



Imbalances in directed multigraphs

S. Pirzada

Department of Mathematics,
University of Kashmir, Srinagar, India
email: sdpirzada@yahoo.co.in

T. A. Naikoo

Department of Mathematics,
University of Kashmir, Srinagar, India
email: tariqnaikoo@rediffmail.com

U. Samee

Department of Mathematics,
University of Kashmir, Srinagar, India
email: pzsamee@yahoo.co.in

A. Iványi

Department of Computer Algebra,
Eötvös Loránd University, Hungary
email: tony@compalg.inf.elte.hu

Abstract. In a directed multigraph, the imbalance of a vertex v_i is defined as $b_{v_i} = d_{v_i}^+ - d_{v_i}^-$, where $d_{v_i}^+$ and $d_{v_i}^-$ denote the outdegree and indegree respectively of v_i . We characterize imbalances in directed multigraphs and obtain lower and upper bounds on imbalances in such digraphs. Also, we show the existence of a directed multigraph with a given imbalance set.

1 Introduction

A directed graph (shortly digraph) without loops and without multi-arcs is called a simple digraph [2]. The imbalance of a vertex v_i in a digraph as b_{v_i} (or simply b_i) = $d_{v_i}^+ - d_{v_i}^-$, where $d_{v_i}^+$ and $d_{v_i}^-$ are respectively the outdegree and indegree of v_i . The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \dots, f_n]$ with $f_1 \geq f_2 \geq \dots \geq f_n$ is feasible if the sum of its elements is zero, and satisfies $\sum_{i=1}^k f_i \leq k(n - k)$, for $1 \leq k < n$.

2010 Mathematics Subject Classification: 05C20

Key words and phrases: Digraph, imbalance, outdegree, indegree, directed multigraph, arc.

The following result [3] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1 *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \dots, b_n]$ with $b_1 \geq b_2 \geq \dots \geq b_n$ is an imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^k b_i \leq k(n-k),$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Corollary 1 *A sequence of integers $B = [b_1, b_2, \dots, b_n]$ with $b_1 \leq b_2 \leq \dots \leq b_n$ is an imbalance sequence of a simple digraph if and only if*

$$\sum_{i=1}^k b_i \geq k(k-n),$$

for $1 \leq k < n$ with equality when $k = n$.

Various results for imbalances in simple digraphs and oriented graphs can be found in [4], [5].

2 Imbalances in r-graphs

A multigraph is a graph from which multi-edges are not removed, and which has no loops [2]. If $r \geq 1$ then an r -digraph (shortly r -graph) is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. Clearly 1-digraph is an oriented graph. Let D be an f -digraph with vertex set $V = \{v_1, v_2, \dots, v_n\}$, and let d_v^+ and d_v^- respectively denote the outdegree and indegree of vertex v . Define b_{v_i} (or simply b_i) = $d_{v_i}^+ - d_{v_i}^-$ as imbalance of v_i . Clearly, $-r(n-1) \leq b_{v_i} \leq r(n-1)$. The imbalance sequence of D is formed by listing the vertex imbalances in non-decreasing order.

Let u and v be distinct vertices in D . If there are f arcs directed from u to v and g arcs directed from v to u , we denote this by $u(f-g)v$, where $0 \leq f, g, f+g \leq r$.

A double in D is an induced directed subgraph with two vertices u , and v having the form $u(f_1f_2)v$, where $1 \leq f_1, f_2 \leq r$, and $1 \leq f_1 + f_2 \leq r$, and f_1 is the number of arcs directed from u to v , and f_2 is the number of arcs directed from v to u . A triple in D is an induced subgraph with three vertices u , v , and w having the form $u(f_1f_2)v(g_1g_2)w(h_1h_2)u$, where $1 \leq f_1, f_2, g_1, g_2, h_1, h_2 \leq r$, and $1 \leq f_1 + f_2, g_1 + g_2, h_1 + h_2 \leq r$, and the meaning of $f_1, f_2, g_1, g_2, h_1, h_2$ is similar to the meaning in the definition of doubles. In D an oriented triple is an induced subdigraph with three vertices. An oriented triple is said to be transitive if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. An r -graph is said to be transitive if all its oriented triples are transitive. In particular, a triple C in an r -graph is transitive if every oriented triple of C is transitive.

The following observation can be easily established and is analogous to Theorem 2.2 of Avery [1].

Lemma 1 *If D_1 and D_2 are two r -graphs with same imbalance sequence, then D_1 can be transformed to D_2 by successively transforming (i) appropriate oriented triples in one of the following ways, either (a) by changing the intransitive oriented triple $u(1-0)v(1-0)w(1-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$, which has the same imbalance sequence or vice versa, or (b) by changing the intransitive oriented triple $u(1-0)v(1-0)w(0-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$, which has the same imbalance sequence or vice versa; or (ii) by changing a double $u(1-1)v$ to a double $u(0-0)v$, which has the same imbalance sequence or vice versa.*

The above observations lead to the following result.

Theorem 2 *Among all r -graphs with given imbalance sequence, those with the fewest arcs are transitive.*

Proof. Let B be an imbalance sequence and let D be a realization of B that is not transitive. Then D contains an intransitive oriented triple. If it is of the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation $i(a)$ of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$ with the same imbalance sequence and three arcs fewer. If D contains an intransitive oriented

triple of the form $u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation $i(b)$ of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$ same imbalance sequence but one arc fewer. In case D contains both types of intransitive oriented triples, they can be transformed to transitive ones with certainly lesser arcs. If in D there is a double $u(1-1)$, by operation (ii) of Lemme 4, it can be transformed to $u(0-0)$, with same imbalance sequence but two arcs fewer. \square

The next result gives necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of some r -graph.

Theorem 3 *A sequence $B = [b_1, b_2, \dots, b_n]$ of integers in non-decreasing order is an imbalance sequence of an r -graph if and only if*

$$\sum_{i=1}^k b_i \geq rk(k-n), \quad (1)$$

with equality when $k = n$.

Proof. Necessity. A multi subdigraph induced by k vertices has a sum of imbalances $rk(k-n)$.

Sufficiency. Assume that $B = [b_1, b_2, \dots, b_n]$ be the sequence of integers in non-decreasing order satisfying conditions (1) but is not the imbalance sequence of any r -graph. Let this sequence be chosen in such a way that n is the smallest possible and b_1 is the least with that choice of n . We consider the following two cases.

Case (i). Suppose equality in (1) holds for some $k \leq n$, so that

$$\sum_{i=1}^k b_i = rk(k-n),$$

for $1 \leq k < n$.

By minimality of n , $B_1 = [b_1, b_2, \dots, b_k]$ is the imbalance sequence of some r -graph D_1 with vertex set, say V_1 . Let $B_2 = [b_{k+1}, b_{k+2}, \dots, b_n]$. Consider,

$$\begin{aligned}
 \sum_{i=1}^f b_{k+i} &= \sum_{i=1}^{k+f} b_i - \sum_{i=1}^k b_i \\
 &\geq r(k+f)[(k+f) - n] - rk(k-n) \\
 &= r(k_2 + kf - kn + fk + f_2 - fn - k_2 + kn) \\
 &\geq r(f_2 - fn) \\
 &= rf(f-n),
 \end{aligned}$$

for $1 \leq f \leq n-k$, with equality when $f = n-k$. Therefore, by the minimality for n , the sequence B_2 forms the imbalance sequence of some r -graph D_2 with vertex set, say V_2 . Construct a new r -graph D with vertex set as follows.

Let $V = V_1 \cup V_2$ with, $V_1 \cap V_2 = \phi$ and the arc set containing those arcs which are in D_1 and D_2 . Then we obtain the r -graph D with the imbalance sequence B , which is a contradiction.

Case (ii). Suppose that the strict inequality holds in (1) for some $k < n$, so that

$$\sum_{i=1}^k b_i > rk(k-n),$$

for $1 \leq k < n$. Let $B_1 = [b_1 - 1, b_2, \dots, b_{n-1}, b_n + 1]$, so that B_1 satisfy the conditions (1). Thus by the minimality of b_1 , the sequences B_1 is the imbalances sequence of some r -graph D_1 with vertex set, say V_1). Let $b_{v_1} = b_1 - 1$ and $b_{v_n} = a_n + 1$. Since $b_{v_n} > b_{v_1} + 1$, there exists a vertex $v_p \in V_1$ such that $v_n(0-0)v_p(1-0)v_1$, or $v_n(1-0)v_p(0-0)v_1$, or $v_n(1-0)v_p(1-0)v_1$, or $v_n(0-0)v_p(0-0)v_1$, and if these are changed to $v_n(0-1)v_p(0-0)v_1$, or $v_n(0-0)v_p(0-1)v_1$, or $v_n(0-0)v_p(0-0)v_1$, or $v_n(0-1)v_p(0-1)v_1$ respectively, the result is an r -graph with imbalances sequence B , which is again a contradiction. This proves the result. \square

Arranging the imbalance sequence in non-increasing order, we have the following observation.

Corollary 2 *A sequence $B = [b_1, b_2, \dots, b_n]$ of integers with $b_1 \geq b_2 \geq \dots \geq b_n$ is an imbalance sequence of an r -graph if and only if*

$$\sum_{i=1}^k b_i \leq rk(n-k),$$

for $1 \leq k \leq n$, with equality when $k = n$.

The converse of an r -graph D is an r -graph D' , obtained by reversing orientations of all arcs of D . If $B = [b_1, b_2, \dots, b_n]$ with $b_1 \leq b_2 \leq \dots \leq b_n$ is the imbalance sequence of an r -graph D , then $B' = [-b_n, -b_{n-1}, \dots, -b_1]$ is the imbalance sequence of D .

The next result gives lower and upper bounds for the imbalance b_i of a vertex v_i in an r -graph D .

Theorem 4 *If $B = [b_1, b_2, \dots, b_n]$ is an imbalance sequence of an r -graph D , then for each i*

$$r(i - n) \leq b_i \leq r(i - 1).$$

Proof. Assume to the contrary that $b_i < r(i - n)$, so that for $k < i$,

$$b_k \leq b_i < r(i - n).$$

That is,

$$b_1 < r(i - n), b_2 < r(i - n), \dots, b_i < r(i - n).$$

Adding these inequalities, we get

$$\sum_{k=1}^i b_k < ri(i - n),$$

which contradicts Theorem 3.

Therefore, $r(i - n) \leq b_i$.

The second inequality is dual to the first. In the converse r -graph with imbalance sequence $B = [b'_1, b'_2, \dots, b'_n]$ we have, by the first inequality

$$\begin{aligned} b'_{n-i+1} &\geq r[(n - i + 1) - n] \\ &= r(-i + 1). \end{aligned}$$

Since $b_i = -b'_{n-i+1}$, therefore

$$b_i \leq -r(-i + 1) = r(i - 1).$$

Hence, $b_i \leq r(i - 1)$. □

Now we obtain the following inequalities for imbalances in r -graphs.

Theorem 5 *If $B = [b_1, b_2, \dots, b_n]$ is an imbalance sequence of an r -graph with $b_1 \geq b_2 \geq \dots \geq b_n$, then*

$$\sum_{i=1}^k b_i^2 \leq \sum_{i=1}^k (2rn - 2rk - b_i)^2,$$

for $1 \leq k \leq n$ with equality when $k = n$.

Proof. By Theorem 3, we have for $1 \leq k \leq n$ with equality when $k = n$

$$\text{rk}(n - k) \geq \sum_{i=1}^k b_i,$$

implying

$$\sum_{i=1}^k b_i^2 + 2(2rn - 2rk)\text{rk}(n - k) \geq \sum_{i=1}^k b_i^2 + 2(2rn - 2rk) \sum_{i=1}^k b_i,$$

from where

$$\sum_{i=1}^k b_i^2 + k(2rn - 2rk)^2 - 2(2rn - 2rk) \sum_{i=1}^k b_i \geq \sum_{i=1}^k b_i^2,$$

and so we get the required

$$\begin{aligned} & b_1^2 + b_2^2 + \dots + b_k^2 + (2rn - 2rk)^2 + (2rn - 2rk)^2 + \dots + (2rn - 2rk)^2 \\ & \quad - 2(2rn - 2rk)b_1 - 2(2rn - 2rk)b_2 - \dots - 2(2rn - 2rk)b_k \\ & \geq \sum_{i=1}^k b_i^2, \end{aligned}$$

or

$$\sum_{i=1}^k (2rn - 2rk - b_i)^2 \geq \sum_{i=1}^k b_i^2.$$

□

The set of distinct imbalances of vertices in an r -graph is called its imbalance set. The following result gives the existence of an r -graph with a given imbalance set. Let $p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n$ denote the greatest common divisor of $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n$.

Theorem 6 *If $P = \{p_1, p_2, \dots, p_m\}$ and $Q = \{-q_1, -q_2, \dots, -q_n\}$ where $p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n$ are positive integers such that $p_1 < p_2 < \dots < p_m$ and $q_1 < q_2 < \dots < q_n$ and $(p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n) = t, 1 \leq t \leq r$, then there exists an r -graph with imbalance set $P \cup Q$.*

Proof. Since $(p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n) = t, 1 \leq t \leq r$, there exist positive integers f_1, f_2, \dots, f_m and g_1, g_2, \dots, g_n with $f_1 < f_2 < \dots < f_m$ and $g_1 < g_2 < \dots < g_n$ such that

$$p_i = tf_i$$

for $1 \leq i \leq m$ and

$$q_i = tg_i$$

for $1 \leq j \leq n$.

We construct an r -graph D with vertex set V as follows.

Let

$$V = X_1^1 \cup X_2^1 \cup \dots \cup X_m^1 \cup X_1^2 \cup X_1^3 \cup \dots \cup X_1^n \cup Y_1^1 \cup Y_2^1 \cup \dots \cup Y_m^1 \cup Y_1^2 \cup Y_1^3 \cup \dots \cup Y_1^n,$$

with $X_i^j \cap X_k^l = \phi$, $Y_i^j \cap Y_k^l = \phi$, $X_i^j \cap Y_k^l = \phi$ and

$$|X_i^1| = g_i, \text{ for all } 1 \leq i \leq m,$$

$$|X_1^i| = g_i, \text{ for all } 2 \leq i \leq n,$$

$$|Y_i^1| = f_i, \text{ for all } 1 \leq i \leq m,$$

$$|Y_1^i| = f_1, \text{ for all } 2 \leq i \leq n.$$

Let there be t arcs directed from every vertex of X_i^1 to each vertex of Y_i^1 , for all $1 \leq i \leq m$ and let there be t arcs directed from every vertex of X_1^i to each vertex of Y_1^i , for all $2 \leq i \leq n$ so that we obtain the r -graph D with imbalances of vertices as under.

For $1 \leq i \leq m$, for all $x_i^1 \in X_i^1$

$$b_{x_i^1} = t|Y_i^1| - 0 = tf_i = p_i,$$

for $2 \leq i \leq n$, for all $x_1^i \in X_1^i$

$$b_{x_1^i} = t|Y_1^i| - 0 = tf_1 = p_1,$$

for $1 \leq i \leq m$, for all $y_i^1 \in Y_i^1$

$$b_{y_i^1} = 0 - t|X_i^1| = -tg_i = -q_i,$$

and for $2 \leq i \leq n$, for all $y_1^i \in Y_1^i$

$$b_{y_1^i} = 0 - t|X_1^i| = -tg_i = -q_i.$$

Therefore imbalance set of D is $P \cup Q$. □

Acknowledgement

The research of the fourth author was supported by the project TÁMOP-4.2.1/B-09/1/KMR-2010-0003 of Eötvös Loránd University.

The authors are indebted for the useful remarks of the unknown referee.

References

- [1] P. Avery, Score sequences of oriented graphs, *J. Graph Theory*, **15** (1991), 251–257.
- [2] J. L. Gross, J. Yellen, *Handbook of graph theory*, CRC Press, London/New York, 2004.
- [3] D. Mubayi, T. G. Will and D. B. West, Realizing degree imbalances in directed graphs, *Discrete Math.*, **239** (2001), 147–153.
- [4] S. Pirzada, T. A. Naikoo and N. A. Shah, Imbalances in oriented tripartite graphs, *Acta Math. Sinica*, (2010) (to appear).
- [5] S. Pirzada, On imbalances in digraphs, *Kragujevac J. Math.*, **31** (2008), 143–146.

Received: June, 2010