



# Generalized GCD matrices

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**Abstract.** Let  $f$  be an arithmetical function. The matrix  $[f(i, j)]_{n \times n}$  given by the value of  $f$  in greatest common divisor of  $(i, j)$ ,  $f((i, j))$  as its  $i, j$  entry is called the greatest common divisor (GCD) matrix. We consider the generalization of this matrix where the elements are in the form  $f(i, (i, j))$ .

## 1 Introduction

The classical Smith determinant was introduced in 1875 by H. J. Smith [14] who also proved that

$$\det[(i, j)]_{n \times n} = \begin{vmatrix} (1, 1) & (1, 2) & \cdots & (1, n) \\ (2, 1) & (2, 2) & \cdots & (2, n) \\ \cdots & \cdots & \cdots & \cdots \\ (n, 1) & (n, 2) & \cdots & (n, n) \end{vmatrix} = \varphi(1) \cdot \varphi(2) \cdots \varphi(n), \quad (1)$$

where  $(i, j)$  represents the greatest common divisor of  $i$  and  $j$ , and  $\varphi(n)$  being the Euler's totient function.

The GCD matrix with respect to  $f$  is

$$[f(i, j)]_{n \times n} = \begin{bmatrix} f((1, 1)) & f((1, 2)) & \cdots & f((1, n)) \\ f((2, 1)) & f((2, 2)) & \cdots & f((2, n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n, 1)) & f((n, 2)) & \cdots & f((n, n)) \end{bmatrix}$$

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If we consider the GCD matrix  $[f(i, j)]_{n \times n}$  where

$$f(n) = \sum_{d|n} g(d),$$

H. J. Smith proved that

$$\det[f(i, j)]_{n \times n} = g(1) \cdot g(2) \cdots g(n).$$

For  $g = \varphi$

$$f(i, j) = \sum_{d|(i, j)} \varphi(d) = (i, j).$$

this formula reduces to (1). Many generalizations of Smith determinants have been presented in literature, see [7, 1, 10, 13, 5].

If we consider the GCD matrix  $[f(i, j)]_{n \times n}$  where  $f(n) = \sum_{d|n} g(d)$  Pólya and Szegő [12] proved that

$$[f(i, j)]_{n \times n} = G \cdot C^T, \tag{2}$$

where  $G$  and  $A$  are lower triangular matrices given by

$$g_{ij} = \begin{cases} g(j), & j | i \\ 0, & \text{otherwise} \end{cases}$$

and

$$c_{ij} = \begin{cases} 1, & j | i \\ 0, & \text{otherwise} \end{cases}$$

L. Carlitz [4] in 1960 gave a new form of (2)

$$[f(i, j)]_{n \times n} = C_n \text{diag}(g(1), g(2), \dots, g(n)) C_n^T, \tag{3}$$

where  $C_n = [c_{ij}]_{n \times n}$ ,

$$c_{ij} = \begin{cases} 1, & \text{if } j | i \\ 0, & \text{if } j \nmid i \end{cases},$$

$D = [d_{ij}]_{n \times n}$  diagonal matrix

$$d_{ij} = \begin{cases} g(i), & \text{ha } i = j \\ 0, & \text{ha } i \neq j \end{cases}.$$

From (3) it follows that the value of the determinant is

$$\det[f(i, j)]_{n \times n} = g(1)g(2) \cdots g(n). \tag{4}$$

There are a quite a few generalized forms of GCD matrices, which can be found in several references [2, 11, 3, 8, 9, 6].

In this paper we study matrices which have as variables the greatest common divisor and the indices:

$$[f(i, j)]_{n \times n} = \begin{bmatrix} f(1, (1, 1)) & f(1, (1, 2)) & \cdots & f(1, (1, n)) \\ f(2, (2, 1)) & f(2, (2, 2)) & \cdots & f(2, (2, n)) \\ \cdots & \cdots & \cdots & \cdots \\ f(n, (n, 1)) & f(n, (n, 2)) & \cdots & f(n, (n, n)) \end{bmatrix}.$$

## 2 Generalized GCD matrices

**Theorem 1** For a given arithmetical function  $g$  let

$$f(n, m) = \sum_{d|n} g(d) - \sum_{d|(n, m)} g(d).$$

Then

$$[f(i, j)]_{n \times n} = C_n \text{diag}[g(1), g(2), \dots, g(n)] D_n^T,$$

where  $C_n = [c_{ij}]_{n \times n}$ ,

$$c_{ij} = \begin{cases} 1, & \text{if } j | i \\ 0, & \text{if } j \nmid i \end{cases},$$

$D_n = [d_{ij}]_{n \times n}$ ,

$$d_{ij} = \begin{cases} 1, & \text{if } j \nmid i \\ 0, & \text{if } j | i \end{cases}.$$

**Proof.** After multiplication, the general element of

$$A = [a_{ij}]_{n \times n} = C \text{diag}[g(1), g(2), \dots, g(n)] D^T$$

is

$$a_{ij} = \sum_{\substack{k | i \\ k \nmid j}} g(k) = \sum_{d|n} g(d) - \sum_{d|(n, m)} g(d) = f(i, j).$$

□

### Particular cases

1. If  $g(n) = \varphi(n)$  then

$$f(n, m) = \sum_{d|n} \varphi(d) - \sum_{d|(n, m)} \varphi(d) = n - (n, m).$$

We have the following decomposition:

$$[i - (i, j)]_{n \times n} = \begin{bmatrix} 1 - (1, 1) & 1 - (1, 2) & \cdots & 1 - (1, n) \\ 2 - (2, 1) & 2 - (2, 2) & \cdots & 2 - (2, n) \\ \cdots & \cdots & \cdots & \cdots \\ n - (n, 1) & n - (n, 2) & \cdots & n - (n, n) \end{bmatrix}$$

2. If  $g(n) = 1$  then

$$f(n, m) = \tau(n) - \tau(n, m)$$

and

$$[\tau(i) - \tau(i, j)]_{n \times n} = C_n \text{diag}(1, 1, \dots, 1) D_n^T.$$

3. Let  $g(n) = \mu(n)$ . From

$$f(n, m) = \sum_{d|n} \mu(d) - \sum_{d|(n, m)} \mu(d) = \begin{cases} 0, & n = 1 \\ 0, & n > 1, m > 1, (n, m) > 1 \\ -1, & \text{otherwise} \end{cases} .$$

we have

$$[f(i, j)]_{n \times n} = C_n \text{diag}(\mu(1), \mu(2), \dots, \mu(n)) D_n^T.$$

4. For  $g(n) = n$ ,  $f(n, m) = \sigma(n) - \sigma((n, m))$  and

$$[f(i, j)]_{n \times n} = C_n \text{diag}(1, 2, \dots, n) D_n^T.$$

**Remarks**

1. Due to the fact that the first line of the matrix  $[f(i, j)]_{n \times n}$  contains only 0-s, the determinant of the matrix will always be 0.

2. We can determine the value of the matrix associated with  $f$ , if the function  $f$  is of the form

$$f(n, m) = h(n) - h((n, m)).$$

By using the Möbius inversion formula, we get

$$g(n) = \sum_{d|n} \mu(d)h\left(\frac{n}{d}\right),$$

consequently by using Theorem 1, the matrix can be decomposed according to the function  $h(n)$ :

$$[f(i, j)]_{n \times n} = C_n \text{diag}[(\mu * h)(1), (\mu * h)(2), \dots, (\mu * h)(n)] D_n^T.$$

**Theorem 2** For a given arithmetical function  $g$  let

$$f(i, j) = \sum_{k=1}^n g(k) - \sum_{d|i} g(d) - \sum_{d|j} g(d) + \sum_{d|(i,j)} g(d).$$

Then

$$[f(i, j)]_{n \times n} = D_n \text{diag}[g(1), g(2), \dots, g(n)] D_n^T,$$

where  $D_n = [d_{ij}]_{n \times n}$ ,

$$d_{ij} = \begin{cases} 1, & \text{if } j \nmid i \\ 0, & \text{if } j | i \end{cases}.$$

**Proof.** After multiplication, the general element of the matrix

$$A = [a_{ij}]_{n \times n} = D_n \text{diag}[g(1), g(2), \dots, g(n)] D_n^T$$

is

$$\begin{aligned} a_{ij} &= \sum_{\substack{k \nmid n \\ k \nmid m}} g(k) = \sum_{k=1}^n g(k) - \sum_{k|n \text{ OR } k|m} g(k) = \\ &= \sum_{k=1}^n g(k) - \sum_{k|n} g(k) - \sum_{k|m} g(k) + \sum_{k|(n,m)} g(k) = f(i, j). \end{aligned}$$

□

**Particular cases**

1. If  $g(n) = \varphi(n)$  then

$$f(i, j) = \sum_{k=1}^n \varphi(k) - i - j + (i, j),$$

$$[f(i, j)]_{n \times n} = D_n \text{diag}[\varphi(1), \varphi(2), \dots, \varphi(n)] D_n^T.$$

2. If  $g(n) = 1$  then

$$f(i, j) = n - \tau(i) - \tau(j) + \tau(i, j)$$

and

$$[f(i, j)]_{n \times n} = D_n \text{diag}(1, 1, \dots, 1) D_n^T.$$

3.  $g(n) = n$ . Then

$$f(i, j) = \frac{n(n+1)}{2} - \sigma(n) - \sigma(m) + \sigma((n, m))$$

and

$$[f(i, j)]_{n \times n} = D_n \text{diag}(1, 2, \dots, n) D_n^T.$$

Another generalization is the following:

**Theorem 3** For a given arithmetical function  $g$  let

$$f(i, j) = \sum_{k=1}^n g(k) - \sum_{d|i} g(d) - \sum_{d|j} g(d) + \sum_{d|(i,j)} g(d).$$

We define the following  $A = [a_{ij}]_{n \times n}$  matrix

$$a_{ij} = \begin{cases} f(i, j), & \text{if } i, j > 1 \\ g(1) + f(i, j), & \text{if } i = 1 \text{ or } j = 1 \end{cases}.$$

Then

$$A = D'_n \text{diag}[g(1), g(2), \dots, g(n)] D_n'^T,$$

where  $D'_n = [d'_{ij}]_{n \times n}$ ,

$$d'_{ij} = \begin{cases} 1, & \text{if } i = j = 1 \\ d_{ij}, & \text{if } ij \neq 1 \end{cases}.$$

**Proof.**

We calculate the general element of the matrix

$$B = [a_{ij}]_{n \times n} = D'_n \text{diag}[g(1), g(2), \dots, g(n)] D_n'^T.$$

If  $i > 1$  or  $j > 1$  we have

$$\begin{aligned} b_{ij} &= \sum_{\substack{k \nmid n \\ k \nmid m}} g(k) = \sum_{k=1}^n g(k) - \sum_{k|n \text{ or } k|m} g(k) = \\ &= \sum_{k=1}^n g(k) - \sum_{k|n} g(k) - \sum_{k|m} g(k) + \sum_{k|(n,m)} g(k) = a_{ij}. \end{aligned}$$

If  $i = j = 1$

$$b_{11} = g(1) = a_{11}.$$

□

### Particular cases

1. If  $g(n) = \varphi(n)$  then

$$a_{ij} = \begin{cases} \sum_{k=1}^n \varphi(k) - i - j + (i, j), & \text{if } i, j > 1 \\ \sum_{k=1}^n \varphi(k) - i - j + (i, j) + 1, & \text{if } i = 1 \text{ or } j = 1 \end{cases}$$

2. If  $g(n) = 1$  then

$$a_{ij} = \begin{cases} n - \tau(i) - \tau(j) + \tau(i, j), & \text{if } i, j > 1 \\ n - \tau(i) - \tau(j) + \tau(i, j) + 1, & \text{if } i = 1 \text{ or } j = 1 \end{cases}$$

The following problems remain open:

**Problem 1** Let  $F(n, m)$  be an arithmetical function with two variables. Determine the structure and the determinant of modified GCD matrices  $A = [a(i, j)]_{n \times n}$

$$a_{ij} = F(i, (i, j)).$$

**Problem 2** Determine the structure and the determinant of modified GCD matrices  $A = [a(i, j)]_{n \times n}$ , where

$$a_{ij} = F(n, i, j, (i, j)).$$

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