1)]_{n\times n} = \begin{bmatrix} (2,2) & (2,3) & \cdots & (2,n+1) \\ (3,2) & (3,3) & \cdots & (3,n+1) \\ \cdots & \cdots & \cdots & \cdots \\ (n+1,2) & (n+1,3) & \cdots & (n+1,n+1) \end{bmatrix}$$

⇝ The modified GCD matrix associated with f

$$[f(i+1, j+1)]_{n\times n} = \begin{bmatrix} f((2,2)) & f((2,3)) & \cdots & f((2,n+1)) \\ f((3,2)) & f((3,3)) & \cdots & f((3,n+1)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n+1,2)) & f((n+1,3)) & \cdots & f((n+1,n+1)) \end{bmatrix}$$

GCUD matrices

⇝ unitary divisor

$$d \parallel n \Longleftrightarrow d \mid n, \ \left(d, \frac{n}{d}\right) = 1.$$

⇝ greatest common unitary divisor (**gcud**)

$$(i, j)^{**} = d$$

⇝ the **GCUD matrix** associated with f

$$[f(i,j)^{**}]_{n\times n} = \begin{bmatrix} f((1,1)^{**}) & f((1,2)^{**}) & \cdots & f((1,n)^{**}) \\ f((2,1)^{**}) & f((2,2)^{**}) & \cdots & f((2,n)^{**}) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)^{**}) & f((n,2)^{**}) & \cdots & f((n,n)^{**}) \end{bmatrix}$$

⇝ the **modified GCUD matrix** associated with f

$$[f(i+1,j+1)^{**}]_{n\times n} = \begin{bmatrix} f((2,2)^{**}) & \cdots & f((2,n+1)^{**}) \\ f((3,2)^{**}) & \cdots & f((3,n+1)^{**}) \\ \cdots & \cdots & \cdots \\ f((n+1,2)^{**}) & \cdots & f((n+1,n+1)^{**}) \end{bmatrix}$$

$\rightsquigarrow$ **Pr. 1.** Let f and g be two arithmetical functions. Determine the conditions for which we can calculate the Hadamard product and the determinant of Hadamard product of $[f(i,j)]_{n\times n}$ and $[g(i,j)]_{n\times n}$.

$\rightsquigarrow$ **Pr. 2.** If we know $\det[f(i,j)]_{n\times n}$ and $\det[g(i,j)]_{n\times n}$, is it possible to calculate the determinant of Hadamard product matrix?

$\rightsquigarrow$ **Pr. 3.** If we known

$$\det\left[[f(i,j)]_{n\times n}\circ[g(i,j)]_{n\times n}\right]_{n\times n}$$

is it possible to determine, under some conditions, the value of $\det[f(i,j)]_{n\times n}$?

$\rightsquigarrow$ **Pr. 4.** Let f and g be two arithmetical functions. Determine the conditions for which we can calculate the Hadamard product and the determinant od Hadamard product of $[f(i,j)^{**}]_{n\times n}$ and $[g(i,j)^{**}]_{n\times n}$.

⇝ **Pr. 5.** Determine the structure and the determinant of generalized GCD matrices.

⇝ **Pr. 6.** Let f be an arithmetical function. Determine the structure and the determinant of modified GCD matrices.

⇝ **Pr. 7.** Let f be an arithmetical function. Determine the structure and the determinant of modified GCUD matrices.

⇝ **Pr. 8.** Determine the inverse of Möbius matrices.

⇝ **Pr. 9.** Determine the number of 1-s in Möbius matrices.

Theorem 1. [A.B.] Let h and g be two arith. func., g totally multiplicative. If $f(n) = \sum_{d|n} h(d) g\left(\frac{n}{d}\right)$, then

1.

$$\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = C \text{ diag}\left(\frac{h(1)}{g(1)}, \frac{h(2)}{g(2)}, \dots, \frac{h(n)}{g(n)}\right) C^T,$$

where $C = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & \text{ha} \quad j \mid i \\ 0, & \text{ha} \quad j \not| i \end{cases},$$

2.

$$\det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{h(1)}{g(1)} \frac{h(2)}{g(2)} \cdots \frac{h(n)}{g(n)},$$

3. Exists $H(n)$ and $G(n)$ arithmetical functions such that

$$\det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{\det[H(i,j)]}{\det[G(i,j)]}.$$

Examples

1. If $g(n) = n$ then $f(n) = \sum_{d|n} h(d)\frac{n}{d}$.

$$\left[\frac{f(i,j)}{(i,j)}\right]_{n\times n} = C \text{ diag}\left(\frac{h(1)}{1}, \frac{h(2)}{2}, \ldots, \frac{h(n)}{n}\right) C^T$$

$$\det\left[\frac{f(i,j)}{(i,j)}\right]_{n\times n} = \det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{h(1)h(2)\cdots h(n)}{n!}$$

2. If $g(n) = \frac{1}{n}$ then $f(n) = \sum_{d|n} h(d)\frac{d}{n}$.

$$[f(i,j)(i,j)]_{n\times n} = C \text{ diag}\left(h(1)1, h(2)2, \ldots, h(n)n\right) C^T$$

$$\det[f(i,j)(i,j)]_{n\times n} = \det\left[[f(i,j)]_{n\times n} \circ [(i,j)]_{n\times n}\right]_{n\times n} = h(1)\cdots h(n)n!$$

Remark. If we want to apply this theorem to given f and g, we have

$$h(n) = \sum_{d|n} \mu(d) g(d) f\left(\frac{n}{d}\right),$$

$$\left[\frac{f(i,j)}{g(i,j)}\right]_{n\times n} = C \text{ diag}\left(\frac{f(1)}{g(1)}, \ldots, \frac{\sum_{d|n} \mu(d) g(d) f\left(\frac{n}{d}\right)}{g(n)}\right) C^T,$$

Examples: 1. If f power free arithmetical function $(f(p^\alpha) = f(p))$

$$\det\left[f(i,j)(i,j)\right]_{n\times n} = \prod_{k=1}^{n} \varphi(k) f(k)$$

$$\det\left[\gamma((i,j))(i,j)\right]_{n\times n} = \prod_{k=1}^{n} \varphi(k) \gamma(k)$$

2. For a power GCD matrix and determinant we have

$$[(i,j)^s]_{n\times n} = C \text{ diag}\Big(J_s(1),\ J_s(2), \ldots, J_s(n)\Big)\ C^T,$$

Generalized GCD matrices

Theorem 2. [A.B. 2008]
For a given arithmetical function g let

$$f(n,m) = \sum_{d|n} g(d) - \sum_{d|(n,m)} g(d).$$

Then

$$[f(i,j)]_{n\times n} = C \text{ diag}[g(1), g(2), \ldots, g(n)]D^T,$$

where $C = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & \text{if} \quad j \mid i \\ 0, & \text{if} \quad j \nmid i \end{cases},$$

$D = [d_{ij}]_{n\times n}$,

$$d_{ij} = \begin{cases} 1, & \text{if} \quad j \nmid i \\ 0, & \text{if} \quad j \mid i \end{cases}.$$

Examples

1. $g(n) = \varphi(n)$

$$f(n,m) = \sum_{d|n} \varphi(d) - \sum_{d|(n,m)} \varphi(d) = n - (n,m).$$

$$[i - (i,j)]_{n\times n} = C \ \mathrm{diag}\Big(\varphi(1),\ \varphi(2), \ldots, \varphi(n)\Big)\ D^T.$$

2. $g(n) = 1.$

$$f(n,m) = \tau(n) - \tau(n,m)$$

$$[\tau(i) - \tau(i,j)]_{n\times n} = C \ \mathrm{diag}\Big(1,\ 1, \ldots, 1\Big)\ D^T.$$

3. $g(n) = \mu(n).$

$$f(n,m) = \sum_{d|n} \mu(d) - \sum_{d|(n,m)} \mu(d) = \begin{cases} 0, & n = 1 \\ 0, & n > 1, m > 1, (n,m) > 1 \\ -1, & \text{otherwise} \end{cases}.$$

$$[f(i,j)]_{n\times n} = C \ \mathrm{diag}\Big(\mu(1),\ \mu(2), \ldots, \mu(n)\Big)\ D^T.$$

Remarks

1. If

$$f(n,m) = h(n) - h\Big((n,m)\Big)$$

then

$$[f(i,j)]_{n\times n} = C \text{ diag}[(\mu * h)(1), (\mu * h)(2), \ldots, (\mu * h)(n)]D^{T}.$$

2. The structure of the generalized GCD matrix still cannot be exactly determined.

$$f(n,m) = g\Big(n, (n,m)\Big)$$

- **Modified GCD matrices**

Theorem 3. [A.B. 2010]
For a given arithmetical function g let

$$f(n) = \sum_{d|n} g(d).$$

Then

$$[f(i+1, j+1)]_{n\times n} = M \text{ diag}[g(1), g(2), \ldots, g(n)] M^T + g(n+1) U,$$

where $M = [m_{ij}]_{n\times n}$,

$$m_{ij} = \begin{cases} 1, & \text{if} \quad j \mid i+1 \\ 0, & \text{if} \quad j \nmid i+1 \end{cases},$$

$U = [u_{ij}]_{n\times n}$,

$$u_{ij} = \begin{cases} 1, & \text{if} \quad j = i = n \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 4. [A.B. 2010]
Let $g(n)$ be an arithmetical function ($g(n) \neq 0$), and

$$f(n) = \sum_{d|n} g(d).$$

Then

$$\det[f(i+1, j+1)]_{n\times n} = g(1)g(2)\cdots g(n+1) \sum_{k=1}^{n+1} \frac{\mu^2(k)}{g(k)}.$$

▲ **Examples**

1. $g(n) = \varphi(n)$ then $f(n) = n$ and

$$[(i+1, j+1)]_{n\times n} = M \text{ diag}[\varphi(1), \varphi(2), \ldots, \varphi(n)]M^T + \varphi(n+1)U,$$

$$\det[(i+1,j+1)]_{n\times n} = \varphi(1)\cdots\varphi(n+1) \sum_{k=1}^{\infty} \sum_{p_1\cdots p_k \le n+1} \frac{1}{(p_1-1)\cdots(p_k-1)}.$$

2. $g(n) = 1$ then $f(n) = \tau(n)$ and

$$[\tau(i+1, j+1)]_{n\times n} = M \text{ diag}[1, 1, \ldots, 1]M^T + U,$$

$$\det[\tau(i+1, j+1)]_{n\times n} = \sum_{k=1}^{n+1} \mu^2(n) = Q_2(n+1).$$

3. If $g(n) = \beta(n) = \sum_{i=1}^{n} (i, n)$, the Pillai function, then

$$f(n) = \sum_{d|n} \beta(n) = n\tau(n),$$

$$[(i+1, j+1)\tau(i+1, j+1)]_{n\times n} = M \text{ diag}\Big(\beta(1),\ \beta(2), \ldots, \beta(n)\Big)\ M^T + \beta(n+1)U,$$

$$\det[(i+1,j+1)\tau(i+1,j+1)]_{n\times n} = \beta(1)\beta(2)\cdots\beta(n+1)\sum_{k=1}^{n+1}\frac{\mu^2(k)}{\beta(k)} =$$

$$= \sum_{k=1}^{\infty}\sum_{p_1\cdots p_k\le n+1}\frac{1}{(2p_1-1)\cdots(2p_k-1)}.$$

4. $g(n) = n$ then $f(n) = \sigma(n)$ and

$$[\sigma(i+1,j+1)]_{n\times n} = M\ \mathrm{diag}\Big(1,\ 2,\ldots,n\Big)\ M^T + (n+1)U,$$

$$\det[\sigma(i+1,j+1)]_{n\times n} = (n+1)!\sum_{k=1}^{n+1}\frac{\mu^2(k)}{k}..$$

- **Modified GCUD matrices**

Theorem 5. [A.B. 2010]
For a given arithmetical function g let

$$f(n) = \sum_{d \| n} g(d).$$

Then

$$[f(i+1, j+1)]_{n\times n} = M^* \text{ diag}[g(1), g(2), \ldots, g(n)] {M^*}^T + g(n+1)U,$$

where $M^* = [m^*_{ij}]_{n\times n}$,

$$m^*_{ij} = \begin{cases} 1, & \text{if} \quad j \| i+1 \\ 0, & \text{if} \quad j \not\| i+1 \end{cases},$$

$U = [u_{ij}]_{n\times n}$,

$$u_{ij} = \begin{cases} 1, & \text{if} \quad j = i = n \\ 0, & \quad \text{otherwise} \end{cases}.$$

- **Möbius matrices**

⇝ Classical Möbius matrices and inverses

$C = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & \text{if} \quad j \mid i \\ 0, & \text{if} \quad j \not| \, i \end{cases},$$

$C^{-1} = [\mu(i,j)]_{n\times n}$,

$$\mu(i,j) = \begin{cases} (-1)^k, & \text{if} \quad \frac{j}{i} = p_1 p_2 \ldots p_k \\ 1, & \text{if} \quad i = j \\ 0, & \text{otherwise} \end{cases},$$

$\rightsquigarrow$ Unitary Möbius matrices and inverses

$C^* = [c^*_{ij}]_{n \times n}$,

$$c^*_{ij} = \begin{cases} 1, & \text{if} \quad j \parallel i \\ 0, & \text{if} \quad j \not\parallel i \end{cases},$$

$C^{*-1} = [\mu^*(i,j)]_{n \times n}$,

$$\mu^*(i,j) = \begin{cases} (-1)^k, & \text{if} \quad i \parallel j,\ \frac{j}{i} = p_1^{\alpha_i} p_2^{\alpha_2} \dots p_k^{\alpha_k} \\ 1, & \text{if} \quad i = j \\ 0, & \text{otherwise} \end{cases},$$

$\rightsquigarrow$ Modified Möbius matrices (2007)

$M = [m_{ij}]_{n\times n}$,

$$m_{ij} = \begin{cases} 1, & \text{if} \quad j \mid i+1 \\ 0, & \text{if} \quad j \nmid i+1 \end{cases},$$

$$\det M = \mu(n+1)$$

Theorem 5. [A. B. 2009]
If $n = p - 1$ the inverse of M_n is $M_n^{-1} = [m'(i,j)]_{n\times n}$, where

$$m'(i,j) = \begin{cases} \mu(i+1,j), & \text{ha} \quad i \le n \\ 1, & \text{ha} \quad j = i+1 \\ -\mu(j), & \text{ha} \quad i = n \end{cases},$$

Problem. Determine the inverse of the Möbius matrix M_n if $n+1$ is the product of prime numbers, $n+1 = p_1 p_2 \cdots p_k$, $k \ge 2$.

$\rightsquigarrow$ Modified unitary Möbius matrices (A. B., 2009)

$M^* = [m^*_{ij}]_{n\times n}$,

$$m^*_{ij} = \begin{cases} 1, & \text{if} \quad j \parallel i+1 \\ 0, & \text{if} \quad j \not\parallel i+1 \end{cases},$$

$$\det M^* = \mu^*(n+1),$$

where

$$\mu^*(n) = \begin{cases} 1, & \text{if} \quad n = 1 \\ (-1)^k, & \text{if} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \end{cases}.$$

The number of 1-s and -1-s in Möbius matrices

Theorem 6. [A. B. 2009]
The number of 1-s in C_n is

$$c_n = \sum_{k=1}^{n} \tau(k),$$

and the number of 1-s in C_n^{-1} is

$$d_n = 1 + \frac{1}{2} \sum_{k=2}^{n} \tau^*(k).$$

Remark.

$$\lim_{n \to \infty} \frac{c_n}{d_n} = \frac{\pi^2}{3}.$$

Theorem 7. [A. B. 2009]
The number of 1-s in C_n^* is

$$c_n = \sum_{k=1}^{n} \tau^*(k),$$

and the number of 1-s in C_n^{*-1} is

$$d_n = 1 + \frac{1}{2} \sum_{k=2}^{n} \tau^*(k).$$

Final observations

▲ Other type matrices:

• LCU and LCUM matrices

• Mixed modified GCD and LCM matrices

$$\left[\frac{(i+1,j+1)^s}{[i+1,j+1]^r}\right]_{n\times n}$$

• The Hadamard product of modified GCD matrices

▲ Generalizations and connections:

• Modified GCD matrices on factor closed sets and GCD closed sets

• GCD matrices on posets (meet semilatices)

• Other connections to combinatorics (for example discrepancy of matrices)

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