COMBINATORIAL PROPERTIES OF MATRICES OF ZEROS AND ONES

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1. Introduction. This paper is concerned with a matrix A of m rows and n columns, all of whose entries are 0's and 1's. Let the sum of row i of A be denoted by r_i $(i = 1, \ldots, m)$ and let the sum of column i of A be denoted by s_i $(i = 1, \ldots, n)$. It is clear that if τ denotes the total number of 1's in A

$$\tau = \sum_{i=1}^{m} r_i = \sum_{i=1}^{n} s_i.$$

With the matrix A we associate the *row sum vector*

$$R = (r_1, \ldots, r_m),$$

where the ith component gives the sum of row i of A. Similarly, the *column sum vector* S is denoted by

$$S = (s_1, \ldots, s_n).$$

We begin by determining simple arithmetic conditions for the construction of a (0, 1)-matrix A having a given row sum vector R and a given column sum vector S. This requires the concept of majorization, introduced by Muirhead. Then we apply to the elements of A an elementary operation called an interchange, which preserves the row sum vector R and column sum vector S, and prove that any two (0, 1)-matrices with the same R and S are transformable into each other by a finite sequence of such interchanges. The results may be rephrased in the terminology of finite graphs or in the purely combinatorial terms of set and element. Applications to Latin rectangles and to systems of distinct representatives are studied.

2. Maximal matrices and majorization. Let

$$\delta_i = (1, \ldots, 1, 0, \ldots, 0)$$

be a vector of n components with 1's in the first r_i positions, and 0's elsewhere. A matrix of the form

$$\bar{A} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_m \end{bmatrix}$$

is called *maximal*, and we refer to $\bar{A}$ as the *maximal form* of A. The maximal $\bar{A}$ may be obtained from A by a rearrangement of the 1's in the rows of A. Also by inverse row rearrangements one may construct the given A from $\bar{A}$.

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Let $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$ and $\bar{S} = (\bar{s}_1, \ldots, \bar{s}_n)$ be the row sum and column sum vectors of $\bar{A}$. Evidently

$$R = \bar{R}.$$

Moreover, it is clear that the row sum vector R uniquely determines $\bar{A}$, and hence $\bar{S}$. Indeed, $\tau = \sum r_i = \sum \bar{s}_i$ constitute conjugate partitions of τ.

Consider two vectors $S = (s_1, \ldots, s_n)$ and $S^* = (s_1^*, \ldots, s_n^*)$, where the s_i and s_i^* are nonnegative integers. The vector S is *majorized* by S^*,

$$S \prec S^*,$$

provided that with the subscripts renumbered (**5**; **3**):

$$(1) \qquad s_1 \geqslant \ldots \geqslant s_n, \; s_1^* \geqslant \ldots \geqslant s_n^* ;$$

$$(2) \qquad s_1 + \ldots + s_i \leqslant s_1^* + \ldots + s_i^* , \qquad i = 1, \ldots, n-1;$$

$$(3) \qquad s_1 + \ldots + s_n = s_1^* + \ldots + s_n^* .$$

For the vectors S and $\bar{S}$ associated with the matrices A and $\bar{A}$, respectively, we prove that

$$S \prec \bar{S}.$$

We renumber the subscripts of the s_i of A so that

$$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n.$$

For $\bar{A}$, we already have

$$\bar{s}_1 \geqslant \bar{s}_2 \geqslant \ldots \geqslant \bar{s}_n.$$

Now A must be formed from $\bar{A}$ by a shifting of 1's in the rows of $\bar{A}$. But for each $i = 1, \ldots, n-1$, the total number of 1's in the first i columns of $\bar{A}$ cannot be increased by a shifting of 1's in the rows of $\bar{A}$. Hence

$$s_1 + \ldots + s_i \leqslant \bar{s}_1 + \ldots + \bar{s}_i,$$

$i = 1, \ldots, n-1$. Moreover,

$$s_1 + \ldots + s_n = \bar{s}_1 + \ldots + \bar{s}_n,$$

whence we conclude that $S \prec \bar{S}$.

THEOREM 2.1[1]. *Let the matrix $\bar{A}$ be maximal and have column sum vector $\bar{S}$. Let S be majorized by $\bar{S}$. Then by rearranging 1's in the rows of $\bar{A}$, one may construct a matrix A having column sum vector S.*

Without loss of generality, we may assume that the column sums of A satisfy $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n$. We construct the desired A inductively by columns by a rearrangement of the 1's in the rows of $\bar{A}$.

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[1]*Added in proof.* The author has been informed recently that Theorem 2.1 was obtained independently by Professor David Gale. His investigations concerning this theorem and certain generalizations are to appear in the Pacific Journal of Mathematics.

By hypothesis, $S \prec \bar{S}$, whence $s_1 \leqslant \bar{s}_1$. If $s_1 = \bar{s}_1$, we leave the first column of $\bar{A}$ unchanged. Suppose that $s_1 < \bar{s}_1$. We may rearrange 1's in the rows of $\bar{A}$ to obtain s_1 1's in the first column, unless

$$\bar{s}_2 > s_1, \ldots, \bar{s}_n > s_1.$$

But if these inequalities hold, then

$$\bar{s}_1 + \ldots + \bar{s}_n > ns_1 \geqslant s_1 + \ldots + s_n = \bar{s}_1 + \ldots + \bar{s}_n,$$

which is a contradiction.

Let us suppose then that the first t columns of A have been constructed, and let us proceed to the construction of column $t + 1$. We have then given an m by n matrix

$$[\eta_1, \ldots, \eta_t, \quad \eta_{t+1}, \ldots, \eta_n],$$

where the number of 1's in column η_i is s_i $(i = 1, \ldots, t)$. Let the number of 1's in column η_j be s'_j $(j = t + 1, \ldots, n)$. We may suppose that

$$s'_{t+1} \geqslant s'_{t+2} \geqslant \ldots \geqslant s'_n.$$

Two cases arise.

Case 1. $$s_{t+1} < s'_{t+1}.$$

In this case, remove 1's from column η_{t+1} by row rearrangements, and place the 1's in columns $\eta_{t+2}, \ldots, \eta_n$. If sufficiently many 1's may be removed from η_{t+1} in this manner, then we are finished. Suppose then that there remain e 1's in column $t + 1$, with

$$s_{t+1} < e \leqslant s'_{t+1},$$

and that no further 1's may be removed by this procedure. Then there must exist an integer $w \geqslant 0$ such that

$$s_{t+1} + \ldots + s_n = (n - t)e + w.$$

But

$$\begin{aligned} &s_{t+1} < e, \\ &s_{t+2} \leqslant s_{t+1} < e, \\ &\cdot \\ &\cdot \\ &\cdot \\ &s_n < e. \end{aligned}$$

Therefore

$$(n - t)e + w = s_{t+1} + \ldots + s_n < (n - t)e,$$

which is a contradiction.

Case 2. $$s'_{t+1} < s_{t+1}.$$

By row rearrangements, insert 1's into column η_{t+1} from columns $\eta_{t+2}, \ldots, \eta_n$. If sufficiently many 1's may be inserted in this manner, then we are finished. Suppose then that there remain e 1's in column $t+1$ with

$$s'_{t+1} \leqslant e < s_{t+1},$$

and that no further 1's may be inserted by our procedure.

Let the matrix at this stage of the construction process be denoted by

$$[e_{rs}].$$

If now

$$e_{r,t+1} = 0,$$

then

$$e_{rj} = 0, \qquad (j = t+1, \ldots, n).$$

Suppose that some

$$e_{rj} = 1, \qquad j \geqslant t+2.$$

Then either

$$e_{rk} = 1, \qquad (k = 1, \ldots, t+1),$$

or else for some k, $1 \leqslant k \leqslant t$,

$$e_{rk} = 0.$$

Consider the case in which $e_{rk} = 0$. Since $s_k \geqslant s_{t+1} > e$, there must exist

$$e_{pk} = 1, \quad e_{p,t+1} = 0.$$

Interchanging $e_{rj} = 1$ and $e_{rk} = 0$, and interchanging $e_{pk} = 1$ and $e_{p,t+1} = 0$, we see that $s_1, \ldots, s_t$ are left unaltered, and that e is increased by 1.

Continue to increase e by transformations of this variety. Suppose that all such transformations have been applied and that e still satisfies

$$s'_{t+1} \leqslant e < s_{t+1}.$$

But now it is no longer possible to move a 1 from columns $t+2, \ldots, n$ into columns $1, 2, \ldots, t+1$. This means that

$$s_1 + \ldots + s_t + e = \bar{s}_1 + \ldots + \bar{s}_t + \bar{s}_{t+1}.$$

But then

$$s_1 + \ldots + s_{t+1} \leqslant \bar{s}_1 + \ldots + \bar{s}_{t+1} = s_1 + \ldots + s_t + e,$$

whence $s_{t+1} \leqslant e$, which is a contradiction. This completes the proof.

The preceding theorem has a variety of applications. For example, let the (0, 1)-matrix A of m rows and n columns contain exactly $\tau = km$ 1's, where k is a positive integer. Let the column sum vector of A be $S = (s_1, \ldots, s_n)$. Then there exists an m by n matrix A^* composed of 0's and 1's with exactly k 1's in each row, and column sum vector S. For let $\bar{A}$ be m by n, with all 1's in the first k columns and 0's elsewhere. If $\bar{S}$ denotes the column sum vector of $\bar{A}$, then $S \prec \bar{S}$, and the desired A^* may be constructed from $\bar{A}$.

In this connection we mention the following result arising in the study of the completion of Latin rectangles (**1**; **7**). Let A be a (0, 1)-matrix of r rows and n columns, $1 \leqslant r < n$. Let there be k 1's in each row of A, and let the column sums of A satisfy $k - (n - r) \leqslant s_i \leqslant k$. Then $n - r$ rows of 0's and 1's may be adjoined to A to obtain a square matrix with exactly k 1's in each row and column (**7**). To prove this it suffices to construct an $n - r$ by n matrix A^* of 0's and 1's with exactly k 1's in each row, and column sum vector $(k - s_1, \ldots, k - s_n)$. By the remarks of the preceding paragraph, such a construction is always possible.

3. Interchanges. We return now to the m by n matrix A composed of 0's and 1's, with row sum vector R and column sum vector S. We are concerned with the 2 by 2 submatrices of A of the types

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An *interchange* is a transformation of the elements of A that changes a specified minor of type A_1 into type A_2, or else a minor of type A_2 into type A_1, and leaves all other elements of A unaltered. Suppose that we apply to A a finite number of interchanges. Then by the nature of the interchange operation, the resulting matrix A^* has row sum vector R and column sum vector S.

THEOREM 3.1. *Let A and A^* be two m by n matrices composed of* 0'*s and* 1'*s, possessing equal row sum vectors and equal column sum vectors. Then A is transformable into A^* by a finite number of interchanges.*

The proof is by induction on m. For $m = 1$ and 2, the theorem is trivial. The induction hypothesis asserts the validity of the theorem for two (0, 1)-matrices of size $m - 1$ by n.

We attempt to transform the first row of A into the first row of A^* by interchanges. If we are successful, the theorem follows at once from the induction hypothesis. Suppose that we are not successful and that we denote the transformed matrix by A'. For notational convenience, we simultaneously permute the columns of A' and A^* and designate the first row of A' by

$$(\delta_r, \eta_s, \delta_t, \eta_t)$$

and the first row of A^* by

$$(\delta_r, \eta_s, \eta_t, \delta_t).$$

Here δ_r and δ_t are vectors of all 1's with r and t components, respectively, and η_s and η_t are 0 vectors with s and t components, respectively. Thus we have been successful in obtaining agreement between the two rows in the positions labelled δ_r and η_s, but have been unable to obtain agreement in the positions labelled δ_t and η_t. We may suppose, moreover, that these 2t positions of disagreement are the minimal number of disagreements obtainable among all attempts to transform the first row of A into the first row of A^* by interchanges.

Let A'_{m-1} and A^*_{m-1} denote the matrices composed of the last $m-1$ rows of A' and A^*, respectively. The row sum vectors of A'_{m-1} and A^*_{m-1} are equal. Also corresponding columns of A'_{m-1} and A^*_{m-1} below the positions labelled δ_r and η_s have equal sums. Let α_i denote the $(r+s+i)$th column of A'_{m-1}, and let β_i denote the $(r+s+t+i)$th column of A'_{m-1}, where $i = 1, \ldots, t$. Let $\alpha_1^*, \ldots, \alpha_t^*$ and $\beta_1^*, \ldots, \beta_t^*$ denote the corresponding columns of A^*_{m-1}. Let a_i, b_i, a_i^*, b_i^* denote the column sums of $\alpha_i, \beta_i, \alpha_i^*, \beta_i^*$, respectively.

Now in A'_{m-1} we cannot have simultaneously a 0 in the position determined by row j and column α_i and a 1 in the position determined by row j and column β_i. For if this were the case, we could perform an interchange and reduce the $2t$ disagreements in the first row of A'. Hence $a_i \geqslant b_i$. Moreover, $a_i^* = a_i + 1$ and $b_i^* = b_i - 1$, whence

$$a_i^* - b_i^* = a_i - b_i + 2 \geqslant 2.$$

In A^*_{m-1}, consider columns α_i^* and β_i^*. There exists a row of A^*_{m-1} that has a 1 in column α_i^* and a 0 in column β_i^*. Replace the 1 by 0 and the 0 by 1, and let such a replacement be made for each $i = 1, \ldots, t$. We obtain in this way a new matrix $\tilde{A}_{m-1}$ whose row and column sum vectors are equal to those of A'_{m-1}. By the induction hypothesis, we may transform A'_{m-1} into $\tilde{A}_{m-1}$ by interchanges. However, these interchanges applied to A' will allow us to perform further interchanges and make the first rows of the transformed A' and A^* coincide. Hence the theorem follows.

Let $\mathfrak{A}$ denote the class of all (0, 1)-matrices of m rows and n columns, with row sum vector R and column sum vector S. The *term rank* ρ of A in $\mathfrak{A}$ is the order of the greatest minor of A with a nonzero term in its determinant expansion **(6)**. This integer is also equal to the minimal number of rows and columns that contain collectively all of the nonzero elements of A **(4)**. A (0, 1)-matrix $A = [a_{rs}]$ may be considered an incidence matrix distributing n elements $x_1, \ldots, x_n$ into m sets $S_1, \ldots, S_m$. Here $a_{ij} = 1$ or 0 according as x_j is or is not in S_i. From this point of view the term rank of a matrix is a generalization of the concept of a system of distinct representatives for subsets $S_1, \ldots, S_m$ of a finite set N **(2)**. Indeed, the subsets $S_1, \ldots, S_m$ possess a system of distinct representatives if and only if $\rho = m$.

THEOREM 3.2. *Let $\tilde{\rho}$ be the minimal and $\bar{\rho}$ the maximal term rank for the matrices in $\mathfrak{A}$. Then there exists a matrix in $\mathfrak{A}$ possessing term rank ρ, where ρ is an arbitrary integer on the range*

$$\tilde{\rho} \leqslant \rho \leqslant \bar{\rho}.$$

For an interchange applied to a matrix in $\mathfrak{A}$ either changes the term rank by 1 or else leaves it unaltered. But by Theorem 3.1, we may transform the matrix of term rank $\bar{\rho}$ into the matrix of term rank $\tilde{\rho}$. This implies that there exists a matrix in $\mathfrak{A}$ of term rank ρ.

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