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Growing perfect cubes

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Abstract

An (n, a, b)-perfect double cube is a $b \times b \times b$ sized n-ary periodic array containing all possible $a \times a \times a$ sized n-ary array exactly once as subarray. A growing cube is an array whose $c_j \times c_j \times c_j$ sized prefix is an (n_j, a, c_j)-perfect double cube for $j = 1, 2, \ldots$, where $c_j = n_j^{v/3}$, $v = a^3$ and $n_1 < n_2 < \cdots$. We construct the smallest possible perfect double cube (a $256 \times 256 \times 256$ sized 8-ary array) and growing cubes for any a.

Key words: de Bruijn array, perfect map, colouring
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1 Introduction

Cyclic sequences in which every possible sequence of a fixed length occurs exactly once have been studied for more than a hundred years [6]. The same problem, which can be applied to position localization, was extended to arrays [5].

Let $\mathbb{Z}$ be the set of integers. For $u,\ v \in \mathbb{Z}$ we denote the set $\{j \in \mathbb{Z} \mid u \leq j \leq v\}$ by $[u..v]$ and the set $\{j \in \mathbb{Z} \mid j \geq u\}$ by $[u..\infty]$. Let $d \in [1..\infty]$ and $k,\ n \in [2..\infty]$, $b_i,\ c_i,\ j_i \in [1..\infty]$ $(i \in [1..d])$ and $a_i,\ k_i \in [2..\infty]$ $(i \in [1..d])$. Let $\mathbf{a} = \langle a_1, a_2, \ldots, a_d \rangle$, $\mathbf{b} = \langle b_1, b_2, \ldots, b_d \rangle$, $\mathbf{c} = \langle c_1, c_2, \ldots, c_d \rangle$, $\mathbf{j} = \langle j_1, j_2, \ldots, j_d \rangle$

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[1] Dedicated to the sixtieth birthday of Miklós Simonovits

and $\mathbf{k} = \langle k_1, k_2, \ldots, k_d \rangle$ be vectors of length d, $\mathbf{n} = \langle n_1, n_2, \ldots \rangle$ an infinite vector with $2 \leq n_1 < n_2 < \cdots$.

Definition 1 *A d dimensional n-ary array A is a mapping $A : [1..\infty]^d \rightarrow [0, n-1]$. If there exist a vector $\mathbf{b}$ and an array M such that*

$$\forall \mathbf{j} \in [1..\infty]^d : A[\mathbf{j}] = M[(j_1 \bmod b_1) + 1, (j_2 \bmod b_2) + 1, \ldots, (j_d \bmod b_d) + 1],$$

then A is a ***b-periodic*** *array and M is a period of A. The* **a***-sized* ***subarrays*** *of A are the* **a***-periodic n-ary arrays.*

Although our arrays are infinite we say that a **b**-periodic array is **b**-sized.

Definition 2 ***Indexset*** *A_{index} of a* **b***-periodic array A is the Cartesian product*

$$A_{index} = \times_{i=1}^{d} [1..b_i].$$

Definition 3 *A d dimensional* **b***-periodic n-ary array A is called* ***$(n, d, \mathbf{a}, \mathbf{b})$-perfect****, if all possible n-ary arrays of size* **a** *appear in A exactly once as a subarray.*

Here n is the alphabet size, d gives the number of dimensions of the "window" and the perfect array M, the vector **a** characterizes the size of the window, and the vector **b** is the size of the perfect array M.

Definition 4 *An $(n, d, \mathbf{a}, \mathbf{b})$-perfect array A is called* ***c-cellular****, if c_i divides b_i for $i \in [1..d]$. A cellular array consists of $b_1/c_1 \times b_2/c_2 \times \cdots \times b_d/c_d$ disjoint subarrays of size* **c***, called* ***cells.*** *In each cell the element with smallest indices is called the* ***head*** *of the cell. The contents of the cell is called* ***pattern.***

Definition 5 *The product of the elements of a vector* $\mathbf{a}$ *is called the* ***volume*** *of the vector and is denoted by* $|\mathbf{a}|$. *The number of elements of perfect array* M *is called the volume of* M *and is denoted by* $|M|$.

Definition 6 *If* $b_1 = b_2 = \cdots = b_d$, *then the* $(n, d, \mathbf{a}, \mathbf{b})$*-perfect array* A *is called* ***symmetric****. If* A *is symmetric and* $a_1 = a_2 = \cdots = a_d$, *then* A *is called* ***doubly symmetric.*** *If* A *is doubly symmetric and*

(1) $d = 1$, *then* A *is called a* ***double sequence****;*

(2) $d = 2$, *then* A *is called a* ***double square****;*

(3) $d = 3$, *then* A *is called a* ***double cube****.*

According to this definition, all perfect sequences are doubly symmetric. In the case of symmetric arrays we use the notion $(n, d, \mathbf{a}, b)$ and in the case of doubly symmetric arrays we use (n, d, a, b) instead of $(n, d, \mathbf{a}, \mathbf{b})$.

The first known result originates from Flye-Sainte [6] who proved the existence of $(2, 1, a, 2^a)$-perfect sequences for all possible values of a in 1894.

One dimensional perfect arrays are often called de Bruijn [4] or Good [7] sequences. Two dimensional perfect arrays are called also perfect maps [16] or de Bruijn tori [8–10].

De Bruijn sequences of even length – introduced in [11] – are useful in construction of perfect arrays when the size of the alphabet is an even number and the window size is 2×2. Their definition is as follows.

Definition 7 *If* n *is an even integer then an* $(n, 1, 2, n^2)$*-perfect sequence* $M = (m_1, m_2, \ldots, m_{n^2})$ *is called* ***even****, if* $m_i = x$, $m_{i+1} = y$, $x \neq y, m_j = y$ *and* $m_{j+1} = x$ *imply* $j - i$ *is even.*

Iványi and Tóth [11] and later Hurlbert and Isaak [9] provided a constructive proof of the existence of even sequences.

Definition 8 ***Lexicographic indexing*** *of an array* $M = [m_{j_1 j_2 \ldots j_d}] = [m_{\mathbf{j}}]$ $(1 \le j_i \le b_i)$ *for* $i \in [1..d]$ *means that the index* $I(m_{\mathbf{j}})$ *is defined as*

$$I(m_{\mathbf{j}}) = j_1 - 1 + \sum_{i=2}^{d} \left((j_i - 1) \prod_{m=1}^{i-1} b_m \right).$$

The concept of perfectness can be extended to infinite arrays in various ways. In *growing arrays* [9] the window size is fixed, the alphabet size is increasing and the prefixes grow in all d directions.

Definition 9 *Let* a *and* d *be positive integers with* $a \ge 2$ *and* $\mathbf{n} = \langle n_1, n_2, \ldots \rangle$ *be a strictly increasing sequence of positive integers. An array* $M = [m_{i_1 i_2 \ldots i_d}]$ *is called* $(\mathbf{n}, d, a)$***-growing****, if the following conditions hold:*

(1) $M = [m_{i_1 i_2 \ldots i_d}]$ $(1 \le i_j < \infty)$ *for* $j \in [1..d]$*;*

(2) $m_{i_1 i_2 \ldots i_d} \in [0..n-1]$*;*

(3) the prefix $M_k = [m_{i_1 i_2 \ldots i_d}]$ $(1 \le i_j \le n_k^{a^d/d}$ *for* $j \in [1..d])$ *of* M *is* $(n_k, d, a, n_k^{a^d/d})$*-perfect array for* $k \in [0..\infty]$.

For the growing arrays we use the terms growing sequence, growing square and growing cube.

Definition 10 *For* $a, n \in [2..\infty]$ *the* ***new alphabet size*** $N(n, a)$ *is*

$$N(n,a) = \begin{cases} n, & if\ any\ prime\ divisor\ of\ a\ divides\ n, \\ nq, & otherwise, \end{cases} \qquad (1)$$

where q *is the product of the prime divisors of* a *not dividing* n.

Note, that alphabet size n and new alphabet size N have the property that $n \mid N$, furthermore, $n = N$ holds in the most interesting case $d = 3$ and

$n = a_1 = a_2 = a_3 = 2.$

The aim of this paper is to prove the existence of a double cube. As a side-effect we show that there exist $(\mathbf{n}, d, a)$-growing matrices for any n, d and a.

2 Necessary condition and earlier results

Since in the period M of a perfect array A each element is the head of a pattern, the volume of M equals the number of the possible patterns. Since each pattern – among others the pattern containing only zeros – can appear only once, any size of M is greater then the corresponding size of the window. So we have the following necessary condition [2,9]: If M is an $(n, d, \mathbf{a}, \mathbf{b})$-perfect array, then

$$|\mathbf{b}| = n^{|\mathbf{a}|} \quad (2)$$

and

$$b_i > a_i \text{ for } i \in [1..d]. \quad (3)$$

Different construction algorithms and other results concerning one and two dimensional perfect arrays can be found in the fourth volume of *The Art of Computer Programming* written by D. E. Knuth [12]. E.g. a (2,1,5,32)-perfect array [12, page 22], a 36-length even sequence whose 4-length and 16-length prefixes are also even sequences [12, page 62], a (2,2,2,4)-perfect array [12, page 38] and a (4,2,2,16)-perfect array [12, page 63].

It is known [4,12] that in the one-dimensional case the necessary condition (2) is sufficient too. There are many construction algorithms, like the ones of Cock [2], Fan, Fan, Ma and Siu [5], Martin [14] or any algorithm for constructing of directed Euler cycles [13].

Chung, Diaconis and Graham [1] posed the problem to give a necessary and sufficient condition of the existence of $(n, 2, \mathbf{a}, \mathbf{b})$-perfect arrays.

The conditions (2) and (3) are sufficient for the existence of (2,2,**a**,**b**)-perfect arrays [5] and (n,2,a,b)-perfect arrays [15]. Paterson in [16] supplied further sufficient conditions.

Hurlbert and Isaak [9] gave a construction for one and two dimensional growing arrays.

3 Algorithms for constructing growing de Bruijn arrays

In the construction of perfect de Bruijn arrays we use the following algorithms.

Algorithm Martin [14] generates de Bruijn sequences. Its inputs are the alphabet size n and the window size a. Its output is an n-ary perfect sequence of length n^a. The output begins with a zeros and always continues with the maximal permitted element of the alphabet.

Algorithm Even [9] produces even de Bruijn sequences.

Algorithm Mesh [9,11] produces doubly symmetric cellular perfect arrays when n is even, $d = 2$, $a_1 = 2$ and $a_2 = 2$. The input of algorithm Mesh is an even alphabet size n and an even de Bruijn sequence $e_1, e_2, \ldots, e_{n^2}$, the output is an $(n, 2, n^2, n^2)$-perfect array P, whose elements are calculated by the meshing function [11]:

$$P_{ij} = \begin{cases} e_j, & \text{if } i + j \text{ is even}, \\ e_i, & \text{if } i + j \text{ is odd}, \end{cases} \tag{4}$$

Algorithm Shift [2] is a widely usable algorithm to construct perfect arrays. We use it to transform cellular $(N, d, a, \mathbf{b})$-perfect arrays into $(N, d+1, a, \mathbf{c})$-perfect arrays.

We introduce 3 new algorithms.

Cellular results cellular perfect arrays. Its input data are n, d and $\mathbf{a}$, its output is an $(N, d, \mathbf{a}, \mathbf{b})$-perfect array, where $b_1 = N^{a_1}$ and $b_i = N^{a_1 a_2 \ldots a_i - a_1 a_2 \ldots a_{i-1}}$ for $i = 2, 3, \ldots, d$. Cellular consists of five parts:

(1) *Calculation* (line 1 in the pseudocode) determining the new alphabet size N using formula (1);
(2) *Walking* (lines 2–3) if $d = 1$, then construction of a perfect symmetric sequence S_1 using algorithm Martin (walking in a de Bruijn graph);
(3) *Meshing* (lines 4–6) if $d = 2$, N is even and $a = 2$, then first construct an N-ary even perfect sequence $\mathbf{e} = \langle e_1, e_2, \ldots, e_{N^2} \rangle$ using Even, then construct an $N^2 \times N^2$ sized N-ary square S_1 using meshing function (4);
(4) *Shifting* (lines 7–12) if $d > 1$ and (N is odd or $a > 2$), then use Martin once, then use Shift $d - 1$ times, receiving a perfect array P;
(5) *Combination* (lines 13–16) if $d > 2$, N is even and $a = 2$, then construct an even sequence with Even, construct a perfect square by Mesh and finally use of Shift $d - 2$ times, results a perfect array P.

Colour transforms cellular perfect arrays into larger cellular perfect arrays.

Its input data are

- $d \geq 1$ – the number of dimensions;
- $N \geq 2$ – the size of the alphabet;
- $\mathbf{a}$ – the window size;
- $\mathbf{b}$ – the size of the cellular perfect array A;
- A – a cellular $(N, d, \mathbf{a}, \mathbf{b})$-perfect array.
- $k \geq 2$ – the multiplication coefficient of the alphabet;
- $\langle k_1, k_2, \ldots, k_d \rangle$ – the extension vector having the property $k^{|\mathbf{a}|} = k_1 \times k_2 \times \cdots \times k_d$.

The *output* of COLOUR is

- a (kN)-ary cellular perfect array P of size $\mathbf{b} = \langle k_1 a_1, k_2 a_2, \ldots, k_d a_d \rangle$.

COLOUR consists of three steps:

(1) *Blocking:* (line 1) arranging $k^{|\mathbf{a}|}$ copies (blocks) of a cellular perfect array A into a rectangular array R of size $\mathbf{k} = k_1 \times k_2 \times \cdots \times k_d$ and indexing the blocks lexicographically (by $0,\ 1, \ldots, k^{|\mathbf{a}|} - 1$);
(2) *Indexing:* (line 2) the construction of a lexicographic indexing scheme I containing the elements $0, 1, \ldots k^{|a|} - 1$ and having the same structure as the array R, then construction of a colouring matrix C, transforming the elements of I into k-ary numbers consisting of $|\mathbf{a}|$ digits;
(3) *Colouring:* (lines 3-4) colouring R into a symmetric perfect array P using the colouring array C that is adding the N-fold of the j-th element of C to each cell of the j-th block in R (considering the elements of the cell as lexicographically ordered digits of a number).

The output P consists of blocks, blocks consist of cells and cells consists of elements. If $e = P[\mathbf{j}]$ is an element of P, then the lexicographic index of the block containing e is called the ***blockindex*** of e, the lexicographic index of the cell containing e is called the ***cellindex*** and the lexicographic index of e in the cell is called ***elementindex***. E.g. the element $S_2[7, 6] = 2$ in Table 3 has blockindex 5, cellindex 2 and elementindex 1.

Finally, algorithm GROWING generates a prefix S_r of a growing array G. Its input data are r, the number of required doubly perfect prefixes of the growing array G, then n, d and $\mathbf{a}$. It consists of the following steps:

(1) *Initialization*: construction of a cellular perfect array P using CELLULAR;
(2) *Resizing:* if the result of the initialization is not doubly symmetric, then construction of a symmetric perfect array S_1 using COLOUR, otherwise we take P as S_1;
(3) *Iteration*: construction of the further $r - 1$ prefixes of the growing array G repeatedly, using COLOUR.

4 Examples of constructing growing arrays using colouring

In this section particular constructions are presented.

4.1 Construction of growing sequences

As the first example let $n = 2,\ a = 2$ and $r = 3$. CELLULAR calculates $N = 2$ and MARTIN produces the cellular (2,1,2,4)-perfect sequence $P = 00|11$.

Since P is symmetric, $S_1 = P$. Now GROWING chooses multiplication coefficient $k = n_2/n_1 = 2$, extension vector $\mathbf{k} = \langle 4 \rangle$ and uses COLOUR to construct a 4-ary perfect sequence.

COLOUR arranges $k_1 = 4$ copies into a 4 blocks sized arrray receiving

$$R = 00|11 \ || \ 00|11 \ || \ 00|11 \ || \ 00|11. \tag{5}$$

COLOURING receives the indexing scheme $I = 0\ 1\ 2\ 3$, and the colouring matrix C transforming the elements of I into a digit length k-ary numbers: $C = 00 \ || \ 01 \ || \ 10 \ || \ 11$.

Finally we colour the matrix R using C – that is multiply the elements of C by n_1 and adding the j-th $(j = 0, 1, 2, 3)$ block of $C_1 = n_1 C$ to both cells of the j-th copy in R:

$$S_2 = 00|11 \ || \ 02|13 \ || \ 20|31 \ || \ 22|33. \tag{6}$$

Since $r = 3$, we use COLOUR again with $k = n_3/n_2 = 2$ and get the (8,1,2,64)-perfect sequence S_3 repeating S_2 4 times, using the same indexing array I and colouring array $C' = 2C$.

Another example is $a = 2,\ n = 3$ and $r = 2$. To guarantee the cellular property now we need a new alphabet size $N = 6$. Martin produces a (6,1,2,36)-perfect sequence S_1, then COLOUR results a (12,1,2,144)-perfect sequence S_2.

4.2 Construction of growing squares

Let $n = a = 2$ and $r = 3$. Then $N(2, 2) = 2$. We construct the even sequence $W_4 = e_1e_2e_3e_4 = 0\ 0\ 1\ 1$ using EVEN and the symmetric perfect array A in

Table 1

a) A (2,2,4,4)-square

column/row	1	2	3	4
1	0	0	0	1
2	0	0	1	0
3	1	0	1	1
4	0	1	1	1

b) Indexing scheme I of size 4×4

column/row	1	2	3	4
1	0	1	2	3
2	4	5	6	7
3	8	9	10	11
4	12	13	14	15

Table 1.a using the meshing function (4). Since A is symmetric, it can be used as S_1. Now the greatest common divisor of a and a^d is 2, therefore indeed $n_1 = N^{2/2} = 2$.

Growing chooses $k = n_1/N = 2$ and Colour returns the array R repeating the array A $k^2 \times k^2 = 4 \times 4$ times.

Colour uses the indexing scheme I containing k^4 indices in the same 4×4 arrangement as it was used in R. Table 1.b shows I.

Transformation of the elements of I into 4-digit k-ary form results the colouring matrix C represented in Table 2.

Colouring of array R using the colouring array $2C$ results the (4,2,2,16)-square S_2 represented in Table 3.

In the next iteration Colour constructs an 8-ary square repeating S_2 4×4 times, using the same indexing scheme I and colouring by $4C$. The result is S_3, a $(8, 2, 2, 64)$-perfect square.

4.3 Construction of growing cubes

If $d = 3$, then the necessary condition (2) is $b^3 = (n)^{a^3}$ for double cubes, implying n is a cube number or a is a multiple of 3. Therefore, either $n \geq 8$ and then $b \geq 256$, or $a \geq 3$ and so $b \geq 512$, that is, the smallest possible

Table 2

Binary colouring matrix C of size 8×8

column/row	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	1	1	0	1	1
3	0	1	0	1	0	1	0	1
4	0	0	0	1	1	0	1	1
5	1	0	1	0	1	0	1	0
6	0	0	0	1	1	0	1	1
7	1	1	1	1	1	1	1	1
8	0	0	0	1	1	0	1	1

perfect double cube is the (8, 3, 2, 256)-cube.

As an example, let $n = 2$, $a = 2$ and $r = 2$. Cellular computes $N = 2$, Mesh constructs the $(2, 2, 2, 4)$-perfect square in Table 1.a, then Shift uses Martin with $N = 16$ and $a = 1$ to get the shift sizes for the layers of the $(2, 3, 2, \mathbf{b})$-perfect output P of Cellular, where $\mathbf{b} = \langle 4, 4, 16 \rangle$. Shift uses P as zeroth layer and the jth ($j \in [1 : 15]$) layer is generated by cyclic shifting of the previous layer downwards by w_i (div 4) and right by w_i (mod 4), where $\mathbf{w} = \langle 0\ 15\ 14\ 13\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \rangle$. 8 layers of P are shown in Table 4.

Let A_3 be a $4 \times 4 \times 16$ sized perfect, rectangular matrix, whose 0. layer is the matrix represented in Table 1, and the $(2, 3, \mathrm{a}, \mathrm{b})$-perfect array P in Table 4, where $\mathrm{a} = (2, 2, 2)$ and $\mathrm{b} = (4, 4, 8)$.

Growing uses Colour to retrieve a doubly symmetric cube. $n_1 = 8$, thus

Table 3

A (4,2,2,16)-square generated by colouring

column/row	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
2	0	0	1	0	0	2	1	2	2	0	3	0	2	2	3	2
3	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1
4	0	1	1	1	0	3	1	3	2	1	3	1	2	3	3	3
5	0	2	0	3	0	2	0	3	0	2	0	3	0	2	0	3
6	0	0	1	0	0	2	1	2	2	0	3	0	2	2	3	2
7	1	2	1	3	1	2	1	3	1	2	1	3	1	2	1	3
8	0	1	1	1	0	3	1	3	2	1	3	1	2	3	3	3
9	2	0	2	1	2	0	2	1	2	0	2	1	2	0	2	1
10	0	0	1	0	0	2	1	2	2	0	3	0	2	2	3	2
11	3	0	3	1	3	0	3	1	3	0	3	1	3	0	3	1
12	0	1	1	1	0	3	1	3	2	1	3	1	2	3	3	3
13	2	2	2	3	2	2	2	3	2	2	2	3	2	2	2	3
14	0	0	1	0	0	2	1	2	2	0	3	0	2	2	3	2
15	3	2	3	3	3	2	3	3	3	2	3	3	3	2	3	3

Table 4

8 layers of a (2,3,2,16)-perfect array

Layer 0	Layer 1	Layer 2	Layer 3	Layer 4	Layer 5	Layer 6	Layer 7
0 0 0 1	0 0 0 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1
0 0 1 0	0 0 1 0	0 0 0 1	0 0 0 1	0 1 1 1	0 1 1 1	0 1 1 1	0 1 1 1
1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1
0 1 1 1	0 1 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	1 0 1 1	0 1 1 1

$b = 256$, $k = n_1/N = 4$ and $\mathbf{k} = \langle 256/4, 256/4, 256/64\rangle$, that is we construct the matrix R repeating P $64 \times 64 \times 16$ times.

I has the size $64 \times 64 \times 16$ and $I[i_1, i_2, i_3] = 64^2(i_1 - 1) + 64(i_2 - 1) + i_3 - 1$. Colour gets the colouring matrix C by transforming the elements of I into 8-digit 4-ary numbers – and arrange the elements into $2 \times 2 \times 2$ sized cubes in lexicographic order – that is in order (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1). Finally colouring results a double cube S_1.

S_1 contains 2^{24} elements therefore it is presented only in electronic form (on the homepage of the corresponding author).

If we repeat the colouring again with $k = 2$, then we get a 64-ary $65536 \times 64536 \times 64536$ sized double cube S_2.

5 Proof of the main result

The main result of this paper can be formulated as follows.

Theorem 11 *If $n \geq 2$, $d \geq 1$, $a \geq 2$, $n_j = N^{dj/\gcd(d,a^d)}$ with $N = N(n,a)$ given by (1) for $j \in [0..\infty]$, then there exists an $(\mathbf{n}, d, a)$-growing array.*

The proof is based on the following lemmas.

Lemma 12 (Cellular lemma) *If* $n \geq 2,\ d \geq 1$ *and* $a \geq 2$*, then algorithm* CELLULAR *produces a cellular* $(N, d, a, \mathbf{b})$*-perfect array* A*, where* N *is determined by formula (1),* $b_1 = N^a$ *and* $b_i = N^{a^i - a^{i-1}}$ $(i \in [2..d])$.

Proof. It is known that algorithms EVEN + MESH and MARTIN + SHIFT result perfect outputs.

Since MESH is used only for even alphabet size and for 2×2 sized window, the sizes of the constructed array are even numbers and so the output array is cellular.

In the case of SHIFT we exploit that all prime divisors of a divide the new alphabet size N, and $b_i = N^{(a-1)(a^{i-1})}$ and $(a-1)(a^{i-1}) \geq 1$. □

Lemma 13 (Indexing lemma) *If* $n \geq 2,\ d \geq 2,\ k \geq 2,\ C$ *is a* d *dimensional* $\mathbf{a}$*-cellular array with* $|\mathbf{b}| = k^{|\mathbf{a}|}$ *cells and each cell of* C *contains the corresponding cellindex as an* $|\mathbf{a}|$ *digit* k*-ary number, then any two elements of* C *having the same elementindex and different cellindex are heads of different patterns.*

Proof. Let P_1 and P_2 be two such patterns and let us suppose they are identical. Let the head of P_1 in the cell have cellindex g and head of P_2 in the cell have cellindex h (both cells are in array C). Let $g - h = u$.

We show that $u = 0 \pmod{k^{|b|}}$. For example in Table 2 let the head of P_1 be $(2, 2)$ and the head of P_2 be $(2, 6)$. Then these heads are in cells with cellindex 0 and 2 so here $u = 2$.

In both cells, let us consider the position containing the values having local value 1 of some number (in our example they are the elements (3,2) and (3,6) of C.) Since these elements are identical, then $k|u$. Then let us consider the positions with local values k (in our example they are (3,1) and (3,5).) Since these elements are also identical so $k^2|u$. We continue this way up to the elements having local value $k^{|b|}$ and get $k^{|b|}|u$, implying $u = 0$.

This contradicts to the conditon that the patterns are in different cells. □

Lemma 14 (Colouring lemma) *If* $k \geq 2$, $k_i \in [2..\infty]$ $(i \in [1..d])$, A *is a cel-*

lular $(n, d, \mathbf{a}, \mathbf{b})$*-perfect array, then algorithm* COLOUR$(N, d, \mathbf{a}, k, \mathbf{k}, A, S)$ *produces a cellular* $(kN, d, \mathbf{a}, \mathbf{c})$*-perfect array* P*, where* $\mathbf{c} = \langle k_1a_1, k_2a_2, \ldots, k_da_d\rangle$.

Proof. The input array A is N-ary, therefore R is also N-ary. The colouring array C contains the elements of $[0..N(k-1)]$, so elements of P are in $[0..kN-1]$.

The number of dimensions of S equals to the number of dimensions of P that is, d.

Since A is cellular and c_i is a multiple of b_i $(i \in [1..d])$, P is cellular.

All that has to be shown is that the patterns in P are different.

Let's consider two elements of P as heads of two windows and their contents – patterns p and q. If these heads have different cellindex, then the considered patterns are different due to the periodicity of R. E.g. in Table 3 $P[11, 9]$ has cellindex 8, the pattern headed by $P[9, 11]$ has cellindex 2, therefore they are different (see parity of the elements).

If two heads have identical cellindex but different blockindex, then the indexing lemma can be applied. □

Proof of the main theorem. Lemma 18 implies that the first call of COLOUR in line 10 of GROWING results a doubly symmetric perfect output S_1. In every iteration step (in lines 14–16 of GROWING) the nzeroth block of S_i is the same as S_{i-1}, since the zeroth cell of the colouring array is filled up with zeros.

Thus S_1 is transformed into a doubly symmetric perfect output S_r having the required prefixes $S_1,\ S_2,\ \ldots, S_{r-1}$. □

6 Final remarks

The proposed definitions and algorithms can be extended for arbitrary $\mathbf{a}$.

Among others, the following problems are open: existence of $(6, 2, 5, \mathbf{b})$-perfect array with $\mathbf{b} = \langle 2 \cdot 3^8, 2^8 \cdot 3\rangle$ or with $\mathbf{b} = \langle 2 \cdot 3^{24}, 2^{24} \cdot 3\rangle$ and the existence of a (2,3,3,512)-perfect array (it would be the second smallest double cube).

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7 Pseudocodes of the algorithms used

The algorithms are written using the pseudocode of [3]. The running time of these algorithms is determined by the number of the elements of the generated perfect array – e.g. Growing needs $\Theta((n_r)^{a^3})$ time.

Since we deal only with the construction of symmetric perfect arrays, the window is always symmetric.

7.1 Pseudocode of the algorithm Growing

Input parameters of Growing are n, d, a and r, the output is a doubly symmetric perfect array S_r, which is the rth prefix of an $(\mathbf{n}, d, a)$-growing array.

Growing(n, d, a, r, S_r)

Cellular(n, d, a, N, P)
calculation of N using formula (1)
if P is symmetric **then** $S_1 \leftarrow P$
if P is not symmetric **then**
    $n_1 \leftarrow N^{d/\gcd(d,a^d)}$
    $k \leftarrow n_1/N$
    $k_1 \leftarrow (n_1)^{a^d/3}/N^a$
    for $i \leftarrow 2$ **to** d
        $k_i \leftarrow (n_1)^{a^d/d}/N^{a^i - a^{i-1}}$
        Colour$(n_1, d, a, k, \mathbf{k}, P, S_1)$
$k \leftarrow N^d/\gcd(d, a^d)$
for $i \leftarrow 1$ **to** d
    $k_i \leftarrow (n_2)^{a^d/d}/N^{a^i - a^{i-1}}$
for $i \leftarrow 2$ **to** r
$n_i \leftarrow N^{di/\gcd(d,a^d)}$
    Colour$(n_i, d, \mathbf{a}, k, \mathbf{k}, S_{i-1}, S_i)$
return S_r

7.2 Pseudocode of the algorithm Cellular

This is an extension and combination of the known algorithms Shift, Martin, Even and Mesh.

```
Cellular(n, d, a, N, A)

N ← N(n, a)
if d = 1 then Martin(N, d, a, A)
return A
if d = 2 and a = 2 and N is even then
                                        Mesh(N, a, A)
                                        return A
if N is odd or a ≠ 2 then
                              Martin(N, a, P_1)
                              for i ← 1 to d − 1
                                  Shift(N, i, P_i, P_{i+1})
                                  A ← P_1
                              return A
Mesh(N, a, P_1)
for i ← 2 to d − 1
    Shift(N, i, P_i, P_{i+1})
A ← P_d
return P_d
```

7.3 Pseudocode of the algorithm Martin

The following effective implementation of Martin is taken from [**?**].

```
Martin(n, a, w)

for i ← 0 to n^(a−1) − 1
    C[i] ← n − 1
for i ← 1 to a
    w[i] ← 0
for i ← a + 1 to n^a
    k ← w[i − a + 1]
    for j ← 1 to a − 1
        k ← kn + w[i − a + j]
    w[i] ← C[k]
    C[k] ← C[k] − 1
```

return P

7.4 Pseudocode of the algorithm Shift

Shift(N, d, a, P_d, P_{d+1})

Martin$(N^{a^d}, a-1, \mathbf{w})$
for $j \leftarrow 0$ **to** $N^{a^d - a^{d-1}} - 1$
  transform w_i to an a^d digit N-ary number
  produce the $(j+1)$-st layer of the output P_{d+1} by multiple shifting the jth layer of P_d by the transformed number (the first a digits give the shift size for the first direction, then the next $a^2 - a$ digits in the second direction etc.)
return P_{d+1}

7.5 Pseudocode of the algorithm Even

If N is even, then this algorithm generates the N^2-length prefix of an even growing sequence [9].

Even$(N, \mathbf{w})$

if $N = 2$ **then**
    $w[1] \leftarrow 0$
    $w[2] \leftarrow 0$
    $w[3] \leftarrow 1$
    $w[4] \leftarrow 1$
    return w
for $i = 1$ **to** $N/2 - 1$
  for $j = 0$ **to** $2i - 1$
    $w[4i^2 + 2j + 1] \leftarrow j$
  for $j = 0$ **to** $i - 1$
    $w[4i^2 + 2 + 4j] \leftarrow 2i$
  for $j = 0$ **to** $i - 1$
    $w[4i^2 + 4 + 4j] \leftarrow 2i + 1$
  for $j = 0$ **to** $4i - 1$
    $w[4i^2 + 4i + 1 + j] \leftarrow w[4i^2 + 4i - j]$
  $w[4i^2 + 8i + 1] \leftarrow 2i + 1$
  $w[4i^2 + 8i + 2] \leftarrow 2i$
  $w[4i^2 + 8i + 3] \leftarrow 2i$

$\quad w[4i^2 + 8i + 4] \leftarrow 2i + 1$
return w

7.6 *Pseudocode of the algorithm* MESH

The following implementation of MESH is taken from [11].

MESH$(N, \mathbf{w}, S)$

for $i \leftarrow 1$ **to** N^2
$\quad$ **for** $j \leftarrow 1$ **to** N^2
$\quad\quad$ **if** $i + j$ is even **then** $S[i, j] \leftarrow w[i]$
$\quad\quad\quad\quad\quad\quad$ **else** $S[i, j] \leftarrow w[j]$
return S

7.7 *Pseudocode of the algorithm* COLOUR

Input parameters are $N,\ d,\ a,\ k,\ \mathbf{k}$, a cellular $(N, d, a, \mathbf{b})$-perfect array A, the output is a $(kN, d, \mathbf{a}, \mathbf{c})$-perfect array P, where $\mathbf{c} = \langle a_1k_1, a_2k_2, \ldots, a_dk_d\rangle$.

COLOUR$(N, d, \mathbf{a}, k, \mathbf{k}, A, P)$

arrange the copies of P into an array R of size
$\quad k_1 \times k_2 \times \cdots \times k_d$ blocks
construct a lexicographic indexing scheme I containing the elements
$\quad$ of $[0..k^{a^d} - 1]$ and having the same structure as R
construct an array C transforming the elements of I into k-ary
$\quad$ numbers of v digits and multiplying them by N
produce the output S adding the j-th $(j \in [0..k^{a^d} - 1])$ element of C
$\quad$ to each cell of the j-th block in R for each block of R
return S

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Growing perfect cubes

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Abstract

An (n, a, b)-perfect double cube is a $b \times b \times b$ sized n-ary periodic array containing all possible $a \times a \times a$ sized n-ary array exactly once as subarray. A growing cube is an array whose $c_j \times c_j \times c_j$ sized prefix is an (n_j, a, c_j)-perfect double cube for $j = 1, 2, \ldots$, where $c_j = n_j^{v/3}$, $v = a^3$ and $n_1 < n_2 < \cdots$. We construct the smallest possible perfect double cube (a $256 \times 256 \times 256$ sized 8-ary array) and growing cubes for any a.

Key words: de Bruijn array, perfect map, colouring
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