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ON SCORE SETS FOR TOURNAMENTS

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We prove that each set of four or five nonnegative integers is a score set of a tournament.

The score set S of a tournament T, a complete oriented graph, is the set of scores (outdegrees) of the vertices of T. In [2] Reid conjectured that each finite, nonempty set S of nonnegative integers is the score set of some tournament and proved it for the cases $|S| = 1, 2, 3$, or if S is an arithmetic or geometric progression. In this note we will verify Reid's conjecture for the cases $|S| = 4, 5$.

It is well known, see, i.e., [1, p. 61] or [3, p. 176], that nonnegative integers $s_1 \leqslant \cdots \leqslant s_n$ are the scores of a tournament with n vertices iff

$$\sum_{i=1}^{k} s_i \geqslant \binom{k}{2}, \quad 1 \leqslant k \leqslant n-1, \quad \text{and} \quad \sum_{i=1}^{n} s_i = \binom{n}{2}.$$

Let $S = \{t_1, \ldots, t_p\}$ be a nonempty set of nonnegative integers with $t_1 < \cdots < t_p$, then S is a score set iff there exist p positive integers $m_1, \ldots, m_p$ such that

$$\sum_{i=1}^{k} m_i t_i \geqslant \binom{M(k)}{2}, \quad 1 \leqslant k \leqslant p-1, \qquad \sum_{i=1}^{p} m_i t_i = \binom{M(p)}{2}, \tag{1}$$

where

$$M(k) = \sum_{i=1}^{k} m_i, \quad 1 \leqslant k \leqslant p,$$

because only the inequalities in the above mentioned formula for those values of k, for which $s_k < s_{k+1}$ hold, need to be checked [2, p. 608].

In our proofs we often have to combine two regular tournaments, say X with $2x+1$ and Y with $2y+1$ vertices, to a $2(x+y+1)$-tournament. This is possible if we orient each arc between X and Y in the direction of Y.

Theorem 1. *Let be a, b, c, d four nonnegative integers with $bcd > 0$. Then there exists a tournament T with score set $S = \{a, a+b, a+b+c, a+b+c+d\}$.*

Proof. Let $a + b \geqslant 2a + 1$. Then we can construct the demanded tournament T out of a regular tournament with $2a+1$ vertices and a tournament with score set $\{b-a-1, c+b-a-1, c+d+b-a-1\}$ which exists by Theorem 6 in [2].

Therefore we can assume

$$b \leq a. \tag{1.1}$$

The cardinality of a tournament with a two element score set $\{a, b\}$ is at most $2(a+b)$ following the construction in Theorem 1 in [2]. Therefore we can assume

$$c \leq a + b - 1. \tag{1.2}$$

The cardinality of a 3-tournament with score set $\{a, b, c\}$ is at most $\max\{2(a+b)+1, 2(b+c)\}$ following the construction in Lemma 4 resp. Lemma 5 in [2]. Therefore $a+b+c+d \geq \max\{2(a+b)+1, 2(b+c)\}$ implies the existence of our tournament T. Hence it can be assumed for $c > a$ that $a + d \leq b + c - 1$ or for $c \leq a$ that $c + d \leq a + b$. Combining this with (1.1) and (1.2), we have

$$d \leq a + b - 1. \tag{1.3}$$

(I) In the first part of the proof we assume that

$$b + 2c + 2d \geq 2(a+1). \tag{1.4}$$

We choose a regular tournament with $2a+1$ vertices and a $(b+c+d-a-1)$-regular tournament and combine both to a tournament with score set $\{a, a+b+c+d\}$ and score-sequence

$$(\underbrace{a, \ldots, a}_{2a+1}, \underbrace{a+b+c+d, \ldots, a+b+c+d}_{2(b+c+d-a)-1}).$$

This is possible because $b+c+d \geq a+1$ holds by (1.4).

Let $m_1 b = m_2 d$ with $m_1 = d$, $m_2 = b$. Then we add b to each of m_1 scores a and subtract d from each of m_2 scores $a+b+c+d$ to obtain the sequence

$$(\underbrace{a, \ldots, a}_{2a+1-m_1}, \underbrace{a+b, \ldots, a+b}_{m_1}, \underbrace{a+b+c, \ldots, a+b+c}_{m_2},$$

$$\underbrace{a+b+c+d, \ldots, a+b+c+d}_{2(b+c+d-a)-1-m_2}).$$

Since $2a \geq m_1 = d$ (see 1.3)) and $2b + 2c + 2d \geq 2a + 1 + b$ (see (1.4)) it follows that each score appears at least one time.

To check that our score-sequence is realizable we only have to prove:

$$\begin{aligned}
&(2a+1-m_1)a + m_1(a+b) + m_2(a+b+c) \geq (2a+1+m_2)(a+\tfrac{1}{2}m_2)\\
&\Leftrightarrow m_1 b + m_2 b + m_2 c \geq \tfrac{1}{2}m_2(2a+1+m_2)\\
&\Leftrightarrow 2d + 2b + 2c \geq 2a + 1 + b,
\end{aligned}$$

which is a consequence of our assumption (1.4).

(II) Now we take a $(a+b)$-regular tournament with $2a+2b+1$ vertices and examine the realizability of the score-sequence

$$(\underbrace{a,\ldots,a}_{m_1},\underbrace{a+b,\ldots,a+b}_{\substack{2(a+b)+1\\-m_1-m_2-m_3}},\underbrace{a+b+c,\ldots,a+b+c}_{m_2},$$
$$\underbrace{a+b+c+d,\ldots,a+b+c+d}_{m_3}),$$

with $m_1 b = m_2 c + m_3(c+d)$. For this we choose $m_1 = 2c+d$, $m_2 = m_3 = b$, and assume

$$2a+1 \geqslant 2(c+d), \tag{1.5}$$

which implies $2a+2b \geqslant 2c+d+2b$.

We have to check the following inequalities (see (1)):

(a) $m_1 a \geqslant \binom{m_1}{2} \Leftrightarrow 2a+1 \geqslant 2c+d$, which follows from (1.5).

(b) $m_1 a + (2(a+b)+1-m_1-m_2-m_3)(a+b) = (2c+d)a + (2a+1-2c-d)(a+b) \geqslant (2a+1)$, which holds at once for $2a+1 \geqslant 2c+d$.

(c) $m_1 a + (2(a+b)+1-m_1-m_2-m_3)(a+b) + m_2(a+b+c)$
$= (2c+d)a + (2a+1-2c-d)(a+b) + b(a+b+c)$
$\geqslant (2a+b+1)(a+\frac{1}{2}b)$
$\Leftrightarrow (2a+1+b-2c-d)b + bc \geqslant \frac{1}{2}(2a+b+1)b$
$\Leftrightarrow 2(2a+1+b-2c-d) + 2c \geqslant 2a+b+1$
$\Leftrightarrow 2a+1+b \geqslant 2c+2d$, which follows from (1.5).

Hence the sequence is realizable.

Thus Theorem 1 is proved, because (1.4) or (1.5) holds for each set of nonnegative integers $\{a, b, c, d\}$. □

The proof of the following theorem is more complicated and demands some new ideas.

We have to check the realizability of several score-sequences. In particular we have to insure a positive number of vertices with each score. Second, when manipulating known score-sequences we have to insure balance, i.e., the amount subtracted from some scores must equal the amount added to other scores, which means that the equality in (1) holds. And we have to check the four inequalities which have to be satisfied (see (1)). These conditions will be denoted by (pos), (bal) and (ine).

Theorem 2. *Let be a, b, c, d, e five nonnegative integers with $bcde > 0$. Then there exists a tournament T with score set $S = \{a, a+b, a+b+c, a+b+c+d, a+b+c+d+e\}$.*

Proof. As we have mentioned in the proof of Theorem 1, the following inequalities may be assumed to hold:

$$b \leqslant a, \tag{2.1}$$

$$c \leqslant a + b - 1, \tag{2.2}$$

$$d \leqslant a + b - 1, \tag{2.3}$$

where

$$a + d \leqslant b + c - 1, \quad \text{if } c > a, \quad \text{or} \tag{2.3.1}$$

$$c + d \leqslant a + b, \quad \text{if } c \leqslant a, \text{ hold.} \tag{2.3.2}$$

Using the proof of Theorem 1 we can assume

$$1 + a + b + c + d + e \leqslant \max\{2a + 2b + 1, 2(b + c + d)\},$$

which implies

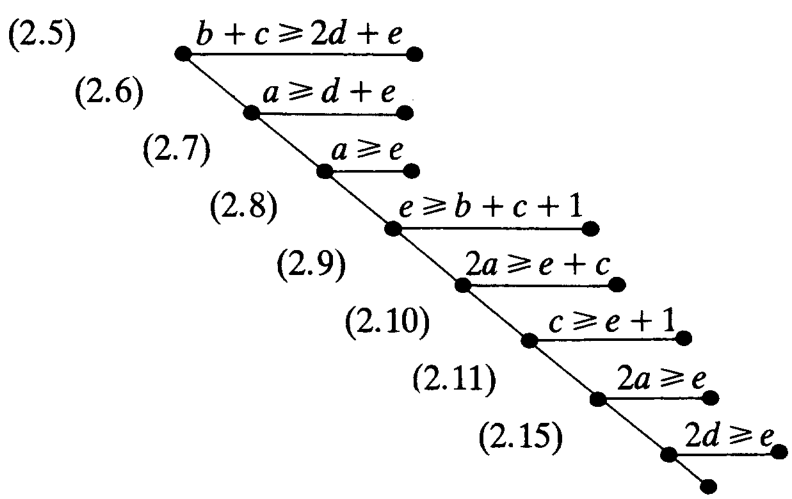
$$1 + a + e \leqslant b + c + d, \quad \text{if } a < c + d, \quad \text{or} \tag{2.4.1}$$

$$c + d + e \leqslant a + b, \quad \text{if } a \geqslant c + d. \tag{2.4.2}$$

Thus we can assume

$$e \leqslant a + b + c - 2. \tag{2.4}$$

Now the proof follows a sequence of inequalities as we show in the decision tree below



If one of the inequalities holds, then Theorem 2 will be shown to follow. Thus we assume the opposite inequality (i.e., $a + 1 \leqslant e$ in (2.7)), going to the next case until we have finished the proof.

(I) We choose a regular tournament with $2(a + b + c) + 1$ vertices and investigate the realizability of the following score-sequence:

$$(\underbrace{a, \ldots, a}_{m_1}, \underbrace{a + b, \ldots, a + b}_{m_2}, \underbrace{a + b + c, \ldots, a + b + c}_{2(a+b+c)+1-m_1-m_2-m_3-m_4},$$

$$\underbrace{a + b + c + d, \ldots, a + b + c + d}_{m_3}, \underbrace{a + b + c + d + e, \ldots, a + b + c + d + e}_{m_4}).$$

If this sequence is realizable, then the following must hold: $m_1(b+c)+m_2c = m_3d+m_4(d+e)$.

(A) At first we assume $b+c \geqslant 2d+e$. Then we choose $m_1(b+2c-d-e) = m_3d$, with $m_3 = b+2c-d-e$ and $m_1 = m_2 = m_4 = d$, (bal).

(a) $2a+1 \geqslant m_1 = d$ holds at once using (2.3).

(b) $m_1a+m_1(a+b) \geqslant d(2d-1) \Leftrightarrow 1+2a+b \geqslant 2d$, which follows from (2.2) and our assumption.

(c) $m_1a+m_1(a+b)+(2(a+b+c)+1-3m_1-m_3)(a+b+c) \geqslant (2(a+b+c)+1-m_1-m_3)(a+b+c-\frac{1}{2}(m_1+m_3))$

$\Leftrightarrow \frac{1}{2}(m_1+m_3)(2(a+b+c)+1-m_1-m_3) \geqslant m_1(b+2c)$

$\Leftrightarrow 2(a+b+c)+1 \geqslant m_1+m_3+2m_1(b+2c)/(m_1+m_3) = b+2c-e+2((m_1+m_3)d+m_1e)/(m_1+m_3)$

$\Leftrightarrow 2a+b+1+e \geqslant 2d+2de/(b+2c-e)$,

which holds if $2a+b+1 \geqslant 2d+e$, a consequence of our assumption.

(d) $m_1a+m_1(a+b)+(2(a+b+c)+1-3m_1-m_3)(a+b+c)+m_3(a+b+c+d) \geqslant (2(a+b+c)+1-m_1)(a+b+c-\frac{1}{2}m_1)$

$\Leftrightarrow m_3d+\frac{1}{2}m_1(2(a+b+c)+1-m_1) \geqslant m_1(b+2c)$

$\Leftrightarrow 2(a+b+c)+1 \geqslant m_1+2(m_1(b+2c)-m_3d)/m_1 = 3d+2e$,

an immediate consequence of our assumption.

At last we have to prove:

$$2(a+b+c) \geqslant b+2c-d-e+3d \Leftrightarrow 2a+b \geqslant 2d-e,$$

which also follows from our assumption, (pos).

Thus we can assume in the following:

$$1+b+c \leqslant 2d+e. \tag{2.5}$$

(B) Now we choose $m_2c = m_1(2d+e-b-c)$, with $m_1 = m_3 = m_4 = c$, $m_2 = (2d+e-b-c)$, (bal).

This is possible because $2d+e \geqslant b+c+1$ holds (see (2.5)). Let us assume $a \geqslant d+e$.

(a) $2a+1 \geqslant m_1 = c$ holds at once ((2.2)).

(b) $m_1a+m_2(a+b) \geqslant (m_1+m_2-1)(\frac{1}{2}(m_1+m_2))$

$\Leftrightarrow 2a+1+2bm_2/(m_1+m_2) \geqslant 2d+e-b$,

which follows from our assumption.

(c) $m_1a+m_2(a+b)+(2(a+b+c)+1-3m_1-m_2)(a+b+c) \geqslant (2(a+b+c)+1-2m_1)(a+b+c-m_1)$

$\Leftrightarrow 2(a+b+c)+1-3m_1-m_2 \geqslant b$

$\Leftrightarrow 2a+2b+1 \geqslant 2d+e$,

which is an implication of $a \geqslant d+e$.

(d) $m_1a+m_2(a+b)+(2(a+b+c)+1-3m_1-m_2)(a+b+c)+m_1(a+b+c+d) \geqslant (2(a+b+c)+1-m_1)(a+b+c-\frac{1}{2}m_1)$

$\Leftrightarrow m_1d+\frac{1}{2}m_1(2(a+b+c)+1-m_1) \geqslant m_1(b+c)+m_2c$

$\Leftrightarrow 2(a+b+c)+1 \geqslant c+2d+2e$,

which also holds.

At last $2a+2b+2c+1 \geq 3c+2d+e-b-c+1 \Leftrightarrow 2a+3b+1 \geq 2d+e+1$ follows at once, (pos).

Hence we can assume in the following:

$$1+a \leq d+e. \tag{2.6}$$

(II) From (2.6) we see that $b+c+d+e \geq a+1$, so that we can combine a regular tournament with $2a+1$ vertices and a regular tournament with $2(b+c+d+e-a)-1$ vertices (see Theorem 1) and investigate the realizability of score-sequences like the following:

$$(\underbrace{a, \ldots, a,}_{\substack{2a+1\\-m_1-m_2}} \underbrace{a+b, \ldots, a+b,}_{m_1} \underbrace{a+b+c, \ldots, a+b+c,}_{m_2}$$

$$\underbrace{a+b+c+d, \ldots, a+b+c+d,}_{m_3} \underbrace{a+b+c+d+e, \ldots, a+b+c+d+e}_{2(b+c+d+e)-2a-1-m_3}).$$

(A) Let us assume $a \geq e$. Then we choose $m_1 = m_2 = e$, $m_3 = 2b+c$, such that $m_1(2b+c) = m_3 e$, (bal).
(2.6) implies $2(d+e) \geq 2a+1$ such that $2(b+c+d+e) \geq 2a+2+2b+c$ holds at once, (pos).

The first three inequalities we have to prove, (ine), hold at once, because there exist regular tournaments with $2a+1$ vertices. The same holds in the next subcases of II.

(d) $(2a+1-2m_1)a + m_1(a+b) + m_1(a+b+c) + m_3(a+b+c+d)$
$\geq (2a+1+m_3)(a+\frac{1}{2}m_3)$
$\Leftrightarrow (2m_1+m_3)b + (m_1+m_3)c + m_3 d \geq \frac{1}{2}m_3(2a+1+m_3)$
$\Leftrightarrow 2(b+c+d+e) \geq 2a+1+2b+c$ (see above).

Thus we can assume

$$1+a \leq e. \tag{2.7}$$

(B) Let us assume $e \geq b+c+1$. Thus we can choose $m_1 = e-b-c$, $m_2 = m_3 = b$, such that $m_1 b = m_2(e-b-c)$ holds, (bal).

$2a \geq e-b-c+b \Leftrightarrow 2a+c \geq e$, which follows from (2.4), and

$2(b+c+d+e) \geq 2a+2+b$, which follows from (2.7), (pos).

(d) $(2a+1-m_1-m_2)a + m_1(a+b) + m_2(a+b+c) + m_2(a+b+c+d)$
$\geq (2a+1+m_2)(a+\frac{1}{2}m_2)$
$\Leftrightarrow m_1 b + m_2(2(b+c)+d) \geq (2a+1+m_2)\frac{1}{2}m_2$
$\Leftrightarrow 2(e-b-c+2(b+c)+d) \geq 2a+1+b$ follows from (2.7).

Hence

$$e \leq b+c \tag{2.8}$$

can be assumed.

(C) Next let us assume that $2a \geq e + c$. Pick $m_2 = e - b$, $m_1 = m_3 = b + c$. Note that $bm_1 + (b + c)m_2 = em_3$ (bal). Clearly, $m_1 = m_3 > 0$, and by (2.1) and (2.7), $m_2 \geq 1$. By assumption, $2a + 1 - m_1 - m_2 > 0$. To complete the check of (pos), use (2.6) and (2.7) to see that $d + 2e > 2a + 1$ from which we see that $2(b + c + d + e) - 2a - 1 - m_3 > 0$.

(d) $(2a + 1 - m_1 - m_2)a + m_1(a + b) + m_2(a + b + c) + m_1(a + b + c + d)$
$\geq (2a + 1 + m_1)(a + \frac{1}{2}m_1)$
$\Leftrightarrow m_1(2b + c + d) + m_2(b + c) \geq (2a + 1 + m_1)\frac{1}{2}m_1$
$\Leftrightarrow 2(b + c + d + e) \geq 2a + 1 + b + c.$

The last inequality follows from (2.6) and (2.7) as in the above check of (pos).

Thus we can assume

$$2a + 1 \leq e + c. \tag{2.9}$$

(III) With (2.9) we get $c + d + e \geq a + b + 1$, such that we can combine a regular tournament with $2(a + b) + 1$ vertices and a regular tournament with $2(c + d + e - a - b) - 1$ vertices and look at the following score-sequence:

$$(\underbrace{a, \ldots, a}_{m_1}, \underbrace{a + b, \ldots, a + b}_{\substack{2(a+b)+1 \\ -m_1-m_2}}, \underbrace{a + b + c, \ldots, a + b + c}_{m_2},$$
$$\underbrace{a + b + c + d, \ldots, a + b + c + d}_{m_3}, \underbrace{a + b + c + d + e, \ldots, a + b + c + d + e}_{2(c+d+e-a-b)-1-m_3}).$$

Let us assume $c \geq e + 1$. Pick $m_1 = c - e$, $m_2 = m_3 = b$, so that $bm_1 + em_3 = cm_2$ (bal).

$2(a + b) \geq c + b - e$ follows from (2.2) and

$$2(c + d + e) \geq 2a + 3b + 2 \tag{*}$$

from (2.5), (2.9) and our assumption $c \geq e + 1$, (pos).

(a) $2a + 1 \geq c - e$ holds at once.

(b) $m_1a + (2(a + b) + 1 - m_1 - m_2)(a + b) \geq (2(a + b) + 1 - m_2)(a + b - \frac{1}{2}m_2)$
$\Leftrightarrow \frac{1}{2}m_2(2(a + b) + 1 - m_2) \geq m_1b$
$\Leftrightarrow 2a + b + 1 \geq 2(c - e)$ follows from (2.7) and (2.2), because $c - e \leq a + b - a = b$.

(c) $m_1a + (2(a + b) + 1 - m_1 - m_2)(a + b) + m_2(a + b + c) \geq (2(a + b) + 1)(a + b)$
$\Leftrightarrow m_2c \geq m_1b$ holds at once.

(d) $m_1a + (2(a + b) + 1 - m_1 - m_2)(a + b) + m_2(a + b + c) + m_2(a + b + c + d)$
$\geq (2(a + b) + 1 + m_2)(a + b + \frac{1}{2}m_2)$
$\Leftrightarrow m_2(2c + d) \geq m_1b + \frac{1}{2}m_2(2(a + b) + 1 + m_2)$
$\Leftrightarrow 2(c + d + e) \geq 2a + 3b + 1$ (see (*) above).

Therefore we can assume

$$e \geq c. \tag{2.10}$$

(IV) With (2.7) and (2.8) we get $b+c \geqslant a+1$ and from (2.10) $a+d+e \geqslant b+c+1$. Hence we can construct a tournament with score-sequence:

$$(\underbrace{a, \ldots, a}_{2a+1}, \underbrace{a+b+c, \ldots, a+b+c}_{2(b+c)-2a-1}, \underbrace{a+b+c+d+e, \ldots, a+b+c+d+e}_{2(a+d+e-b-c)+1}).$$

Now we add b to each of $m_1 = e$ scores a and subtract e from each of $m_2 = b$ scores $a+b+c+d+e$. Note that $m_1 b = m_2 e$ (bal). This yields the new sequence:

$$(\underbrace{a, \ldots, a,}_{2a+1-m_1} \underbrace{a+b, \ldots, a+b,}_{m_1} \underbrace{a+b+c, \ldots, a+b+c,}_{2(b+c)-2a-1}$$
$$\underbrace{a+b+c+d, \ldots, a+b+c+d,}_{m_2} \underbrace{a+b+c+d+e, \ldots, a+b+c+d+e}_{2(a+d+e-b-c)+1-m_2}).$$

We assume that $2a \geqslant e$. $2(a+d+e) \geqslant 2(b+c)+b+1$ follows from (2.5), (2.10) and (2.1), (pos).

Again only the fourth inequality of (ine) needs to be verified here.

(d) $(2a+1-m_1)a + m_1(a+b) + (2(b+c-a)-1)(a+b+c) + m_2(a+b+c+d) \geqslant (2(b+c)+m_2-1)(b+c+\frac{1}{2}m_2)$

$\Leftrightarrow$ $(2(b+c)+m_2)a + m_2 d \geqslant (2a-m_1)b + 2ac + \frac{1}{2}m_2(2(b+c)+m_2-1)$

$\Leftrightarrow$ $m_2 a + m_1 b + m_2 d \geqslant \frac{1}{2}m_2(2(b+c)+m_2-1)$

$\Leftrightarrow$ $2(a+d+e) \geqslant 2(b+c)+b-1$ (see above).

Thus we can assume in the following

$$e \geqslant 2a+1. \tag{2.12}$$

With (2.11) and (2.8) we get $b+c \geqslant 2a+1$, which implies, using (2.1),

$$c \geqslant a+1. \tag{2.12}$$

(2.12) implies $1+a+d \leqslant b+c$, see (2.3.1), and therefore

$$1+d \leqslant c, \quad \text{using (2.1).} \tag{2.13}$$

Also (2.12) implies $1+a+e \leqslant b+c+d$ (see (2.4.1)), such that

$$1+e \leqslant c+d \tag{2.14}$$

can be deduced using (2.1).

Now we handle the final subcase.

(V) We take a regular tournament with $2(a+b+c+d)+1$ vertices and investigate the score-sequence:

$$(\underbrace{a, \ldots, a,}_{m_1} \underbrace{a+b, \ldots, a+b,}_{m_2} \underbrace{a+b+c, \ldots, a+b+c,}_{m_3}$$
$$\underbrace{a+b+c+d, \ldots, a+b+c+d,}_{\substack{2(a+b+c+d)+1-m_1\\ -m_2-m_3-m_4}} \underbrace{a+b+c+d+e, \ldots, a+b+c+d+e}_{m_4}).$$

The equality $m_1(b+c+d)+m_2(c+d)+m_3d=m_4e$ must hold to insure balance.

(A) We choose $m_1=e-d,\ m_2=d,\ m_3=b,\ m_4=b+c+d$, (bal).
$e\geq d+1$ is an implication of (2.10) and (2.13).
$2(a+b+c+d)\geq 2b+c+d+e$ follows from (2.14), (pos).
(a) $2a+1\geq e-d$ follows from (2.14) and (2.2).
(b) $m_1a+m_2(a+b)\geq(m_1+m_2-1)(\frac{1}{2}(m_1+m_2))$
$\Leftrightarrow\ 1+2a+(2d/e)b\geq e.$

If we assume $2d\geq e$, then this inequality is a consequence of (2.8) and (2.2).
(c) $m_1a+m_2(a+b)+m_3(a+b+c)\geq(m_1+m_2+m_3-1)(\frac{1}{2}(m_1+m_2+m_3))$
$\Leftrightarrow\ 1+2a+2b(b+d)/(e+b)+2bc/(e+b)$
$=1+2a+2b(b+c+d)/(e+d)\geq e+b,$

which follows from (2.8) and (2.2).
(d) $m_1a+m_2(a+b)+m_3(a+b+c)+(2(a+b+c+d)+1-m_1-m_2$
$-m_3-m_4)(a+b+c+d)$
$\geq(2(a+b+c+d)+1-m_4)(a+b+c+d-\frac{1}{2}m_4)$
$\Leftrightarrow\ \frac{1}{2}m_4(2(a+b+c+d)+1-m_4)\geq m_1b+(m_1+m_2)c+(m_1+m_2+m_3)d=m_4e$
$\Leftrightarrow\ 2a+b+c+d+1\geq 2e$, which follows from (2.14), (2.8) and (2.2).

Hence we can assume

$$e\geq 2d+1. \tag{2.15}$$

(B) Now we choose $m_1=d,\ m_2=e-2d,\ m_3=c+d-b,\ m_4=c+d$. This is possible, because (2.15) and (2.12) hold and the score-sequence is balanced.
$2(a+b+c+d)\geq e+2c+d-b\Leftrightarrow 2a+3b+d\geq e$ follows from (2.8), (pos).
(a) $2a+1\geq d$ holds at once, see (2.3).
(b) $m_1a+m_2(a+b)\geq(\frac{1}{2}(m_1+m_2))(m_1+m_2-1)$
$\Leftrightarrow\ 1+2a+2b(e-2d)/(e-d)\geq e-d$, which follows from (2.14) and (2.2).
(c) $m_1a+m_2(a+b)+m_3(a+b+c)\geq(m_1+m_2+m_3-1)(\frac{1}{2}(m_1+m_2+m_3))$
$\Leftrightarrow\ 1+2a+2b(e+c-d-b)/(e+c-b)+2c(c+d-b)/(e+c-b)$
$=1+2a+2b(e-(b+d))/(e+c-d)+2c(c+d)/(e+c-d)\geq e+c-b.$
$e\geq b+d$ follows from (2.11) and (2.15).
$2(c+d)\geq e+c-b$ is an implication of (2.14).

Thus we get $1+2a+c+b\geq e+c$ (see (2.8) and (2.2)).
(d) $m_1a+m_2(a+b)+m_3(a+b+c)+(2(a+b+c+d)+1-m_1-m_2$
$-m_3-m_4)(a+b+c+d)$
$\geq(2(a+b+c+d)+1-m_4)(a+b+c+d-\frac{1}{2}m_4)$
$\Leftrightarrow\ \frac{1}{2}m_4(2(a+b+c+d)+1-m_4)\geq m_1b+(m_1+m_2)c$
$+(m_1+m_2+m_3)d=m_4e$
$\Leftrightarrow\ 2(a+b+c+d)+1\geq 2e+c+d$
$\Leftrightarrow\ 2a+2b+c+d+1\geq 2e$, which follows from (2.8), (2.14) and (2.2).

Hence Theorem 2 is proved. □

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References

[1] J.W. Moon, Topics on Tournaments (Holt, Rinehart and Winston, New York, 1968).
[2] K.B. Reid, Score sets for tournaments, Proc. 9th S-E Conf. Combinatorics, Graph Theory, and Computing, Congressus numerantium XXI, Utilitas Math. (1978) 607–618.
[3] K.B. Reid and L.W. Beineke, Tournaments, Selected Topics in Graph Theory, L.W. Beineke and R.J. Wilson, eds. (Academic Press, New York, 1979).