Maximal tournaments*

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Abstract

We show that the maximum number $f(n,x)$ of closest results in a good x-tournament is $f(n,x) = c(x)n^2 + O(n)$ if x is rational and $f(n,x) = n^{3/2}/2 + O(n)$ if x is irrational. E.g. $c(0) = 1/8$ for chess and $c(1) = 1,5 - \sqrt{2} \sim 0.0858$ for modern football.

MSC. 05C20, 05A15

1 Introduction

Round-robin tournaments are popular in the world of sport (and informatics, sociometry, biology too).

The players often divide a fixed amount of points. E.g. 1 in individual tennis (where the unique possible result is 1:0) and chess (possible results are 1:0 or 1/2:1/2), 2 in traditional football (2:0 or 1:1), 4 in Chess Olympiad for man teams ($4:0,\ 3\ 1/2:1/2,\ 3:1, 2\ 1/2:1\ 1/2,\ 2:2$), 9 or 16 in table tennis team.

In other sports a variable amount of points is divided: e.g. 2 or 3 in modern football (1:1 or 3:0 in Germany; 1:1, 2:0 or 3:0 in Japan) and individual table tennis (2:0, 2:1); 25, 26, 27, 28, 29 or 30 in Bridge Olympiad.

A round-robin **tournament** is a $n \times n$ real matrix $T_n = [t_{ij}]$ $(n \geq 2)$. The elements of the main diagonal t_{ii} equal to zero and the pairs of symmetric elemnts $t_{ij} : t_{ji}$ give the result of the match between $\mathcal{P}_i$ (the i-th player) and $\mathcal{P}_j$ (the j-th player). $t_{ij} = t_{ji}$ means a draw, while $t_{ij} > t_{ji}$ means the win of $\mathcal{P}_i$ against $\mathcal{P}_j$. The sum of the elements of the i-th row

$$s_i = \sum_{k=1}^{n} t_{ik}$$

is called **the score** of the i-th player and the vector $(s_1, s_2, \ldots, s_n)$ is called the score vector of the tournament. The nondecreasingly ordered vector of the scores is denoted by $\mathbf{q} = \langle q_1, q_2, \ldots, q_n \rangle$ and is called **the score sequence** of the tournament.

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A tournament $T_n = [t_{ij}]$ can be represented as a directed weigthed graph consisting of n nodes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ such that each pair of distinct nodes $\mathcal{P}_i$ and $\mathcal{P}_j$ is joined by two directed weighted arcs: the arc from $\mathcal{P}_i$ to $\mathcal{P}_j$ has the weight t_{ij} and the arc from $\mathcal{P}_j$ to $\mathcal{P}_i$ has the weight t_{ji}.

A set $\mathcal{T}_n$ of tournaments T_n is **complete** or **incomplete**. A set $\mathcal{T}_n$ is called k-complete if all matrices contain only nonnegative integer elements, the sum of the symmetric elements is always k and the set contains all possible such matrix. The sets of all tournaments with n players are 1-complete for individual tennis and 2-complete for traditional football (chess also can be considered as 2-complete sport).

We denote by $\mathcal{F}_{nx}$ (where x is a nonnegative real number) the following incomplete set of tournaments with n players: the permitted $f_{ij} : f_{ji}$ pairs are $(2+x) : 0$ and 1:1 and the set contains all possible such tournaments.

In this case $\mathcal{F}_{n1}$ describes chess and $\mathcal{F}_{n1}$ describes modern football (in the following shortly: football) tournaments.

We say that a tournament is n-**good** (shortly: good), if its score sequence contains n scores and they are different [5]. A tournament is called **unique** [9] if its score sequence uniquely determines the matrix. A tournament is called **uniform** [5] if its score sequence is equidistant.

We call **the closest result** of a sport the result where $|t_{ij} - t_{ji}|$ is minimal. So for tennis 1:0, for chess 1/2 : 1/2, for modern football 1:1 and for table tennis 2:1 is the closest result.

A tournament is called n-**maximal** (shortly: maximal) if it is n-good and contains the maximal number of closest results.

The aim of this paper is to determine these maximal numbers for different x-tournaments.

2 Preliminary analysis of x-tournaments

A nondecreasing sequence $\mathbf{q} = \langle q_1, \ldots, q_n \rangle$ is called **realisable** if there exists a tournament T_n whose score sequence equals to $\mathbf{q}$. There are different algorithms for complete tournament sets [2,4,5,6,8] checking in $O(n)$ steps if a given sequence is realisable. In [5] a linear algorithm, in [7] parallel algorithms were presented for similar test of score vectors.

If the answer is affirmative then there are algorithms reconstructing a corresponding tournament in $O(n^2)$ steps [1, 3, 8].

It is easy to get good tournaments: e.g. if the players win against players with smaller index (transitive tournaments). In the case of tennis only the transitive tournaments are maximal.

Let $w_i(T_{nx})$ denote the number of wins of the i-th player of T_{nx}. Then the sum $S(T_{nx})$ of the scores of T_{nx} equals

$$S(T_{nx}) = \sum_{i=1}^{n} q_i(T_{nx}) = n(n-1) + x \sum_{i=1}^{n} w_i(T_{nx}) = n(n-1) + xW(T_{nx}), \quad (1)$$

where W_{nx} denotes the total number of wins in T_{nx}.

The minimal number of wins in n-good x-tournaments is denoted by $f(n,x)$. For x-tournaments the maximal number of closest results equals to $\frac{n(n-1)}{2} - f(n,x)$.

If $x = 0$, then $S(T_{nx}) = n(n-1)$, and if $x > 0$, then

$$W(T_{nx}) = (S(T_{nx}) - n(n-1))/x. \tag{2}$$

The next lemma gives a linear lower bound for $f(n,x)$.

Lemma 2.1 *If $n \geq 2$, then*

$$f(n,x) \geq \lfloor n/2 \rfloor. \tag{3}$$

Proof. If every result is a draw, then all scores are equal to $n-1$. A nondraw result makes at most 2 scores different from $n-1$ so a good score sequence requires the change of at least $n-1$ scores what imply $f(n,x) \geq \lceil (n-1)/2 \rceil$, where the right side equals to $\lfloor n/2 \rfloor$. ■

For 2 players according to Lemma 2.1 we have $f(2,x) \geq 1$. If $\mathcal{P}_1 \to \mathcal{P}_2$ (that is $\mathcal{P}_1$ wins against $\mathcal{P}_2$), then the score sequence is $\mathbf{q} = \langle 0, 2+x \rangle$, what is maximal.

Bound (3) is precise for 3 players too: now $f(3,x) \geq 1$ and if $\mathcal{P}_1 \to \mathcal{P}_3$, then $\mathbf{q} = \langle 1, 2, 3+x \rangle$ is maximal and so $f(3,x) = 1$.

According to Lemma 2.1. $f(4,x) \geq 2$. If 4 different players play in 2 decided matchs, then the scores of winners are equal. But if $\mathcal{P}_1 \to \mathcal{P}_2$ and $\mathcal{P}_2 \to \mathcal{P}_4$, then $\mathbf{q} = \langle 2, 3, 3+x, 4+x \rangle$: if x is positive, then this $\mathbf{q}$ is maximal. If $x = 0$, then 2 wins are not sufficient: we need 3 wins to get a maximal sequence $\mathbf{q} = \langle 1, 2, 4, 5 \rangle$.

If $n \geq 5$, then the bound of Lemma 2.1. is not precise even for positive x.

The following construction uses about the quater of the matches to get good sequences.

Lemma 2.2 *If $n \geq 2$ and $x > 0$, then*

$$f(n,x) \leq \frac{\lfloor n/2 \rfloor (\lfloor n/2 \rfloor + 1)}{2}.$$

Proof. If $n = 2m$, then we divide the players into two subsets: winners $(\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m)$ and losers $(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m)$. $\mathcal{W}_i$ wins against $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i$, and all the remaining matches end with draw. Then $\frac{m(m+1)}{2}$ matches ended with a win and the score sequence is

$$\mathbf{q} = \langle n-1-m, n-1-m+1, \ldots, n-1-1, n-1+x, n-1+2x, \ldots, n-1+mx \rangle.$$

So if $m \geq 1$, then

$$f(2m,x) \leq \frac{m(m+1)}{2}.$$

If $n = 2m+1$, then we can add a new player to the previous tournament who makes only draws. Then we get the same upper bound as for $n = 2m$. ■

3 The basic case x = 1

In this section we suppose $x = 1$. Our analysis becomes simpler if we reduce the points: drawers get zero, winners $1 + x$ and losers -1 point. If a reduced score sequence $\mathbf{r}$ contains k $(0 \leq k \leq n)$ negative scores, then is can be considered as $\mathbf{r} = \langle -n_k, -n_{k-1}, \ldots, -n_1, p_0, p_1, \ldots, p_j \rangle$, where p_0 is nonnegative, $p_1, \ldots, p_j, n_1, \ldots, n_j$ are positive integers.

The scores $p_j,\ p_{j-1},\ \ldots,\ p_1,\ p_0$ form **the nonnegative,** while the scores $-n_1,\ -n_2,\ \ldots, -n_k$ **the negative side** of the sequence. The number of odd scores in the nonnegative side is called **the odd part** of the sequence and is denoted by $\omega(\mathbf{s})$. Then

$$\omega(\mathbf{r}) = \sum_{i=0}^{j} (p_i - 2\lfloor p_i/2 \rfloor).$$

The difference of the sum of scores in the nonnegative side and the odd part is denoted by $\epsilon(\mathbf{s})$ and is called **the even part** of the sequence. Then

$$\epsilon(\mathbf{r}) = \left(\sum_{i=0}^{j} p_i \right) - \omega(\mathbf{r}).$$

The 1-uniform sequences of form $\mathbf{r} = \langle -k, -(k-1), \ldots, -1, 0, 1, \ldots, j \rangle$ are called **ideal**. An ideal sequence is not always realisable and if it is realisable then not always maximal. E.g. the sequence $\langle -1, 0, 1 \rangle$ is not realisable. According to (2) the sequence $\langle -1, 0, 1, 2, 3 \rangle$ needs 5 wins while the sequence $\langle -2, -1, 0, 2, 4 \rangle$ can be realised with 3 wins (now both sequences are realisable).

3.1 Quadratic lower bound for $f(n, 1)$

Analysing the reduced score sequence we get a lower bound of $f(n, 1)$.

Lemma 3.1 *If $n \geq 2$ and $\mathbf{r} = (p_j, \ldots p_0, -n_1, \ldots, -n_k)$ is the reduced score sequence of a given matrix T, then*

$$W(\mathbf{r}(T)) \geq \frac{k(k+1)}{2}.$$

Proof. If the score of a player is $-n_i$, then this player has at least n_i losses since negative number of points can be gathered only by a loss (1 point per match). If the score p_j of a player is an odd number then this player has at least one loss. So we have for any score vector $\mathbf{r}(T)$

$$W(\mathbf{r}(T)) \geq \sum_{i=1}^{k} n_i + \omega(\mathbf{r}(T)).$$

The sum on the right side is minimal , if $\mathbf{r}$ does not contain odd positive number and the absolute value of the negative scores is small as possible, that is $n_i = i$ $(i = 1, 2, \ldots, k)$. ■

Lemma 3.2 *If $n \geq 2$ and $\mathbf{r} = (p_j, \dots p_0, -n_1, \dots, -n_k)$ is a reduced score sequence of a given matrix T, then*

$$W(\mathbf{r}(T)) \geq \frac{j(j+1)}{4} + \left\lfloor \frac{j+1}{2} \right\rfloor /2.$$

Proof. If the score of $\mathcal{P}_i$ in T_{nx} is p_i, then $\mathcal{P}_i$ has at least $\lfloor (p_i+1)/2 \rfloor$ wins since positive number of points can be gathered only by a win (two points per match) and if the score is odd then at least one loss has to be compensated. So for any score vector $\mathbf{s}$ holds

$$W(\mathbf{s}(T)) \geq \epsilon(\mathbf{s})/2 + \omega(\mathbf{s}).$$

The sum on the right side is minimal (for a given j), if the positive scores are small as possible, that is $p_j = j$ (if j odd, then $j+1, j-1, j-2, \dots, 1, 0$ results a minimum). Then the majority of wins required by the positive scores are contained by the first term of the formula in the lemma; for the odd positive scores a half win is contained in the second term. ■

From here we get the following lower bound for $f(n, 1)$.

Lemma 3.3 *If $n \geq 2$, then*

$$f(n,1) \geq \min_{0 \leq j \leq n-1} \max \left(\frac{j(j+1)}{4} + \left\lfloor \frac{j+1}{2} \right\rfloor /2, \frac{(n-j-1)(n-j)}{2} \right). \tag{4}$$

Proof. The previous two lemmas imply this assertation: for a given n we choose the worst (resulting the lowest lower bound) j, and both lower bound has to hold for this j. ■

Evaluating this minimax problem we get a quadratic lower bound of $f(n, 1)$.

Theorem 3.4 *If $n \geq 3$, then there are 3 cases: a) if $\sqrt{2n^2 + 2n}$ is an a integer number, then*

$$f(n,1) \geq \frac{a(a+1)}{4} + \left\lceil \frac{a+1}{2} \right\rceil /2; \tag{5}$$

b) if $\sqrt{2n^2 + 2n + 1}$ is an a integer number, then

$$f(n,1) \geq \frac{a(a+1)}{2} + \left\lceil \frac{a+1}{2} \right\rceil /2; \tag{6}$$

c) if neither $\sqrt{2n^2 + 2n}$ nor $\sqrt{2n^2 + 2n + 1}$ is an integer number, then

$$f(n,1) \geq \min \left(\frac{a(a+1)}{4} + \left\lceil \frac{a+1}{2} \right\rceil /2, \frac{b(b+1)}{2} \right), \tag{7}$$

where

$$a = 2n - \lfloor \sqrt{2n^2 + 2n} \rfloor \tag{8}$$

and

$$b = \lfloor \sqrt{2n^2 + 2n} \rfloor - n - 1. \tag{9}$$

Proof. In Lemma 3.3. the first term is increasing, the second term is decreasing while j (as a real variable) increases from 1 to n. So we get the minimum when the values of the terms are equal. Therefore we get the following quadratic equation:

$$\frac{y(y+1)}{4} + \lfloor \frac{y+1}{2} \rfloor /2 = \frac{(n-y-1)(n-y)}{2}$$

If j is even, then this equation implies

$$y^2 - 4ny + 2n^2 - 2n = 0.$$

The suitable root of this equation is $y = 2n - \sqrt{2n^2 + 2n}$. E.g. for $n = 8$ and $n = 49$ the roots are integer, but usually we get noninteger root: then we have to choose either the lower or the upper integer part of the root. Is is possible that identical values belong to these neighbouring j's.

If j is odd, then we get the equation

$$y^2 - 4ny + 2n^2 - 2n - 1 = 0.$$

Then the corresponding solution $y = 2n - \sqrt{2n^2 + 2n + 1}$. E.g. if $n = 3$ or $n = 20$, then the root is integer , but in the majority of the cases this root also is a noninteger number and have to choose from the integer neighbours. Taking into account the behaviour of the terms we get the minimum occuring in the c) part of the theorem. ∎

3.2 Upper bound for $f(n, 1)$

Let denote the minimum in Lemma 3.3. by $g(n)$, and the corresponding j by m (if there are two values, then we choose the smaller of them).

Lemma 3.5 *If $n \geq 2$, then*

$$f(n, 1) \leq g(n) + n. \tag{10}$$

Proof. If we cover the even part of the score vector $\mathbf{s} = (m, \ldots, 1, 0, -1, \ldots, -n_{n-m-1})$ by negative scores (if $\frac{(n-m-1)(n-m)}{2}$ is not sufficient, then we decrease the negative scores in a uniform way), and cover the odd part by increasing of the largest score, then the scores will be different. ∎

3.3 Maximal 1-tournaments

Now we show that the lower bounds (5), (6), (7) of $f(n, 1)$ in Theorem 3.4. give the correct order (in some cases even the precise value) and also present a construction method of maximal tournaments.

Theorem 3.6.

$$f(n, 1) = (3/2 - \sqrt{2})n^2 + O(n). \tag{11}$$

Proof. Let j that root in the proof of Theorem 3.4. which determines the minimum: this root is a in cases a) and b) and $2n - \lfloor \sqrt{2n^2 + 2n} \rfloor$ or $2n - \lceil \sqrt{2n^2 + 2n} \rceil$ (maybe, both) in case c). Let $\mathbf{r} = (j, j-1, ..., \ldots, 1, 0, -1, \ldots, -k)$ (where $k = n - j - 1$) be the ideal score sequence belonging to j).

We distinguish the following 3 cases: a) there are **many victories** in the sequence $\mathbf{r}$, if

$$\frac{k(k+1)}{2} \leq \frac{j(j+1)}{4} - \left\lceil \frac{j+1}{2} \right\rceil \tag{12}$$

— these are the V (*victory*) type score sequences;

b) there are **many losses in the vector s**, if

$$\frac{k(k+1)}{2} \geq \frac{j(j+1)}{2} \tag{13}$$

— these are the L (*loss*) type score sequences;

c) the vector **s** is **balanced**, if neither condition holds — these are the B (*balanced*) type score sequences.

In case a) the losses of the losers cover at most the even part of the scores of the winners. Then the bound in the theorem is sharp:

$$f(n,1) = \frac{a(a+1)}{4} + \left\lceil \frac{a+1}{2} \right\rceil /2 \tag{14}$$

implying (11).

Adding the additonal victories of the winners to the losses of the losers we get a maximal score sequence: the difference of the right and left side of (12) determines the number of these losses. If this difference equals to zero then we get an ideal maximal solution — otherwise we decrease the negative scores.

In case b) using the losses of the losers we can guarantee $j, j-1, \ldots, 1, 0$ wins for winners. The bound of the theorem is sharp in this case too. Since $k > j$ can not occur, now the ideal solution score sequence is $2j, 2j-2, \ldots, 2, 0, -1, \ldots, -k$.

In case c) we can decrease the number of necessary odd positive scores using the additional losses of the losers but this decreasing usually do not result a maximal solution. We handle case c) solving the following optimization problem.

We seek the positive part of the score sequence in the form $0, 1, 2, \ldots, 2m-2, 2m-1, 2m, 2m+2, 2m+4, \ldots, 2m+2(j-2m)$. We determine the minimal m-et for which the losses of the negative part and the losses necessary for the even positive scores $1, 3, \ldots, 2m-1$ cover the wins needed by the positive side. So we determine

$$\min\left\{m:\ 0 \leq m \leq \lfloor j/2 \rfloor, \binom{k}{2} + m \geq (j-2m)m + \binom{j-2m}{2} + m(m+1)\right\}. \tag{15}$$

The solution is

$$m = \left\lceil \frac{j+1-\sqrt{2k^2+2k-j^2+1}}{2} \right\rceil.$$

In this case

$$f(n,1) = \frac{k(k+1)}{2} + m = \frac{k(k+1)}{2} + \left\lceil \frac{j+1-\sqrt{2k^2+2k-j^2+1}}{2} \right\rceil.$$

If $2m+1=j$, then $(2m+2, 2m, 2m-1, \ldots, 1, 0, -1, \ldots, -k)$ maximal and $f(n,1) = g(n)$. If $2m+1 \neq j$ and the solution of (6) is integer number, that is if

$$\left\lceil \frac{j+1-\sqrt{2k^2+2k-j^2+1}}{2} \right\rceil = \frac{j+1-\sqrt{2k^2+2k-j^2+1}}{2},$$

then $f(n,1) = \frac{k(k+1)}{2} + m$ and the maximal score sequence is $((2m+2(j-2m), (2m+2(j-2m)-2, \ldots, 2m, 2m-1, \ldots, 1, 0, -1, \ldots, -(n-m-1))$.

If $j \neq 2m+1$ and (15) does not hold, then $f(n,1) > g(n)$ and

$$r(n) = \frac{k(k+1)}{2} + m - \left(m(j-2m) + \frac{(j-2m)(j-2m+1)}{2} + m(m+1) \right)$$

wins are missing from the supposed form, therefore the maximal sequence is $(2m+2(j-2m)+2r(n), (2m+2(j-2m)-2, \ldots, 2m, 2m-1, \ldots, 1, 0, -1, \ldots, -(n-m-1)$ és $f(n) = \frac{k(k+1)}{2} + m + r(n)$.

In this case we also get the required order of $f(n,1)$. ■

Table 1.1 contains quantities connected with Theorem 3.6. The meaning of u and l is as follows: if $\sqrt{2n^2+2n}$ is an integer a, then $u = 2n - a$ (and in the column of l — occurs); if $\sqrt{2n^2+2n+1}$ is an integer a, then $l = 2n - a$ (and in the column of u — occurs); if neither root is integer, then $l = 2n - \lceil \sqrt{2n^2+2n} \rceil$ and $u = 2n - \lfloor \sqrt{2n^2+2n} \rfloor$. The definition of the functions p and q is as follows:

$$p(m) = \frac{m(m+1)}{4} + \left\lfloor \frac{m+1}{2} \right\rfloor /2$$

és

$$q(m) = \frac{(n-m-1)(n-m)}{2}.$$

The j values resulting the minimum occuring in the theorem and the numbers from $p(l), q(l), p(u), q(u)$ giving $g(n)$ are written using **bold** digits. In the case of balanced sequences m and $r(n)$ are also given.

n	l	u	$p(l)$	$q(l)$	$p(u)$	$q(u)$	$g(n)$	type of $\mathbf{s}$	m	$r(n)$
3	**1**	—	**1**	**1**	—	—	1	L	–	–
4	1	**2**	1	1	**2**	1	2	V	–	–
5	**2**	3	2	**3**	4	1	3	L	–	–
6	2	**3**	2	6	**4**	3	4	B	1 $(2m+1=u)$	–
7	**3**	**4**	4	**6**	**6**	3	6	V,L	–	–
8	—	4	—	—	**6**	**6**	6	B	1 $(2m+1 \neq l)$	—
9	4	**5**	7	10	**9**	6	9	L	–	–
10	**5**	6	9	**10**	12	6	10	B	1	–
11	5	**6**	9	15	**12**	10	12	B	2	1
12	**6**	7	12	**15**	14	10	15	B	1	–
13	6	**7**	12	21	**16**	15	16	B	3	2
14	7	**8**	16	21	**20**	15	20	V	–	–
15	8	**9**	20	**21**	25	15	21	B	2	3
16	8	**9**	20	28	**25**	21	25	B	4 $(2m+1=u)$	–
17	**9**	10	25	**28**	30	21	28	B	3	4
18	9	**10**	25	36	**30**	28	30	B	4	1
19	**10**	**11**	30	**36**	**36**	28	36	V,B	3	3
20	**11**	—	**36**	**36**	—	—	36	B	4	2
21	**11**	—	**36**	**36**	—	—	36	B	4	2
25	13	**14**	49	66	**56**	55	56	B	5	–
26	14	**15**	56	66	**64**	55	64	V	–	–
30	16	**17**	72	91	**81**	78	81	B	7	2
35	19	**20**	100	120	**110**	105	110	B	9	3
40	22	**23**	132	149	**144**	132	144	B	10	—
45	25	**26**	169	190	**142**	171	182	B	12	–
49	—	**28**	—	—	**210**	**210**	182	B	12	6
50	28	**29**	210	231	**225**	210	225	V	–	–
55	31	**32**	256	276	**272**	253	272	V	–	–
60	34	**35**	306	325	**324**	300	324	V	–	–
80	46	**47**	552	561	**576**	528	576	B	–	–
100	57	**58**	841	903	**870**	861	870	B	25	–
120	**11**	—	**36**	**36**	—	—	36	B	4	2

Table 1.1. Data connected with Theorem 1.6. for $n = 1, 2, \ldots, 21$ and 25, 26, 30, 35, 40, 45, 49, 50, 55, 60, 80, 100, 120 players

If (12) is an equality, then the maximal sequence is ideal. An example is when $n = 4$: according to the theorem $j = 2$ and $k = 1$. The winners need $\frac{k(k+1)}{2}$ wins for the even part of their scores — adding 1 win necessary for the even part we get 2 by the theorem. Similar case is $n = 50$. Now the theorem results $j = 29, k = 20$, so the loserss have 210 losses. The even part of the scores of winners requires exactly 210 wins. Adding 15 wins necessary for the 15 odd scores we get 225 corresponding to the theorem.

If (12) is an inequality, then we divide the additional losses needed by the even part at the end of the sequence (giving 1 loss each loser).

Case b) occurs rarely: an example is $n = 3, n = 5$ and $n = 7$ (if $j = 3$).

The smallest example for case c) is $n = 6$: now $j = 3,\ k = 2,\ f(6,1) \geq g(6) = 4$. Substituting the values of j and k we get $m = 1$: then $j = 2m+1$. Now the sequence $\mathbf{s}$ = (4, 2, 1, 0, -1, -2, -3) is maximal and $f(6,1) = g(6) = 4$.

The next example for case c) is c) $n = 8$: according to Theorem 1.6. $j = 4,\ k = 3,\ f(8,1) \geq g(8) = 6$. Then $m = 1$ and $2m+1 < j$. Now the maximal score sequence is (6, 4, 2, 1, 0, -1, -2, -3), $f(8,1) = 7$ and $f(8,1) = g(8) + 1$.

4 $x = 0$ or $x \geq 2$ (x integer)

If $x = 0$, then the construction of Lemma 1.2. results a maximal score sequence. Therefore

$$f(n,0) = \frac{\lfloor n/2 \rfloor (\lfloor n/2 \rfloor + 1)}{2} = n^2/8 - \rho(n),$$

where $\rho(2m) = 0$ and $\rho(2m+1) = 1/8$.

In case $x = 1$ the additional point guarantees, that the value of one win plus one loss is greater then the value of two draws. If $x = 2$, then one win plus two losses are better, then three draws. Therefore the positive scores can be different so, that three players have $1, 2, \ldots$ wins, resp. but one of them always have also one, and one of them always has two losses too. E.g. for 10 players the score sequence (-3, -2, -1, 0, 1, 2, 3, 5, 6, 9) can be constructed using only 10 wins so $f(10,2) \leq 10 < f(10,1) = 11$.

Here the inequality of Lemma 1.5. is modified so, that we divide the firts term by 6, and the second term is about j (since every 3 players have 0 + 1 + 2 losses).

5 x is irrational

For the simplicity let us suppose that $n = b^2$ for some integer b. Then the players can form a $b \times b$ array and we can suppose that $\mathcal{P}_{ij}$ wins i times and loses j times. In this cases the scores are different and the total number of wins equals $b^2(b-1)/2 = O(n^{3/2})$.

6 Numerical results

In column $\mathbf{s}$ of Table 1.2. * denotes the missing elements, in column $f(n,1) - g(n)$! denotes the ideal solution and = such balanced case where $2m + 1 = j$.

n	$f(n,1)$	$f(n,1)-g(n)$	**s**
2	1	–	(2, *, -1)
3	1	0	(2, *, 0, -1)
4	2	0 !	(2, 1, 0, -1)
5	3	0	(4, *, 2, *, 0, -1, -2)
6	4	0 =	(4, *, 2, 1, 0, -1, -2)
7	6	0	(6, *, 4, *, 2, *, 0, -1, -2, -3), (4, 3, 2, 1, 0, -1, *, -3)
8	7	1	(6, *, 4, *, 2, 1, 0, -1, -2, -3)
9	9	0 !	(5, 4, 3, 2, 1, 0, -1, -2, -3)
10	11	1	(8, *, 6, *, 4, *, 2, 1, 0, -1, -2, -3, -4)
11	13	1	(8, *, 5, 4, . . . , -4)
12	16	2	(10, *, 8, *, 6, *, 4, *, 2, 1, 0, -1, -2, -3, -4, -5)
13	18	2	(12, *, 8, 5, 6, *, 4, 3, . . . , -5)
14	20	0	(10, *, 8, 7, . . . , -4, *, -6)
15	24	3	(16, *, 8, *, 6, 5, . . . , -6)
16	25	0 =	(10, *, 8, 7, . . . , -6)
17	31	3	(20, *, 10, *, 8, *, 6, 5, . . . , -7)
18	32	2	(14, *, 10, *, 8, 7, . . . , -7)
19	36	0	(12,11,. . . ,-4,*,-6,-7,-8),(20,*,12,*,10,*,8,6,5,. . . ,-8)
20	40	4	(18, *, 12, *, 10, *, 8, 7, . . . , -8)
21	40	!	(12, . . . , -8)
25	60	4	(18, *, 16, *, 14, *, 12, *, 10, 9, . . . , -10)
26	64	0	(15, . . . , -9, *, -11)
30	85	4	(24, *, 18, *, 16, *, 14, 13, . . . , -12)
35	114	4	(28, *, 20, *, 18, 17, . . . , -14)
40	144	0	(44, *, 21, 20, . . . , -16)
45	183	12	(28, *, 26, 24, 23, . . . , -16)
49	222	12	(44, *, 30, *, 28, *,26, *,24, 23, . . . , -20)
50	225	0 !	(29, 28, . . . , -20)
55	272	0	(38, *, 31, 30, . . . , -22)
60	324	0	(35, 34, . . . , -24)
80	579	18	(60,*,54,*,52,*,. . . ,*,36, 35, . . . , -33)
100	886	16	(66, *, 64,*, . . . , *,50, 49, . . . , -41)
120	40	!	(70, . . . , -49)

Table 1.2. Maximal score sequences for $n = 1, 2, \ldots, 21$ and 25, 26, 30, 40, 49, 50, 55, 60, 100, 120 players

7 Asymptotic behaviour

In table 1.3. the connection between $f(n,1)$ and its approximations is characterized.

n	$f(n,1)$	$f(n,1)-(1,5-\sqrt{2})n^2$	$100f(n,1)/\binom{n}{2}$
2	1	0,6568	100
3	1	0,2278	33,33
4	2	0,6272	33,33
5	3	0,6	30
10	11	2,42	24,5
100	886	28	17,08

Table 1.3. The connection among $f(n,1)$ and its approximations

The last column of the table shows how tends the ratio of the number of the necessary wins and the number of the matches to $100(3-2\sqrt{2}) \sim 17,16$ per cents as n tends to infinity.

If x tends to infinity, then the first member in the inequality of Lemma 1.5. lemma increases and the root tends to n . Therefore the coefficient of the quadratic member of $f(n,x)$ tends to zero.

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