Landau's Inequalities for Tournament Scores and a Short Proof of a Theorem on Transitive Sub-Tournaments

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January 4, 2000

Abstract

Ao and Hanson, and Guiduli, Gyárfás, Thomassé and Weidl independently, proved the following result: For any tournament score sequence $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \cdots \leq s_n$, there exists a tournament T on vertex set $\{1, 2, \ldots, n\}$ such that the score of each vertex i is s_i and the sub-tournaments of T on both the even and the odd indexed vertices are transitive in the given order; that is, i dominates j whenever $i > j$ and $i \equiv j \pmod 2$. In this note, we give a much shorter proof of the result. In the course of doing so, we show that the score sequence of a tournament satisfies a set of inequalities which are individually stronger than the well-known set of inequalities of Landau, but collectively the two sets of inequalities are equivalent.

1 Introduction

A *tournament* $T = (V, E)$ of order n is a loopless digraph with vertex set $V = \{1, 2, \ldots, n\}$ such that each pair of distinct vertices i and j is joined by exactly one of the arcs (i, j) and (j, i). If $(i, j) \in E$, we say that i *dominates* j. The *score* s_i, also called the *outdegree* $\deg^+(i)$, of vertex i is the number of vertices dominated by i. The *score sequence* of a tournament is the sequence of scores of its vertices. The vertices of T may be labelled so that $s_1 \leq s_2 \leq \cdots \leq s_n$. In 1953, Landau characterized the score sequence of a tournament.

*Supported by an NSERC Postdoctoral Fellowship.

Lemma 1 (Landau [5]) *A sequence of integers $S = (s_1, s_2, \dots, s_n)$ with $s_1 \le s_2 \le \dots \le s_n$ is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \dots, n\}$,*

$$\sum_{i \in I} s_i \ge \binom{|I|}{2} \tag{1}$$

with equality when $|I| = n$.

Because of the monotonicity assumption $s_1 \le s_2 \le \dots \le s_n$, the inequalities (1), called *the Landau inequalities*, are equivalent to

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2}, \quad (k = 1, 2, \dots, n). \tag{2}$$

Lemma 1 asserts that if a sequence of integers $S = (s_1, s_2, \dots, s_n)$ satisfies the Landau inequalities, then there is a tournament on n vertices with score sequence S. Recently, Guiduli, Gyárfás, Thomassé and Weidl [4] investigated the following general problem: For a given tournament score sequence S, does there exist a tournament having not only this score sequence but also other prescribed properties? They proved that there is always a tournament T on vertex set $\{1, 2 \dots, n\}$ such that $\deg^+(i) = s_i$ with the additional property that the sub-tournaments of T on the even and the odd indexed vertices, respectively, are transitive in the given order; that is, i dominates j whenever $i > j$ and $i \equiv j \pmod 2$. This theorem is a strengthening of one direction of Landau's theorem in that it asserts the existence of a tournament with a score sequence satisfying (1) whose vertices can be partitioned, in a very specific way, into two transitive sub-tournaments of orders that are as equal as possible. A consequence of this result is that in constructing a tournament with a given score sequence S one may fix in advance $\lfloor n^2/4 \rfloor - \lfloor n/2 \rfloor$ arcs, which is roughly half of the total number of arcs. Coincidentally, Ao and Hanson [1] independently obtained the same result by using a different approach. Ao and Hanson went on to use this tournament construction to prove a conjecture of Bagga, Beineke and Harary [2]. We note that both proofs of the above result in [1] and [4] are very long and complicated. In this note, we give a much shorter proof.

2 Landau Inequalities and Transitive Sub-Tournaments

Our proof of the theorem on transitive subtournaments is based on a strengthening of Landau's theorem in the other direction. We show that the score sequence of a tournament of order n satisfies inequalities, (3) below, that are individually stronger than the inequalities (1), although collectively the two sets of inequalities are equivalent.

Theorem 1 *A sequence of integers $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \le s_2 \le \cdots \le s_n$ is a score sequence of a tournament if and only if for every subset $I \subseteq [n]$,*

$$\sum_{i \in I} s_i \ge \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{2} \binom{|I|}{2} \tag{3}$$

with equality when $|I| = n$.

Proof. The sufficiency part of the theorem follows from Lemma 1 since

$$\frac{1}{2} \sum_{i \in I} (i-1) \ge \frac{1}{2} \sum_{i=1}^{|I|} (i-1) = \frac{1}{2} \binom{|I|}{2}.$$

Thus the inequalities (3) imply the inequalities (1).

We now prove that the score vector of a tournament satisfies (3). Suppose that $S = (s_1, s_2, \ldots, s_n)$ is a score sequence of a tournament such that $s_1 \le s_2 \le \cdots \le s_n$. For any subset $I \subseteq [n]$, define

$$f(I) := \sum_{i \in I} s_i - \frac{1}{2} \sum_{i \in I} (i-1) - \frac{1}{2} \binom{|I|}{2}.$$

We choose I firstly to have $f(I)$ minimum and secondly to have $|I|$ minimum.

Claim: $I = \{i : 1 \le i \le |I|\}$.
Otherwise, there exist $i \notin I$ and $j \in I$ such that $j = i + 1$. Then $s_i \le s_j$. Since

$$s_j - \frac{1}{2}(j + |I| - 2) = f(I) - f(I \setminus \{j\}) < 0$$

and

$$s_i - \frac{1}{2}(i + |I| - 1) = f(I \cup \{i\}) - f(I) \ge 0,$$

we have

$$\frac{1}{2}(i + |I| - 1) \le s_i \le s_j < \frac{1}{2}(j + |I| - 2) = \frac{1}{2}(i + |I| - 1),$$

a contradiction. This proves the claim.

Therefore

$$f(I) = \sum_{i=1}^{|I|} s_i - \frac{1}{2} \sum_{i=1}^{|I|} (i-1) - \frac{1}{2} \binom{|I|}{2} = \sum_{i=1}^{|I|} s_i - \binom{|I|}{2} \ge 0,$$

where the last inequality follows from Lemma 1. By the choice of the subset I, Theorem 1 follows. $\square$

We remark that equality can occur often in (3). For instance, equality holds for regular tournaments of odd order n (score sequence equal to $(n-1/2, \ldots, n-1/2)$) whenever $|I| = k$ and $I = \{n-k+1, \ldots, n\}$. In the next section we shall examine the equality case more closely.

We also remark that Theorem 1 is best possible in the sense that, for any real $\epsilon > 0$, the inequality

$$\sum_{i \in I} s_i \geq \left(\frac{1}{2} + \epsilon\right) \sum_{i \in I} (i-1) + \left(\frac{1}{2} - \epsilon\right) \cdot \binom{|I|}{2}$$

fails for some I and some tournaments (for example, regular tournaments).

Now that Theorem 1 gives a lower bound for $\sum_{i \in I} s_i$, it is natural to ask for a upper bound for $\sum_{i \in I} s_i$. The following set of upper bounds for $\sum_{i \in I} s_i$ in Corollary 1 is equivalent to the set of lower bounds for $\sum_{i \in I} s_i$ in Theorem 1.

Corollary 1 *A sequence of integers $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \cdots \leq s_n$ is a score sequence of a tournament if and only if for any subset $I \subseteq [n]$,*

$$\sum_{i \in I} s_i \leq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{4} |I| \left(2n - |I| - 1\right),$$

with equality when $|I| = n$.

Proof. Let $J = [n] \setminus I$. Then

$$\sum_{i \in I} s_i \leq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{4} |I| \left(2n - |I| - 1\right),$$

with equality when $|I| = n$, if and only if

$$\sum_{i \in J} s_i = \binom{n}{2} - \sum_{i \in I} s_i \geq \frac{1}{2} \sum_{i \in J} (i-1) + \frac{1}{2} \binom{|J|}{2},$$

with equality when $|J| = n$. Therefore Corollary 1 follows from Theorem 1. □

Theorem 1 and Corollary 1 extend the following corollary due to Landau in 1953.

Corollary 2 (Landau [5]) *Suppose $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \cdots \leq s_n$ is a score sequence of a tournament. Then, for each i,*

$$\frac{i-1}{2} \leq s_i \leq \frac{n+i-2}{2}.$$

Proof. Choose $I = \{i\}$. Then Corollary 2 follows immediately from Theorem 1 and Corollary 1. $\square$

A *bipartite tournament* is an orientation of a complete bipartite graph. The *score sequence* of a bipartite tournament consists of two sequences of integers, which are the outdegrees of the vertices of each vertex class of the bipartite tournament.

Now we are ready to prove in Theorem 2 the result of Ao and Hanson, and Guiduli *et. al.* We make use of the following lemma of Gale-Ryser.

Lemma 2 (Gale [3], Ryser [6]) *Two sequences of integers $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_m)$ are the score sequence of a bipartite tournament if and only if for any $A \subseteq [n]$ and $B \subseteq [m]$,*

$$\sum_{i\in A} s_i + \sum_{j\in B} t_j \geq |A| \cdot |B|,$$

with equality when $|A| = n$ and $|B| = m$.

Theorem 2 (Ao and Hanson [1], Guiduli *et. al.* **[4])** *For any tournament score sequence $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \cdots \leq s_n$, there exists a tournament T on vertex set $\{1, 2, \ldots, n\}$ such that $\deg^+(i) = s_i$ for each i, and i dominates j whenever $i > j$ and $i \equiv j \pmod 2$.*

Proof. Let $V_1 = \{i : 1 \leq i \leq n \text{ and } i \text{ is odd}\}$ and $V_2 = \{i : 1 \leq i \leq n \text{ and } i \text{ is even}\}$. By Corollary 2, we have $s_i \geq (i-1)/2$. This makes it possible to construct arcs within each V_i in the following way: Let i dominate j whenever $i > j$ and $i \equiv j \pmod 2$. Thus the sub-tournament induced by each V_i is transitive, and $\lfloor (i-1)/2 \rfloor$ arcs coming from each vertex i have been utilized. Now it suffices to prove that the sequences of integers $(s_i - (i-1)/2 : i \text{ is odd})$ and $(s_i + 1 - i/2 : i \text{ is even})$ are the score sequence of a bipartite tournament with vertex set partition V_1 and V_2. Let $A \subseteq \{i : 1 \leq i \leq n \text{ and } i \text{ is odd}\}$ and $B \subseteq \{i : 1 \leq i \leq n \text{ and } i \text{ is even}\}$. Then by Theorem 1,

$$\begin{aligned}
& \textstyle\sum_{i\in A} (s_i - (i-1)/2) + \sum_{i\in B} (s_i + 1 - i/2) \\
= \;& \textstyle\sum_{i\in A\cup B} s_i - \frac{1}{2}\sum_{i\in A\cup B}(i-1) + \frac{1}{2}\sum_{i\in B} 1 \\
\geq \;& \tfrac{1}{2}\binom{|A\cup B|}{2} + \tfrac{1}{2}|B| \\
= \;& \tfrac{1}{2}\binom{|A| + |B|}{2} + \tfrac{1}{2}|B| \geq |A| \cdot |B|,
\end{aligned}$$

with equality when $|A| = |V_1|$ and $|B| = |V_2|$. Therefore Theorem 2 is a consequence of Lemma 2. $\square$

3 Equality in the Strengthened Inequalities

In this section, we investigate what it means for equality to occur in one of the inequalities (3). As showed in the next theorem, when an equality occurs, there are implications concerning the strong connectedness and regularity of everyy tournament with the score sequence S.

For any integers r and s with $r \leq s$, we use $[r, \ldots, s]$ to denote the set of all integers between r and s.

Theorem 3 *Let the sequence of integers $S = (s_1, s_2, \ldots, s_n)$ with $s_1 \leq s_2 \leq \cdots \leq s_n$ be a tournament score sequence. If*

$$\sum_{i \in I} s_i = \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{2} \binom{|I|}{2}$$

for some $I \subseteq [n]$, then one of the following holds:

(1) $I = [1, \ldots, |I|]$ *and* $\sum_{i=1}^{|I|} s_i = \binom{|I|}{2}$.

(2) $I = [t, \ldots, t + |I| - 1]$ *for some* t, $2 \leq t \leq n - |I| + 1$, $\sum_{i=1}^{t+|I|-1} s_i = \binom{t+|I|-1}{2}$ *and* $s_i = (t + |I| - 2)/2$ *for all* $i \leq t + |I| - 1$.

(3) $I = [1, \ldots, r] \cup [r + t, \ldots, t + |I| - 1]$ *for some* r *and* t *such that* $1 \leq r \leq |I| - 1$ *and* $2 \leq t \leq n - |I| + 1$; $\sum_{i=1}^{r} s_i = \binom{r}{2}$, $\sum_{i=1}^{t+|I|-1} s_i = \binom{t+|I|-1}{2}$ *and* $s_i = (r + t + |I| - 2)/2$ *for all* i, $r + 1 \leq i \leq t + |I| - 1$.

Conditions (1), (2), *and* (3) *are equivalent to the assertion that every tournament with the score sequence S has one of the three structures shown in Figure* 1. *(The notation T_r is used to denote a sub-tournament on r vertices, and the double arrows mean that all edges between the two parts go in that direction.)*

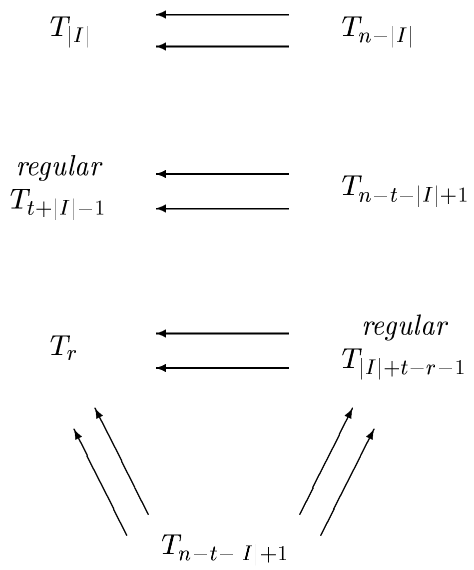

Figure 1.

Proof. For any subset $J \subseteq [n]$, we define $f(J)$ as in Theorem 1. Thus $f(I) = 0$ and $f(J) \geq 0$ for all $J \subseteq [n]$.

We first prove that I is of the form of one of the three types described in Theorem 3. Suppose, on the contrary, that this is not true. Then there are indices i, j, k, l with $j = i+1$ and $l = k+1$ such that $\{i, k\} \cap I = \emptyset$ and $\{j, l\} \subseteq I$. Thus $s_i \leq s_j$ and $s_k \leq s_l$. Since

$$s_j + s_l - \frac{1}{2}(j + l + 2|I| - 5) = f(I) - f(I \setminus \{j, l\}) \leq 0$$

and

$$s_i + s_k - \frac{1}{2}(i + k + 2|I| - 1) = f(I \cup \{i, k\}) - f(I) \geq 0,$$

we have

$$\frac{1}{2}(i + k + 2|I| - 1) \leq s_i + s_k \leq s_j + s_l \leq \frac{1}{2}(j + l + 2|I| - 5) = \frac{1}{2}(i + k + 2|I| - 3),$$

a contradiction. This proves that I satisfies one of (1), (2), or (3). Now the proof is divided into three cases.

Case 1: $I = [1, \ldots, |I|]$. Then $f(I) = \sum_{i=1}^{|I|} s_i - \binom{|I|}{2}$, and so $\sum_{i=1}^{|I|} s_i = \binom{|I|}{2}$.

For the other two cases, we need the following claim:

Claim 1: If there exist indices i and j with $j = i + 1$ such that $i \notin I$ and $j \in I$, then $f(I \cup \{i\}) = f(I \setminus \{j\}) = 0$. In fact, since

$$s_j - \frac{1}{2}(j + |I| - 2) = f(I) - f(I \setminus \{j\}) \leq 0$$

and

$$s_i - \frac{1}{2}(i + |I| - 1) = f(I \cup \{i\}) - f(I) \geq 0,$$

we have

$$\frac{1}{2}(i + |I| - 1) \leq s_i \leq s_j \leq \frac{1}{2}(j + |I| - 2) = \frac{1}{2}(i + |I| - 1),$$

which implies that equalities hold throughout all the above inequalities. Thus $f(I \cup \{i\}) = f(I) = f(I \setminus \{j\}) = 0$.

Case 2: $I = [t, \ldots, t + |I| - 1]$ for some t, $2 \leq t \leq n - |I| + 1$. By applying Claim 1 recursively, we have $f(\{t + |I| - 1\}) = 0$; that is, $s_{t+|I|-1} = (t + |I| - 2)/2$. Since

$$\sum_{i=1}^{t+|I|-1} s_i \geq \binom{t + |I| - 1}{2} \quad \text{and} \quad s_1 \leq s_2 \leq \cdots \leq s_{t+|I|-1},$$

equalities hold throughout all the above inequalities.

Case 3: $I = [1, \ldots, r] \cup [r + t, \ldots, t + |I| - 1]$ for some r and t such that $1 \leq r \leq |I| - 1$ and $2 \leq t \leq n - |I| + 1$. Again, by applying Claim 1 recursively, we have $f([1, \ldots, r]) = f([1, \ldots, t + |I| - 1]) = f([1, \ldots, r] \cup \{t + |I| - 1\}) = 0$. Thus

$$\sum_{i=1}^{r} s_i - \binom{r}{2} = f([1, \ldots, r]) = 0,$$

$$\sum_{i=1}^{t+|I|-1} s_i - \binom{t + |I| - 1}{2} = f([1, \ldots, t + |I| - 1]) = 0,$$

and

$$s_{t+|I|-1} - \frac{1}{2}(|I| + r + t - 2) = f([1, \ldots, r] \cup \{t + |I| - 1\}) - f([1, \ldots, r]) = 0.$$

Thus $s_{t+|I|-1} = (|I| + r + t - 2)/2$. Since

$$\sum_{i=r+1}^{t+|I|-1} s_i = \sum_{i=1}^{t+|I|-1} s_i - \sum_{i=1}^{r} s_i = \binom{t + |I| - 1}{2} - \binom{r}{2} = \frac{1}{2}(|I| + r + t - 2)(|I| - r + t - 1)$$

and $s_{r+1} \leq s_{r+2} \leq \cdots \leq s_{t+|I|-1}$, we have $s_i = (r + t + |I| - 2)/2$ for all i, $r + 1 \leq i \leq t + |I| - 1$.

By combining the above three cases, we complete the proof of Theorem 3. $\square$

4 A Conjecture

We conclude this note with the following conjecture.

Conjecture 1 *Let $n_1, n_2, \ldots, n_t$ be a sequence of integers such that $\sum_{i=1}^{t} n_i = n$ and $n_i \leq (n+1)/2$ for all i. For each tournament score sequence $S = (s_1, s_2, \ldots, s_n)$, there exists a tournament $T = (V, E)$ with score sequence S such that the vertex set V can be partitioned into t pairwise disjoint subsets V_i of cardinality n_i, $1 \leq i \leq t$, where the sub-tournament induced by each V_i is transitive.*

We note that if (1) $n_i \geq (n-1)/2$ for some i, or (2) all n_i are as equal as possible, then Conjecture 1 follows from Theorem 2. We also note that in order to prove Conjecture 1 it suffices to prove it for $t = 3$. For, if $t > 3$ then the index set $[t] = \{1, 2, \ldots, t\}$ can be partitioned into three pairwise disjoint sets I_i, $1 \leq i \leq 3$, such that $m_i := \sum_{j \in I_i} n_j \leq (n+1)/2$ for all $i = 1, 2, 3$. Since a transitive tournament of order m_i can be partitioned into transitive subtournaments of orders $n_j (j \in I_i)$, the conjecture follows from its validity for $t = 3$.

The major difficulty in proving the conjecture for $t = 3$ seems to be in deciding how to partition the vertex set $\{1, 2, \ldots, n\}$ into three sets of the specified cardinalities m_1, m_2, and m_3. In the case $t = 2$, the partition is that determined by the even and odd indexed vertices, respectively. If n is odd, and we consider the score sequence $(n - 1/2, \ldots, n - 1/2)$ of a regular tournament of order n, then the difficulty disappears. In fact, it is not difficult to show that, given integers m_1, m_2, and m_3 with $m_1 \geq m_2 \geq m_3$ and $n = m_1 + m_2 + m_3$, there is a regular tournament T of order n whose vertex set can be partitioned into transitive sub-tournaments of orders m_1, m_2, and m_3. By the comments following Conjecture 1, it may be supposed that $m_1 < (n-1)/2$. We illustrate the construction by specifying the adjacency matrix of T.

Let $J_{a,b}$ denote the a by b matrix of all 1s; let L_k denote the matrix of order k with 0s on and above the main diagonal and 1s below the main diagonal; let M_k denote the matrix of order k with 0s on and below the cross-diagonal (running from lower left to upper right) and 1s above the cross-diagonal; and let P_k be the permutation matrix of order k with 1s on the cross-diagonal and 0s elsewhere. As usual, let I_k denote the identity matrix of order k, and let $O_{a,b}$ denote the a by b zero matrix. Sometimes we omit subscripts on matrices if their size is determined by context. We use X^t to denote the transpose of a matrix X, and X^c to denote $J - X$.

The following is the adjacency matrix of a regular tournament with the required property:

$$\left[\begin{array}{c|c|c|c}
\multicolumn{2}{c|}{L_{m_1}} & \begin{array}{c}(I_{m_2}+L_{m_2})^t\\ \hline O\end{array} & \begin{array}{c|c} \begin{array}{c} J_{m_2,(n-1)/2-m_2} \\ \hline \begin{array}{c|c} J & P_{m_1-m_2}+M_{m_1-m_2}\end{array}\end{array} & \begin{array}{c} O \\ \hline O\end{array}\end{array} \\ \hline
L^t_{m_2} & J_{m_2,m_1-m_2} & L_{m_2} & A \\ \hline
\begin{array}{c} O_{(n-1)/2-m_2,m_2} \\ \hline J\end{array} & \begin{array}{c} O \\ \hline (P_{m_1-m_2}+M_{m_1-m_2})^c \\ \hline J \end{array} & (A^t)^c & L^t_{m_3}
\end{array}\right]$$

The matrix A is a m_2 by m_3 matrix of 0s and 1s with constant row sums equal to $(n-1)/2-(m_1-1)$ and column sums chosen to give a regular tournament. But it follows easily from Lemma 2 that such a matrix A exists.

Acknowledgment. Jian Shen wishes to thank Professor Stéphan Thomassé for some initial discussion on the problem while he visited the University of Lyon, France, in June, 1999. He also thanks Professor Adrian Bondy for hosting the visit.

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