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Density of safe matrices

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Abstract. A binary matrix A of size $m \times n$ is called *r-good,* if it contains in each column at most r 1's; the matrix is called *r-schedulable,* if deleting some zeros the matrix becomes *r-good*; A is called *r-safe*, if the first k $(1 \leq k \leq n)$ columns of the matrix contains at most kr 1's.

Let $\mathbf{Z} = [z_{ij}]_{m \times n}$ be a matrix of independent random variables, having the join distribution $P(z_{ij} = 1) = p$ and $P(z_{ij} = 0) = 1 - p$. For $m \geq 1$ lower and upper bounds are presented for the asymptotic probability of the event that a concrete realization of $\mathbf{Z}$ is 1-schedulable: the lower bound is connected with good, and the upper bound with safe matrices. Further exact formula is given for the critical probabilities $p_{\mathtt{crit}}(m)$ defined as the supremum of probabilities, guaranteeing that the matrix Z is 1-safe with positive probability for arbitrary n.

1 Introduction

A very popular research area is in combinatorics [3, 4, 8, 9, 24] and physics [1, 6, 19, 21] is the random walk in graphs.

In this paper such a mathematical model is investigated which originally was proposed for percolation [5, 8, 9, 17, 18, 21, 25], but is useful also to study of some scheduling problems of parallel processes using resources requireing mutual exclusion [20].

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Estimations of the probability of schedulability of processes are derived using different methods, first of all investigating of asymmrtric random walks accross the x axis.

2 Formulation of the problem

Let m and n be positive integers, let r $(0 \le r \le m)$ be a real number and let

$$\mathbf{Z} = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \dots & & & \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix}$$

be a matrix of independent random variables with the common distribution

$$P(z_{ij} = k) = \begin{cases} p, & \text{if } k = 1 \text{ and } 1 \le i \le m, \ 1 \le j \le n, \\ q = 1 - p, & \text{if } k = 0 \text{ and } 1 \le i \le m, \ 1 \le j \le n. \end{cases}$$

Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

be a concrete realization of $\mathbf{Z}$.

The good, safe and schedulable matrices are defined as follows.

Matrix A is called ***r*-good,** if the number of the 1's is at most r in all columns. The number of different r-good matrices of size $m \times n$ is denoted by $G_r(m, n)$ and the probability that $\mathbf{Z}$ is good is denoted by $g_r(m, n, p)$.

Matrix A is called ***r*-safe**, if

$$\sum_{i=1}^{m} \sum_{j=1}^{k} a_{ij} \le kr \quad (k = 1, 2, \dots, n).$$

The number of different r-safe matrices of size $m \times n$ is denoted by $S_r(m, n)$ and the probability that $\mathbf{Z}$ is safe, is denoted by $s_r(m, n, p)$.

If $a_{ij} = 0$, then it can be deleted from A. Deletion of a_{ij} means that we decrease the second indices of $a_{i,j+1}, \dots, a_{im}$ and add $a_{im} = 0$ to the i-th row of A.

Matrix A is called **Winkler r-schedulable** (shortly r**-schedulable** or r-compatible) if it can be transformed into a r-good matrix B using deletions. The number of different r-schedulable matrices of size $m \times n$ is denoted by $W_r(m,n)$ and the probability that $\mathbf{Z}$ is r-schedulable is denoted by $w_r(m,n,p)$. The function $w_r(m,n,p)$ is called **r-schedulability function.**

The functions $g_r(m,n,r)$, $w_r(m,n,r)$ and $s_r(m,n,r)$ are called the density functions of the corresponding matrices. The **asymptotic density** of the good, safe and schedulable matrices are defined as

$$g_r(m,p) = \lim_{n\to\infty} g_r(m,n,p),$$

$$s_r(m,p) = \lim_{n\to\infty} s_r(m,n,p),$$

$$w_r(m,p) = \lim_{n\to\infty} w_r(m,n,p).$$

The critical probabilities defined as

$$w_{crit,r}(m) = \sup\{p \mid w_r(m,p) > 0\},$$

$$g_{crit,r}(m) = \sup\{p \mid g_r(m,p) > 0\},$$

and

$$s_{crit,r}(m) = \sup\{p \mid s_r(m,p) > 0\}$$

represent special interest for some applications.

The aim of this paper is to characterize the density, asymptotic density and critical probability of good, schedulable and safe matrices.

Starting point of our research is due to Péter Gács [9] proving that $w_1(2,p)$ is positive for p small enough. His proof implies that $w_{crit,1}(2) \geq 10^{-400}$.

We remark that our other [13] paper contains the algorithms of generating and evaluating of all safe and schedulable matrices, and further simulatoin results.

2.1 Interpretation of the problem

Although Winkler model was proposed to study the percolation, we describe a possible interpretation as a model of parallel processes. Let m processes use r units of some resource R. The requirements of the process P_i are modeled by the sequence $a_{i1}, a_{i2}, \ldots, a_{im}$. $a_{ij} = 1$ means that the process P_i needs a unit of the given resource in the jth time unit. $a_{ij} = 0$ means that the process P_i executes some background work in the jth time unit which can be delayed and executed after the last usage of R.

The special case $m = 1$ and $r = 1$ is the well-konown ticket problem [24] or ballot problem [7], while the special case $m = 2$ and $r = 2$ is the Winkler model of percolation [9, 25].

The good matrices are schedulable without deletion of zeros. But some not good matrices are schedulable since using the permitted deletion operation they can be transformed into a good matrix. Safeness is a necessary condition of schedulability. Therefore the number of good matrices gives a lower bound and the number of the safe matrices results an upper bound for the number of the schedulable matrices.

Since we handle the model as a model of informatics, in the sequel we follow the terminology used by Feller [7] in queueing theory.

3 Analysis

In this section we investigate first of all – using different methods – the function of the asymptotic density of 1's as the function of the appearence of ones p and of the number of sequences m.

Some basic properties of the investigated functions ($g_r(m, n, p)$, $w_r(m, n, p)$ és $s_r(m, n, p)$) are the following:

- $n \in \mathbb{N}^+$, $r \in \mathbb{R}$ and $r \in [0, m]$, $p \in \mathbb{R}$ and $p \in [0, 1]$;
- as the functions of n they are monotonically decreasing;
- as the functions of p they are monotonically decreasing;
- as the functions of m they are monotonically decreasing;
- as the functions of r they are monotonically increasing;

In the following we suppose that $r = 1$, that is in the column of the good matrices at most one 1, and in the first k columns of the safe matrices at most k 1's are permitted. Since r equals everywhere to 1, therefore it is omitted as an index.

3.1 Preliminary results

In the further sections we need the following assertions.

Let C_n-nel ($n \in \mathbb{N}^+$) denote the number of binary sequences $a_1, a_2, \ldots, a_{2n}$, containing n ones and n zeros in such a manner that each prefix $a_1, a_2, \ldots, a_k$ ($1 \le k \le 2n$) contains at most so many ones as zeros.

Lemma 1 *If* $n \geq 0$, *then*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is worth to remark that C_n is the n-edik Catalan number, whose explicite form appears in numerous books and papers [2, 14, 16, 24].

Lemma 2 *If* $0 \leq x \leq 1$, *then*

$$f(x) = x\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}(x(1-x))^k = \begin{cases} \frac{x}{1-x}, & \textit{if } 0 \leq x < \frac{1}{2}, \\ 1, & \textit{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Proof. If $m \geq 2$, then the columns containing only 0's are called **white** (W), the columns containing exactly one 1 are called **grey** (G) and the remaining columns are called **black** (B).

If $m \geq 2$, then each column of the matrix A mátrix is white or grey with probability $q^m + mq$, therefore $g(m, n, p) = ((q^m + mq))^n$. If $p > 0$, then

$$g(m, p) = \lim_{n\to\infty} (q^m + mq)^n = 0,$$

so the density of the good matrices tends to zero, when the number of the columns tends to infinity.

If in the case $m = 2$ we delete the white columns from a good matrix, then only grey columns remain in the matrix, that is each row of the matrix is *the complementer* of the other row. ∎

The following simple assertion plays important role in the following.

Lemma 3 *If* $m \geq 2$, *then the good matrices are schedulable, and the schedulable matrices are safe.*

Proof. If in every column of the matrix **A** are at most one 1, then the first k columns contain at most k 1's.

If there is a k $(1 \leq k \leq n)$, that the first k columns of the matrix **A** contains more 1's than k, then according to the pigeonhole principle there is at least one column containing two 1's. If we delete a zero from **A**, then the number of the 1's in the first k columns does nor decrease, therefore **A** is not schedulable. ∎

A useful consequence of this assertion is the following corollary.

Corollary 1 *If* $m \geq 2$, *then*

$$g(m,n,p) \leq w(m,n,p) \leq s(m,n,p),$$
$$g(m,p) \leq w(m,p) \leq s(m,p),$$
$$g_{crit}(m) \leq w_{crit}(m) \leq s_{crit}(m).$$

3.2 Two rows in the matrix

For the simplicity of the notations we analyze the function $u(2,n,p) = 1 - s(2,n,p)$ instead of $s(2,n,p),p) = 1 - s(2,n,p)$. At first we derive a closed formula for $u(2,n,0.5)$.

Lemma 4 *If* $n \geq 1$, *then*

$$u(2,n,0.5) = \sum_{i=1}^{n} \sum_{j=0}^{\lfloor (i-1)/2 \rfloor} 2^{i-1-2j} C_j \binom{i-1}{2j} 4^{n-i}.$$

Proof. Let's classify the possible matrices of size $2 \times n$ according to their first such column, in which the cumulated number of 1's became greater than the number of 0's. This column is called *the deciding column* of the matrix.

The index of the deciding column is $0, 1, \ldots, n-1$ or n. The matrices of the received classes can be further classified according to the number of black columns before the deciding column: the possible values of this number are $0,\ 1, \ldots,\ \lfloor (n-1)/2 \rfloor$.

The outer summing takes into account the deciding columns, while the inner summing does the black columns before the deciding column. The binomial coefficient mirrors the number of possibilities for the placement of the $2j$ black and white columns in the $i-1$ columns preceeding the deciding column. The jth Catalan number C_j gives the number of corresponding sequence of the black and whirt columns. The power of base 2 gives the number of possible arrangements of the grey columns. Finally the power of base 4 takes into account the fact, that the columns after the deciding one can be filled in arbitrary manner – the matrix will be unsafe in any case. ■

It seems that it would be hard to handle the formula (4) for $u(2,n,p)$. Therefore we present a combinatorial method and two ones based on random walks to get the explicit form of $s(2,p)$.

Lemma 5 *If* $0 \leq p \leq 1$, *then*

$$u(2,p) = \begin{cases} \frac{p^2}{q^2}, & \text{if } 0 \leq p < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq p \leq 1. \end{cases}$$

Proof.

Some part of the unsafe matrices is unsafe due to the first black column. The general form of such matrices is $G^a B A^b$, where $a + b + 1 = n$, further G means a grey, B means a black and A means an arbitrary column. The asymptotic fraction of such columns is

$$\sum_{a=0}^{\infty} C_0 (2pq)^a p^2 = \frac{p^2}{1-2pq} C_0 \ .$$

The general form of the following group of the unsafe matrices is $G^a B G^b W G^c B A^d$, where $a + b + c + d + 3 = n$. The fraction of the such matrices asymptotically equals to

$$\sum_{a=0}^{\infty} (2pq)^a p^2 \sum_{b=0}^{\infty} (2pq)^b q^2 \sum_{c=0}^{\infty} (2pq)^c p^2 = \frac{p^2}{1-2pq} C_1 \frac{p^2}{1-2pq} \frac{q^2}{1-2pq} \ .$$

Gerenally, if the $(i+1)$-th black column is deciding, then the the asymptotic contribution of the such matrices to the probability of the unsafe matrices equals to

$$\frac{p^2}{1-2pq} C_i \left(\frac{p^2}{1-2pq} \frac{q^2}{1-2pq} \right)^2 ,$$

and so

$$u(2, p) = \sum_{i=0}^{\infty} \frac{p^2}{1-2pq} C_i \left(\frac{p^2}{1-2pq} \frac{q^2}{1-2pq} \right)^i .$$

Lemma 2 with the substitutions $p^2/(p^2 + q^2) = x$ and $q^2/(p^2 + q^2) = 1 - x$ gives the required formula. ∎

We get a useful method for the investigation of our matrices assigning a random walk across the x-axis to each matrix.

If in the kth point of time the walking point is in the point $P_k(b_k, 0)$, where b_k is the difference of numbers of the 1's and the 0's in the first k columns of the matrix.

If the random walk starts in the origo, then

$$b_k = \begin{cases} -1, & \text{if } \exists k \text{ with } k < \sum_{i=1}^{k} (a_{i1} + a_{i2}), \\ k - \sum_{i=1}^{k} (a_{i1} + a_{i2}) & \text{otherwise.} \end{cases}$$

Another proof of Lemma 5 is the following.

Second proof of Lemma 5. Azt akarjuk meghatározni, hogy az origóból induló pont milyen valószínűséggel nyelődik el a -1 pontban lévő nyelőben. Bár a fehér, illetve fekete oszlop után kettővel változik az addig előfordult nullák és egyesek számának különbsége, az egyszerűség kedvéért azt tételezzük fel, hogy fekete oszlop után p^2 valószínűséggel lépünk egyet balra, fehér oszlop után q^2 valószínűséggel lépünk egyet jobbra, és szürke oszlop után $2pq$ valószínűséggel helyben maradunk.

Az $u(2, p) = x$ jelöléssel a teljes valószínűség tétele alapján azt kapjuk, hogy

$$x = p^2 + 2pqx + q^2x^2.$$

The roots of this equation are

$$x_{1,2} = \frac{1 - 2pq \pm \sqrt{(1 - 2pq)^2 - 4p^2q^2}}{2q^2} = \frac{p^2 + q^2 \pm \sqrt{(p^2 - q^2)^2}}{2q^2} ,$$

from where we get

$$x_1 = \frac{p^2}{q^2}, \text{ and } x_2 = 1. \tag{1}$$

This formula and $s(2, p) = 1 - u(2, p)$ results the required formula. ■

Mivel bennünket elsősorban az elnyelődés valószínűsége érdekel, egy másik bolyongást is hozzárendelhetünk az **Z** mátrixhoz úgy, hogy a szürke oszlopoktól eltekintünk – azok ugyanis az elnyelődés határvalószínűségét nem befolyásolják, csak lassítják a konvergenciát.

Another proof of Lemma 5. is the following.

Third proof of Lemma 5. A szürke oszlopok valószínűségét a fekete és fehér oszlopok között a megfelelő arányban elosztva a balra lépés a és a jobbra lépés b valószínűségére azt kapjuk, hogy

$$a = \frac{p^2}{p^2 + q^2} \quad \text{és} \quad b = \frac{q^2}{p^2 + q^2}. \tag{2}$$

Using these probabilities we get the equation

$$x = a + bx^3.$$

Substituting a and b into the roots of this equation accordig to 2 we get here also the roots corresponding to (1). ■

Finally we present such a method which later can be extended to arbitrary $m \geq 2$ sequances.

Fourth proof of Lemma 5. Jelöljük x_k-val ($k = -1, 0, 1, 2, \ldots$) annak valószínűségét, hogy a k pontból induló bolyongó pont elnyelődik az $x = -1$ helyen. A két egyest tartalmazó, p^2 valószínűséggel előforduló oszlopnak feleltessünk meg balra lépést, a $2pq$ valószínűséggel előforduló vegyes oszlopkhoz tartozzon helybenmaradás és a két nullát tartalmazó, q^2 valószínűséggel előforduló oszlopoknak feleljen meg jobbra lépés.

Then we can write the following system of equations.

$$\begin{array}{lllllll} x_0 & = & q^2x_1 & + & 2qpx_0 & + & p^2, \\ x_1 & = & q^2x_2 & + & 2qpx_1 & + & p^2x_0, \\ x_2 & = & q^2x_3 & + & 2qpx_2 & + & p^2x_1, \\ x_3 & = & q^2x_4 & + & 2qpx_3 & + & p^2x_2, \\ \ldots \end{array} \tag{3}$$

Let

$$G(z) = \sum_{i=1}^{\infty} x_i z^i \tag{4}$$

be the generator function of the sequence $x_0, x_1, x_2, \ldots$. Multiplying the equations beginning with x_i of the system of equations (5) by z^i and summing up the new equations we get the equation

$$G(z) = q\frac{G(z) - x_0}{z} + 2pqG(z) + p^2(1 + zG(z)).$$

From this equation $G(z)$ can be expressed in the form

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = q^2 - p^2z$$

and

$$Q(z) = p^2z^2 + 2pq + p^3 - z.$$

In the zero places x_o with $|x_0| \leq 1$ of the polinom $Q(z)$ according to Cauchy–Hadamard-theorem [23] if must hold $P(z) = 0$. Writing the equation$Q(z) = 0$ in the form

$$(pz + q)^2 = 1$$

we directly get that $z = 1$ is a root of the polinom $Q(z)$. From the equation $P(1) = 0$ we get the root

$$x_0 = \frac{p^2}{q^2}. \qquad \blacksquare$$

Figure 1 shows the part belonging to the interval $p \in [0, 0.5]$ of the curve of the function $s(2, p, 1)$ defined in the interval $[0, 1]$.

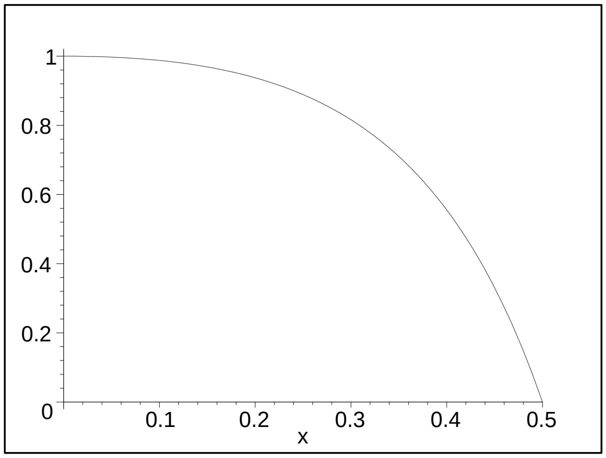

Figure 1: The curve of the schedulability function $s(2, p, 1)$ in the interval $p \in [0, 0.5]$.

A $g(2, p)$ és $s(2, p)$ függvények alapján az $m = 2$ értékhez tartozó kritikus valószínűségekre fennáll

$$0 = g_{crit}(2) \leq w_{crit}(2) \leq s_{crit}(2) = \frac{1}{2} \ .$$

Emlékeztetünk Gács eredményére, amely szerint $w_{crit}(2) \geq 10^{-400}$.

Figure 2 contains the number and fraction of the good, schedulable and safe matrices. The fraction equals to the probability of the corresponding matrices in the case $p = 0.5$ értéknek. The number of the columns of the matrices characzerized in the figure is $1, \ 2, \ \ldots, \ 15$ or 16.

Let $T(m, n)$ denote the number of the binary matrices of size $m \times n$. Then $T(m, n) = 2^{mn}$.

According to our results $G(m, n)/T(m, n)$, $W(m, n)/T(m, n)$, and $S(m, n)/T(m, n)$ too tend to zero when n tends to infinity.

n	$G(2,n)$	$\frac{G(2,n)}{T(2,n)}$	$W(2,n)$	$\frac{W(2,n)}{T(2,n)}$	$S(2,n)$	$\frac{S(2,n)}{T(2,n)}$	$\frac{W(2,n)}{S(2,n)}$
1	3	0.750	3	0.750	3	0.750	1
2	9	0.562	10	0.625	10	0.625	1
3	27	0.452	35	0.547	35	0.547	1
4	81	0.316	124	0.484	126	0.492	0.984
5	243	0.237	444	0.434	462	0.451	0.961
6	729	0.178	1592	0.389	1716	0.419	0.927
7	2187	0.133	5731	0.350	6435	0.393	0.890
8	6561	0.100	20671	0.315	24310	0.371	0.850
9	19683	0.075	74722	0.285	92378	0.352	0.808
10	59049	0.056	270521	0.258	352716	0.336	0.767
11	177147	0.042	980751	0.234	1352078	0.322	0.725
12	531441	0.032	3559538	0.212	5200300	0.310	0.684
13	1594323	0.022	12931155	0.193	20058300	0.299	0.646
14	4782969	0.018	47013033	0.175	77558760	0.289	0.606
15	14348907	0.013	171036244	0.159	300540195	0.280	0.568

Figure 2: Rounded data belonging to the parameters $m = 2$ and $p = 0.5$.

A 3. táblázat $s(2, n, 0.4)$ oszlopában a számoknak az 5/9 határértékhez kell tartaniuk – ami még messze van.

A 6. táblázatban a $s(2, n, 0.35)$ oszlop számainak a $120/169 \sim 0.7101$ határértékhez kell tartaniuk.

3.3 Three rows in the matrix

Az $m = 3$ esetben 3:0, 2:1, 1:2 és 0:3 lehet a nullák és egyesek aránya. A vizsgált mátrixhoz olyan bolyongást rendelünk, amelyikben a csupa egyes oszlop p^3 valószínűségével ugrunk balra kettővel, a két egyest tartalmazó oszlop $3p^2q$ valószínűségével lépünk balra, az egy egyest tartalmazó oszlopok $3p^2$ valószínűségével maradunk helyben és a három nullát tartalmazó oszlop q^3 valószínűségével lépünk jobbra.

A korábban is használt x_k jelölést bevezetve a következő egyenleteket írhatjuk fel:

n	$T(2,n)$	$g(2,n,0.4)$	$w(2,n,0.4)$	$s(2,n,0.4)$	$\frac{w(2,n,0.4)}{s(2,n,0.4)}$
1	4	0.8400	0.8400	0.8400	1
2	16	0.7056	0.7632	0.7632	1
3	64	0.5927	0.7171	0.7171	1
4	256	0.4979	0.6795	0.6862	0.9902
5	1024	0.4182	0.6487	0.6639	0.9771
6	4096	0.3513	0.6206	0.6470	0.9592
7	16384	0.2951	0.5957	0.6339	0.9397
8	65536	0.2479	0.5731	0.6234	0.9193
9	262144	0.2082	0.5524	0.6149	0.8984
10	1048576	0.1749	0.5332	0.6078	0.8773
11	4194304	0.1469	0.5155	0.6019	0.8565
12	16777216	0.1234	0.4988	0.5967	0.8359
13	67108864	0.1037	0.4832	0.5924	0.8157
14	268435456	0.0871	0.4685	0.5886	0.7960
15	1073741824	0.0731	0.4545	0.5854	0.7764
16	4294967296	0.0644	0.4412	0.5825	0.7574
17	169779869184	0.0516	0.4286	0.5800	0.7390

Figure 3: Rounded data belonging to the parameters $m = 2$ and $p = 0.4$.

$$\begin{array}{lcccccccc} x_0 & = & q^3x_1 & + & 3q^2px_0 & + & 3qp^2 & + & p^3, \\ x_1 & = & q^3x_2 & + & 3q^2px_1 & + & 3qp^2x_0 & + & p^3, \\ x_2 & = & q^3x_3 & + & 3q^2px_2 & + & 3qp^2x_1 & + & p^3x_0, \\ x_3 & = & q^3x_4 & + & 3q^2px_3 & + & 3qp^2x_2 & + & p^3x_1, \\ \dots \end{array} \tag{5}$$

Let

$$G(z) = \sum_{i=0}^{\infty} x_n z^n$$

be the generator function of the sequence $x_0, x_1, x_2, \ldots$. Then multiplying the equations of the system (5) with the corresponding powers of z and summing up the received equations we get

$$G(z) = q^3\frac{G(z) - x_0}{z} + 3q^2p\frac{G(z)}{z} + 3qp^2(1 + zG(z)) + p^3\left(1 + z + z^2G(z)\right),$$

n	$T(2,n)$	$g(2,0.35)$	$w(2,0.35)$	$s(2,n,0.35)$	$\frac{w(2,0.35)}{s(2,0.35)}$
1	4	0.8775	0.8775	0.8775	1
2	16	0.7700	0.8218	0.8218	1
3	64	0.6757	0.7901	0.7901	1
4	256	0.5929	0.7645	0.7699	0.9930
5	1024	0.5203	0.7441	0.7561	0.9841
6	4096	0.4565	0.7255	0.7462	0.9723
7	16384	0.4006	0.7094	0.7389	0.9601
8	65536	0.3515	0.6949	0.7334	0.9475
9	262144	0.3085	0.6817	0.7291	0.9350
10	1048576	0.2707	0.6696	0.7258	0.9226
11	4194304	0.2375	0.6585	0.7231	0.9107
12	16777216	0.2084	0.6481	0.7210	0.8989
13	67108864	0.1839	0.6383	0.7192	0.8875
14	268435456	0.1605	0.6291	0.7178	0.8764
15	1073741824	0.1401	0.6204	0.7166	0.8658
16	4294967296	0.1236	0.6122	0.7156	0.????

Figure 4: Rounded data belonging to the parameters $m = 2$ and $p = 0.35$.

from where $G(z)$ can be expressed as the fraction of two polinom:

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = q^3 x_0 - 3qp^2 z - p^3(z + z^2)$$

and

$$Q(z) = p^3 z^3 + 3p^2 q z^2 + 3pq^2 z + q^3 - 1.$$

The equation $Q(z) = 0$ can be transformed into the form

$$(q + pz)^3 = 1$$

from where the root $z_1 = 1$ follows immediately. Dividing the equation $P(1) = 0$ by q^3 and using the substitution $p/q = t$ it follows

$$x_0 = 3t^2 + 2t^3$$

n	$T(3,n)$	$g(3,n,0.5)$	$w(3,n,0.5)$	$s(3,n,0.35)$	$\frac{w(3,n,0.5)}{s(3,n,0.5)}$
1	8	0.5000	0.5000	0.5000	1
2	64	0.2500	0.2969	0.2969	1
3	256	0.1250	0.1914	0.1914	1
4	4096	0.0625	0.???	0.???	???
5	32768	0.	0.	0.	0.????

Figure 5: Rounded data belonging to the parameters $m = 3$ and $p = 0.5$.

n	$T(3,m)$	$g(3,n,0.25)$	$w(3,n,0.5)$	$s(3,n,0.25)$	$\frac{w(3,n,0.25)}{s(3,n,0.25)}$
1	8	0.5000	0.5000	0.5000	1
2	64	0.	0.	0.	1
3	256	0.	0.	0.	1
4	4096	0.	0.	0.	???
5	32768	0.	0.	0.	0.????

Figure 6: Rounded data belonging to the parameters $m = 3$ and $p = 0.25$.

. Az $x_0 = x_0(t)$ függvény értéke a $t = 0$ helyen nulla, és pozitív t-re monoton nő. A $t = p/(1-p)$ értéket visszahelyettesítve és $(1-p)^3$-nel beszorozva a

$$\frac{3p^2}{(1-p)^2} + \frac{2p^3}{(1-p)^3} = 1$$

egyenletet kapjuk, amelyből rendezés után a $p = 1/3$ értéket kapjuk, azaz $s_{crit}(3) = 1/3$.

4 Main result

Az $m \times n$ méretű biztos mátrixok elemzését az $m \geq 4$ esetben az $m = 3$ esethez hasonló módon végezhetjük el.

A bolyongó pont a legalább $b \geq 3$ egyest tartalmazó oszlop esetén $(b-2)$-t ugrik balra, két egyest tartalmazó oszlop esetén egyet lép balra, egy egyest tartalmazó oszlop esetén helyben marad és az m nullát tartalmazó oszlop esetén egyet lép jobbra.

A $(b-2)$-vel balra ugrás valószínűsége $\binom{m}{b}p^{b-2}q^{n-b+2}$, a balra lépésé $\binom{m}{2}p^{m-2}q^2$, a helyben maradásé $\binom{m}{1}pq^{m-1}$ és a jobbra lépésé $\binom{m}{0}q^m$, ezért a következő egyenleteket írhatjuk fel.

$$\begin{array}{lcccccc} x_0 & = & \binom{m}{0}q^m x_1 & + & \binom{m}{1}pq^{m-1}x_0 & + & \binom{m}{2}p^2q^{m-2} \\ & + & \binom{m}{3}p^3q^{m-3} & + & \dots & + & \binom{m}{m}p^m, \\ x_1 & = & \binom{m}{0}q^m x_2 & + & \binom{m}{1}pq^{m-1}x_1 & + & \binom{m}{2}p^2q^{m-2}x_0 \\ & + & \binom{m}{3}p^3q^{m-3} & + & \dots & + & \binom{m}{m}p^m, \\ x_2 & = & \binom{m}{0}q^m x_3 x_3 & + & \binom{m}{1}pq^{m-1}x_2 & + & \binom{m}{2}p^2q^{m-2}x_1 \\ & + & \binom{m}{3}p^3q^{m-3}x_0 & + & \dots & + & \binom{m}{m}p^m, \\ \dots \end{array} \tag{6}$$

Let

$$G(z) = \sum_{i=0}^{\infty} x_n z^n$$

be the generator function of the sequence $x_0, x_1, x_2, \dots$. Then multiplying the equations (6) with the corresponding powers of z and summing up them we get

$$G(z) = \binom{m}{0}q^m \frac{G(z)-x_0}{z} + \binom{m}{1}pq^{m-1}G(z) + \binom{m}{2}p^2q^{m-2}(1+zG(z))$$

$$+\binom{m}{3}p^3q^{m-3}(1+zG+z^2G(z))+\dots+\binom{m}{m}q^m\left(1+z+\dots+z^{m-2}+z^{m-1}G(z)\right),$$

from where one can express $G(z)$ as the fraction of two polinoms:

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = \sum_{i=0}^{m} \binom{m}{i} p^i q^{m-i} z^i - z$$

and

$$Q(z) = \binom{m}{0}q^m x_0 - \sum_{i=0}^{m} \binom{m}{p}^i q^{m-i} \sum_{j=0}^{i-2} z^j.$$

Ahol a nevezőnek legfeljebb 1 abszolut értékű gyöke van, ott a számlálónak a nulla értéket kell felvennie.

A $Q(z) = 0$ egyenletet

$$(q+pz)^m = 1$$

alakra hozhatjuk, ahonnan a $z_1 = 1$ gyököt kapjuk. A $P(1) = 0$ egyenletet q^m-nel osztva az

$$x_0 = \sum_{i=0}^{m} \binom{m}{i} \left(\frac{p}{1-p}\right)^i (i-1)$$

egyenletet kapjuk. Az $x_0 = x_0(p)$ függvény értéke a $p = 0$ helyen nulla, és pozitív p-re monoton nő. Az $x_0 = 1$ egyenletből a $p = 1/m$ értéket kapjuk.

Az $m = 2$ és $m = 3$-ra kapott korábbi eredményeket is figyelembe véve ezzel a következő eredményt kaptuk.

Theorem 1 *If* $m \geq 2$ *and* $0 \leq p \leq 1$, *then*

$$s(m,p) = 1 - \sum_{i=2}^{m} \binom{m}{i} \left(\frac{p}{1-p}\right)^i (i-1) \tag{7}$$

and

$$s_{crit}(m) = \frac{1}{m}.qlabelmain2 \tag{8}$$

5 Summary

We determined the explicit form of the asymptotic density $s(m,p)$ for every number of the rows $m \geq 2$ and probability of 1's p. Further we gave the exact values of the critical probabilities $s_{crit}(m)$ for $m \geq 2$. The value of $s_{crit}(2)$ is 0.5 what is characteristic to several other two dimensional critical probabilities. The further critical probabilities are decreasing when m grows.

A szimulációs vizsgálatok szerint a kritikus ütemezhetőségi valószínűségek közel vannak a kapott felső korlátokhoz: a 2. táblázat a $p = 0.5$, a 3. táblázat a $p = 0.4$, a 6. táblázat pedig a $p = 0.35$ valószínűséghez tartozó adatokat mutatja.

On teh base of the data of the figures we suppose that the bound $p \geq 10^{-400}$ in [9] can be improved, but the analysis of the behaviour of fraction $w(m,p)/s(m,p)$ requires further work.

We are able to give a bit better lower and upper bounds of the investigated $w(_r(m,n,p)$ probabilities, but the more precise characterization of the critical probabilities requires more useful matrices than the good and safe ones.

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