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ANOTHER GENERALISATION OF SMITH'S DETERMINANT

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Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = (x_i, x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S. H.J.S. Smith showed that the determinant of the matrix $[E(n)]$, $E(n) = \{1, 2, \ldots, n\}$, is $\phi(1)\phi(2)\ldots\phi(n)$, where $\phi(x)$ is Euler's totient function. We extend Smith's result by considering sets $S = \{x_1, x_2, \ldots x_n\}$ with the property that for all i and j, (x_i, x_j) is in S.

1. INTRODUCTION

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = (x_i, x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S (see [2]). In [6], Smith showed that if $E(n) = \{1, 2, \ldots, n\}$, then the determinant of $[E(n)]$, det $[E(n)]$, is $\phi(1)\phi(2)\ldots\phi(n)$, where $\phi(x)$ is Euler's totient function. Many generalisations of Smith's result in various directions [**1**, **2**, **3**, **4**, **5**] have been published. In fact, Smith commented that $E(n)$ can be replaced by a factor-closed set. A set S of positive integers is said to be factor-closed if whenever x_i is in S and d divides x_i then d is in S. In [**2**], we considered GCD matrices in the direction of their structure, determinant, and arithmetic in $\mathbf{Z}_n$, the ring of integers modulo n. The purpose of this paper is to give a generalisation of Smith's result in the direction of extending the sets $E(n)$ and factor-closed sets to a larger class of sets.

2. MAIN RESULT

Definition 1. A set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers is said to be *gcd-closed* if for every i, $j = 1, 2, \ldots, n$, (x_i, x_j) is in S.

Clearly every factor-closed set, and hence $E(n)$, is gcd-closed, but not conversely. We present in this section a structure theorem for GCD matrices defined on gcd-closed sets and compute their determinant, thus generalising Smith's result.

It was remarked in [**2**] that the determinant of a GCD matrix defined on a set S is independent of the order of the elements in S.

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PROPOSITION 1. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be gcd-closed with $x_1 < x_2 < \ldots < x_n$. For every i, $j = 1, 2, \ldots, n$, let C_{ij} be the sum*

$$\sum_{x_k | (x_i, x_j)} \left(\sum_{\substack{d | x_k \\ d \nmid x_t \\ t < k}} \phi(d) \right).$$

Then $C_{ij} = (x_i, x_j)$.

PROOF: It is true that

$$(x_i, x_j) = \sum_{d | (x_i, x_j)} \phi(d) \tag{1.1}$$

It is obvious that the sums (1.1) and C_{ij} are non-repetitive; that is, each d is counted only once. Now let x_k divide (x_i, x_j) and d divide x_k. Then d divides (x_i, x_j). Thus every d occuring in C_{ij} occurs in (1.1). Conversely, suppose d divides (x_i, x_j). Since S is gcd-closed, $(x_i, x_j) = x_m$ for some m less than or equal to the minimum of i and j. Hence d divides x_m. Let $k \leqslant m$ be the first integer such that d divides x_k. Then d does not divide x_t for $t < k$. Now $(x_k, x_i) = x_r$ for some $r \leqslant k$. Hence d divides x_r. By the minimality of k, it must be that $r = k$. Thus $x_r = x_k$ and x_k divides x_i. Similarly, x_k divides x_j. Therefore x_k divides (x_i, x_j). This completes the proof. $\square$

THEOREM 1. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be gcd-closed with $x_1 < x_2 < \ldots < x_n$. Then $[S]$ is the product of a lower triangular matrix A and an upper triangular matrix B. Moreover, $\det [S] = \det (A) = a_{11} a_{22} \ldots a_{nn}$, where $a_{ii} = \sum_{\substack{d | x_i \\ d \nmid x_t \\ t < i}} \phi(d)$.*

PROOF: Define $A = (a_{ij})$ via

$$a_{ij} = \begin{cases} \sum_{\substack{d | x_j \\ d \nmid x_t \\ t < j}} \phi(d) & \text{if } x_j \mid x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Define B to be the incidence matrix corresponding to A^T, the transpose of A: if the (i, j)-entry of A^T is 0, then the (i, j)-entry of B is 0; otherwise the (i, j)-entry of B is 1. Thus, if $B = (b_{ij})$, then the (i, j)-entry of AB is equal to $\sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{\substack{x_k | x_i \\ x_k | x_j}} a_{ik}$.

But this is precisely the sum C_{ij} as in Proposition 1. Therefore, the (i, j)-entry of AB is (x_i, x_j). It is obvious that A is lower triangular and B is upper triangular and that $\det (B) = 1$. Hence $\det [S] = \det (A) = a_{11} a_{22} \ldots a_{nn}$, and the proof is complete. $\square$

COROLLARY 1. (Smith) *Let* $S = \{x_1, x_2, \ldots, x_n\}$ *be a factor-closed set. Then* $\det [S] = \phi(x_1)\phi(x_2)\ldots\phi(x_n)$.

It was conjectured in [2] that the converse of the above corollary is true. The following is a partial answer to the conjecture.

COROLLARY 2. *Let* $S = \{x_1, x_2, \ldots, x_n\}$ *be gcd-closed. Then* $\det [S] = \phi(x_1)\phi(x_2)\ldots\phi(x_n)$ *if and only if* S *is factor-closed.*

PROOF: Sufficiency is Corollary 1. Now suppose S is not factor-closed. We note that in Theorem 1, $a_{ii} \geqslant \phi(x_i)$. Since S is not factor-closed, there exist i and d such that $d \neq x_i$, d divides x_i, and d does not divide x_t for $t < i$. Hence $a_{ii} \geqslant \phi(x_i) + \phi(d) > \phi(x_i)$. Thus $a_{11}a_{22}\ldots a_{nn} > \phi(x_1)\phi(x_2)\ldots\phi(x_n)$. ∎

3. REMARKS

In [2] we considered GCD matrices defined on arbritary sets S of positive integers. It was shown that $[S]$ is positive definite and hence $\det [S] > 0$. In a different direction, we considered in [3] another generalisation of the set $E(n)$. Let $D(s,d,n)$ be the arithmetic progression defined as follows:

$$D(s,d,n) = \{s, s+d, s+2d, \ldots, s+(n-1)d\}, \text{ where } (s,d) = 1.$$

Observe that $D(1,1,n) = E(n)$. The following open problem is mentioned in [3].

Problem. What is the value of the determinant of the GCD matrix defined on $D(s,d,n)$?

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