Imbalances of bipartite multitournaments

Shariefuddin Pirzada (Srinagar, India)
Antal Iványi (Budapest, Hungary)
Namir A. Shah (Srinagar, India)

Abstract. A bipartite (a, b, p, q)-tournament is a bipartite tournament in which the parts of the tournament contain p, resp. q vertices and the vertices belonging to different parts of the tournament are connected with at least a and at most b arcs. The imbalance of a vertex is defined as the difference of its out-degree and in-degree. In this paper existence criteria and construction algorithms are presented for bipartite (a, b, p, q)-tournaments having prescribed imbalance sequences and prescribed imbalance sets.

1. Introduction

An actual research topic of graph theory is the characterization of different special graphs (as simple, oriented, bipartite, multipartite, signed and semicomplete graphs, see e.g. [1, 11, 12, 14, 15, 16, 19, 30]), and

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different generalizations (as hypergraphs, hypertournaments, weighted graphs, see e.g. [18, 27, 28]) having prescribed degree properties.

The classical results, as the theorem published by Landau in 1953 [13], and the theorem of Erdős and Gallai published in 1960 [4] contained necessary and sufficient conditions of the existence of a tournament, respectively of a simple graph with prescribed parameters. Later also constructive results appeared as the Havel-Hakimi theorem [7, 8] on simple graphs and the construction algorithm for optimal (a, b, n)-tournaments [10].

The structure of the paper is as follows. Section 2 contains some preliminary results, while Section 3 deals with imbalances of $(0, \infty, p, q)$-tournaments. In Section 4 the reconstruction results of imbalance sequences are discussed, Section 5 is devoted to imbalance sets.

2. Preliminary notions and earlier results

Let a, b and n be nonnegative integers ($b \geq a$, $b > 0$, $n \geq 1$), $\mathcal{T}(a, b.n)$ be the set of directed multigraphs $T = (V, E)$, where $|V| = n$, and elements of each pair of different vertices u, $v \in V$ are connected with at least a and at most b arcs [9]. $T \in \mathcal{T}(a, b, n)$ is called (a, b, n)-*tournament.* $(1, 1, n)$-tournaments are the usual tournaments, and $(0, 1, n)$-tournaments are also called oriented graphs or simple directed graphs [5]. The set $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcup_{b \geq 1,\ n \geq 1} \mathcal{T}(0, b, n).$$

According to this definition $\mathcal{T}$ is the set of the finite directed loopless multigraphs.

For any vertex $v \in V$ let $d(v)^+$ and $d(v)^-$ denote the out-degree and in-degree of x, respectively. Define $f(v) = d(v)^+ - d(v)^-$ as the imbalance of the vertex v. The imbalance sequence of $T \in \mathcal{T}$ is formed by listing the vertex imbalances of the vertices in nonincreasing or nondecreasing order.

The following result due to Avery [1] and Mubayi, Will and West [16] provides a necessary and sufficient condition for a nonincreasing sequence F of integers to be the imbalance sequence of a tournament $T \in \mathcal{T}(0,1,n)$.

Theorem 2.1. *A nonincreasing sequence of integers $F = [f_1, \ldots, f_n]$ is an imbalance sequence of a tournament $T \in \mathcal{T}(0,1,n)$ if and only if*

$$\sum_{i=1}^{k} f_i \leq k(n-k),$$

for $1 \leq k < n$ with equality when $k = n$.

Proof. See [1, 16]. □

Arranging the sequence F in nondecreasing order, we have the following equivalent assertion.

Corollary 2.1. *A nondecreasing sequence of integers $F = [f_1, \ldots, f_n]$ is the imbalance sequence of a $(0,1,n)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq k(k-n)$$

for $1 \leq k < n$, with equality when $k = n$.

The following theorem gives a characterization of imbalance sequences of $(0,b,n)$-tournaments [25].

Theorem 2.2. *If $b \geq 1$, then a nonincreasing sequence $F = [f_1, \ldots, f_n]$ of integers is the imbalance sequence of an $(0,b,n)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq bk(n-k),$$

for $1 \leq k \leq n$ with equality when $k = n$. □

Proof. See [25]. □

In [25] also a construction algorithm of a $(0, b, n)$-tournament can be found. Some other results on imbalances of $(0, b, n)$-tournaments and their special cases can be found in [17, 26, 30, 31].

Reid in 1978 [29] introduced the concept of the score set of $(1, 1, n)$-tournaments as the set of different scores (out-degrees) of the given tournament. At the same time he formulated the conjecture that for any set of nonnegative integers S there exists a tournament T having S as its score set. In the same paper he proved the conjecture for sets containing 1, 2, or 3 elements. Hager in 1986 [6] proved the conjecture for $|S| = 4$ and $|S| = 5$ and Yao in 1989 [32] published a proof of the whole conjecture.

There are some known results on the imbalance sets of $(0, 1, n)$-tournaments (see e.g. [20, 23, 25]).

3. Imbalances in $(0, \infty, p, q)$-tournaments

Let a, b, p and q be nonnegative integers ($b \geq a$, $b > 0$, $p \geq 1$, $q \geq 1$), $\mathcal{B}(a, b, p, q)$ be the set of directed bipartite multigraphs $B = (U \cup V, E)$, where $|U| = p$ and $|V| = q$, and the elements of each pair of vertices $u \in U$ and $v \in V$ are connected with at least a and at most b arcs. Then $B \in \mathcal{B}(a, b, p, q)$ is called (a, b, p, q)-*tournament.* $B \in \mathcal{B}(0, 1, p, q)$ is an oriented bipartite graph and a $(1, 1, p, q)$-tournament is a bipartite tournament.

According to this definition $\mathcal{B}$ is the set of the finite directed bipartite multigraphs.

For any vertex $v \in U \cup V$ of $T \in \mathcal{B}(a, b, p, q)$ let d_v^+ and d_v^- denote the out-degree and in-degree of v, respectively. Define $f_(v) = d(v)^+ - d(v)^-$ and $g(v) = d(v)^+ - d(v)^-$ as the imbalances of the vertex v for $v \in U$, resp. $v \in V$. Then the nonincreasing or nondecreasing sequences $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$ are the imbalance sequences of the (a, b, p, q)-tournament $T = (U \cup V, E)$.

3.1. Basic properties of imbalance sequences

If in an (a,b,p,q)-tournament $B(U \cup V, E)$ there are x arcs directed from vertex $u \in U$ to $v \in V$ and y arcs directed from v to u, with $a \leq x \leq b$, $a \leq y \leq b$ and $a \leq x+y \leq b$, then it is denoted by $u(x-y)v$. We also call $u(i-y)v$ as a *double.* A *tetra* in an (a,b,p,q)-tournament is an induced $(0,1,2,2)$-tournament. Define tetras of the form $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ and $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ to be of α*-type,* and all other tetras to be of β*-type.* An (a,b,p,q)-tournament is said to be of α*-type* or β*-type* according as all of its tetras are of α-type or β-type respectively. We note that an α-type tetra $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ or $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ can be respectively transformed to the β-type tetra $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-0)u_1$ or $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-1)u_1$ and vice-versa with imbalances of the vertices $u_1, u_2 \in U$ and $v_1, v_2 \in V$ remaining unchanged (see Figure 1). We note that a double of the form $u(x-x)v$ can be transformed to the double of the form $u(0-0)v$ making number of arcs lesser by $2x$ while imbalances remaining unchanged.

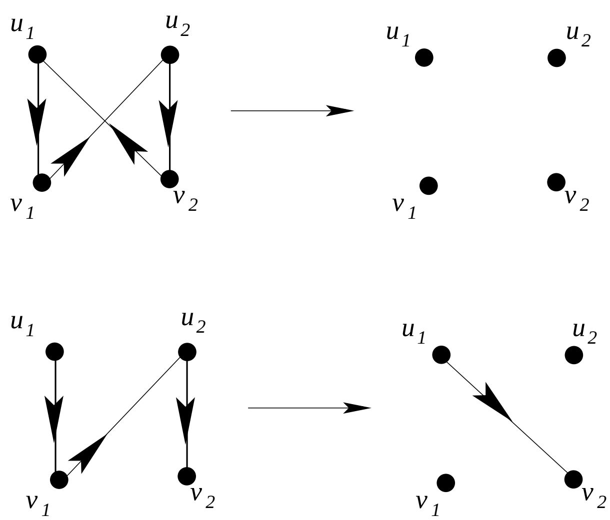

Figure 1. Transformation of an α-type tetra to a β-type tetra.

The above facts lead us to the following assertion.

Lemma 3.1. *Among all (a, b, p, q)-tournaments with given imbalance sequences, those with the fewest arcs are of β-type.*

Proof. Let $B = B(U \cup V, E)$ be an (a, b, p, q)-tournament with imbalance sequences F and G. If B is not of β-type, it contains an oriented tetra of α-type. Thus for $u_1, u_2 \in U$ and $v_1, v_2 \in V$, we have $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$, or $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ as an oriented tetra of α-type in B. Clearly $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ can be changed to $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-0)u_1$ with the same imbalance sequences and four arcs fewer, and $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ can be changed to $u_1(0-0)v_1(0-0)u_2(0-0)v_2(0-1)u_1$ with same imbalance sequences and two arcs fewer. Hence in both cases we obtain a realization $B'(U \cup V, E)$ of F and G with fewer arcs. In case there is a double of the form $u(x-x)v$, it can be transformed to the double of the form $u(0-0)v$ making number of arcs lesser by $2x$. □

A *transmitter* is a vertex whose in-degree is zero. We have the following assertion about the transmitter in a β-type $(0, b, p, q)$-tournament.

Lemma 3.2. *In a β-type $(0, b, p, q)$-tournament with nondecreasing imbalance sequences $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$, either a vertex with imbalance f_p, or a vertex with imbalance g_q, or both may act as transmitters.*

Proof. Let $U = \{u_1, u_2, \ldots, u_p\}$ and $V = \{v_1, v_2, \ldots, v_q\}$ be the parts of a $(0, b, p, q)$-tournament $B(U \cup V, E)$, so that $g(u_p) = f_p$ and $g(v_q) = g_q$. Assume that neither u_p nor v_q is a transmitter. Then there exist some vertices $u_i \in U$ and $v_j \in V$ such that $u_i(1-0)v_q$ and $v_j(1-0)u_p$. Since $g(u_p) \geq g(u_i)$ and $g(v_q) \geq bg(v_j)$, there exist vertices $u_r \in U$ and $v_s \in V$ such that $u_p(1-0)v_s$ and $v_q(1-0)u_r$ (see Figure 2(a)). We have the following possibilities.

Case (i). $v_s(1-0)u_r$ and $u_r(0-0)v_j$. Here $v_j(1-0)u_p(1-0)v_s(1-0)u_r(0-0)v_j$ is a tetra of α-type, a contradiction (see Figure 2(b)).

Case (ii). $v_s(1-0)u_r$ and $u_r(1-0)v_j$. Here $v_j(1-0)u_p(1-0)v_s(1-0)u_r(1-0)v_j$ is a tetra of α-type, a contradiction (see Figure 2(c)).

Case (iii). $u_r(1-0)v_s$ and $v_s(0-0)u_i$. In this case $u_i(1-0)v_q(1-0)u_r(1-0)v_s(0-0)u_i$ is a tetra of α-type, again a contradiction (Figure 2(d)).

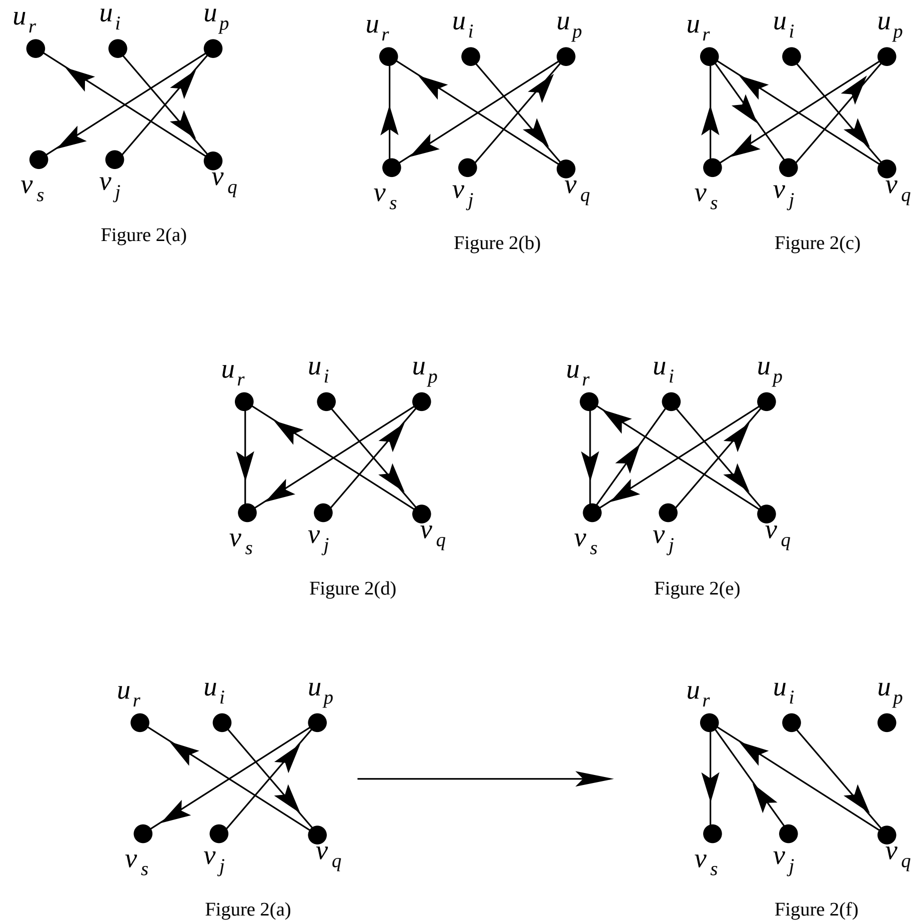

Figure 2. Illustration of the different cases in the proof of Lemma 3.2.

Case (iv). $u_r(1-0)v_s$ and $v_s(1-0)u_i$. Clearly $u_i(1-0)v_q(1-0)u_r(1-0)v_s(1-0)u_i$ is a tetra of α-type, again a contradiction (Figure 2(e)).

Case (v). If $u_r(1-0)v_s$ and $u_i(1-0)v_s$, then $b(u_i) > b(u_p)$, which is a contradiction. Similarly if $v_s(1-0)u_r$ and $v_j(1-0)u_r$, then $b(v_j) > b(v_q)$, again a contradiction.

Case (vi). Finally if $u_r(0-0)v_s$, $u_r(0-0)v_j$ and $u_i(0-0)v_s$, then there is a tetra $v_j(1-0)u_p(1-0)v_s(0-0)u_r(0-0)v_j$ and this can be transformed to the tetra $v_j(0-0)u_p(0-0)v_s(0-1)u_r(0-1)v_j$ and the imbalances remain unchanged (see Figure 2(f)). This means there is an α-type tetra $u_i(1-0)v_q(1-0)u_r(1-0)v_s(0-)u_i$, a contradiction. □

4. Reconstruction of imbalance sequences

This section starts with a necessary and sufficient condition for two sequences F and G to be imbalance sequences of some $(0, b, p, q)$-tournament. Then we deal with minimal reconstruction of imbalance sequences.

4.1. Existence of a realization of an imbalance sequence

The following result is a combinatorial criterion for determining whether some prescribed sequences are realizable as imbalance sequences of a $(0, b, p, q)$-tournament. This is analogous to a result on degree sequences of simple graphs by Erdős and Gallai [4] and a result on bipartite tournaments due to Beineke and Moon [2].

Theorem 4.1. *Let b, p and q be positive integers. Two nonincreasing sequences $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$ of integers are the imbalance sequences of some $(0, b, p, q)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j \leq bk(q-l) + bl(p-k) \tag{4.1}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$, with equality when $k = p$ and $l = q$.

Proof. The necessity follows from the fact that a directed bipartite subgraph of a $(0, b, p, q)$-tournament induced by k vertices from the first part and l vertices from the second part has a sum of imbalances at most $bk(q-l) + bl(p-k)$.

For sufficiency, assume that $F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$ are the sequences of integers in nonincreasing order satisfying conditions 4.1 but are not the imbalance sequences of any $(0, b, p, q)$-tournament. Let these sequences be chosen in such a way that p is the smallest possible and q is the smallest possible among the tournaments with the smallest p, and f_1 is the least with that choice of p and q. We consider the following two cases.

Case (i). Suppose equality in 4.1 holds for some $k \leq p$ and $l < q$, so that

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j = bk(q-l) + bl(p-k).$$

By the minimality of p and q, $F = [f_1, \ldots, f_k]$ and $G = [g_1, \ldots, g_l]$ are the imbalance sequences of some $(0, b, p, q)$-tournament $B_1(U_1 \cup V_1, E_1)$. Let $F_2 = [f_{k+1}, \ldots, f_p]$ and $G_2 = [g_{l+1}, \ldots, g_q]$.

Now,

$$\begin{aligned}
\sum_{i=1}^{f} a_{k+i} + \sum_{j=1}^{g} b_{l+j} &= \sum_{i=1}^{k+f} a_i + \sum_{j=1}^{l+g} b_j - \left(\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \right) \\
&\geq r[2(k+f)(l+g) - (k+f)q - (l+g)p] - r(2kl + kq + lp) \\
&= r(2kl + 2kg + 2fl + 2fg - kq - fq - lp - gp - 2kl + kq + lp) \\
&= r(2fg - fq - gp + 2kg + 2fl) \\
&\geq r(2fg - fq - gp),
\end{aligned}$$

for $1 \leq f \leq p-k$ and $1 \leq g \leq q-l$, with equality when $f = p-k$ and $g = q-l$. So, by the minimality for p and q, the sequences F_2 and G_2 form the imbalance sequences of the $(0, b, p-k, q-l)$-tournament $B_2(U_2 \cup V_2, E_2)$. Now construct a $(0, b, p, q)$-tournament $B(U \cup V, E)$ as follows.

Let $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ with $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$ and the arc set containing those arcs which are between U_1 and V_1 and between U_2 and V_2. Then we obtain a $(0, b, p, q)$-tournament $B(U \cup V, E)$ with the imbalance sequences F and G, which is a contradiction.

Case (ii). Suppose that the strict inequality holds in 4.1 for all $k \neq p$ and $l \neq q$. That is,

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j > bk(q-l) + bl(p-k)$$

for $1 \leq k < p$, $1 \leq l < q$.

Let $F_1 = [f_1 - 1, f_2, \dots, f_{p-1}, f_p + 1]$ and $G_1 = [g_1, \dots, g_q]$, so that F_1 and G_1 satisfy the conditions 4.1. Thus, by the minimality of f_1, the sequences F_1 and G_1 are the imbalances sequences of some $(0, b, p, q)$-tournament $B_1(U_1 \cup V_1)$. Let $f_{u_1} = f_1 - 1$ and $f_{u_p} = f_p + 1$. Since $f_{u_p} > f_{u_1} + 1$, therefore there exists a vertex $v_1 \in V_1$ such that $u_p(0-0)v_1(1-0)u_1$, or $u_p(1-0)v_1(0-0)u_1$, or $u_p(1-0)v_1(1-0)u_1$, or $u_p(0-0)v_1(0-0)u_1$, in $D_1(U_1 \cup V_1, E_1)$ and if these are changed to $u_p(0-1)v_1(0-0)u_1$, or $u_p(0-0)v_1(0-1)u_1$, or $u_p(0-0)v_1(0-0)u_1$, or $u_p(0-1)v_1(0-1)u_1$ respectively, the result is a $(0, b, p, q)$-tournament with imbalances sequences F and G, which is a contradiction proving the result. □

Since $(0, 1, p, q)$-tournaments (oriented graphs) are special (a, b, p, q)-tournaments, the following corollary of Theorem 4.1 gives a necessary and sufficient condition for nonincreasing sequences of integers to be imbalance sequences of some $(0, 1, p, q)$-tournament.

Corollary 4.1. *Two nonincreasing sequences $F = [f_1, \dots, f_p]$ and $G = [g_1, \dots, g_q]$ of integers are the imbalance sequences of some $(0, 1, p, q)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j \le k(q - l) + l(p - k), \tag{4.2}$$

for $1 \le k \le p$, $1 \le l \le q$ with equality when $k = p$ and $l = q$.

Proof. Let us substitute $b = 1$ into (4.1). □

Another simple consequence of Theorem 4.1 is the following assertion: if $F = [f_1, \dots, f_p]$ and $G = [g_1, \dots, g_q]$ are imbalance sequences of a $(0, b, p, q)$-tournament, then

$$\sum_{i=1}^{p} f_i + \sum_{j=1}^{q} g_j = 0. \tag{4.3}$$

From the other side, for arbitrary sequences of integer numbers F and G satisfying (4.3) one can find such a b, that F and G are imbalance sequences of some $(0, b, p, q)$-tournament.

Let F_{max}, G_{max}, and z be defined as follows:

$$F_{max} = \max_{1 \le i \le p} |f_i|,$$

$$G_{max} = \max_{1 \le j \le p} |g_j|,$$

and

$$z = \max(F_{max}, G_{max}).$$

The following assertion gives lower and upper bound for b_{min}.

Lemma 4.1. *If* $p \ge 1$ *and* $q \ge 1$, *then*

$$\max\left(\left\lceil \frac{F_{\max}}{q} \right\rceil, \left\lceil \frac{G_{\max}}{p} \right\rceil\right) \le b_{\min} \le \max(F_{\max}, G_{\max}). \tag{4.4}$$

Proof. From one side it is easy to write a program which constructs a $(0, z, p, q)$-tournament, and even the uniform allocation of the degrees requires

$$b_{min} \ge \max\left(\left\lceil \frac{F_{max}}{q} \right\rceil, \left\lceil \frac{G_{max}}{p} \right\rceil\right). \tag{4.5}$$

□

We are interested in the least possible b allowing the realization of F and G.

4.2. Computation of b_{min}

We are interested in the computation of the minimal value of b, satisfying (4.1) Using Theorem 4.1 we can compute b_{min}.

Let

$$\alpha(k, l) = \sum_{i=1}^{k} f_i + \sum_{j=1}^{l} g_j$$

and

$$\beta(k, l) = bk(q - l) + bl(p - k)$$

for $1 \le i \le p$ and $1 \le j \le q$.

The following theorem allows quickly to compute b_{min}.

Theorem 4.2. *Two nonincreasing sequences* $F = [f_1, \ldots, f_p]$ *and* $G = [g_1, \ldots, g_q]$ *of integers are the imbalance sequences of some* $(0, b, p, q)$*-tournament* B *if and only if* $b \ge b_{min}$, *where*

$$b_{min} = \min_{1 \le k \le p, 1 \le l \le q} \{b \mid \alpha(k, l) \le \beta(k, l)\}. \tag{4.6}$$

Proof. If $k = p$ and $l = q$, then both sides of (4.1) equal to zero, otherwise the right side is positive and a multiple of b, therefore (4.6) holds, if b is sufficiently large.

The following program MINIMAL is based on Theorem 4.2. The pseudocode uses the conventions described in [**?**].

Inputs. p and q: the numbers of the elements in the prescribed imbalance sequences;
b: maximum number of permitted arcs between two vertices $u \in U$ and $v \in V$;
$F = [f_1, \ldots, f_p]$ and $G = [g_1, \ldots, g_q]$: given nonincreasing sequences of integers.

Output. b_{min}: the minimal number of allowed arcs between two vertices belonging to different parts of B.

Working variables. i, j: cycle variables;
S: actual sum of the imbalances;
$L = \alpha(k, l)$: the actual value of the left side of (4.2);
$R = \alpha(k, l)$: the actual value of the right side of (4.2).

MINIMAL(b, p, q, F, G, b_{min})

```
S ← 0
F_max ← max(|f_1|, |f_p|)
G_max ← max(|g_1|, |g_q|)
b_min ← max(⌈F_max / q⌉, ⌈G_max / p⌉)
for i ← 1 to p
    S ← S + f_i
    L ← S
    for j ← 1 to q
        L ← S + g_j
        R ← b_min[i(q − j) + j(p − i)]
        if L > R
            b_min ← b_min + 1
        if b_min == max(F_max, G_max)
            return b_min
return b_min
```

MINIMAL computes b_{min} in all cases in $\Theta(pq)$ time.

4.3. Reconstruction of imbalance sequences

The next result provides a useful recursive test whether given sequences of integers in nondecreasing order are the imbalance sequences of some $(0, b, p, q)$-tournament.

Theorem 4.3. *Let* $F = [f_1, \ldots, f_p]$ *and* $G = [g_1, \ldots, g_q]$ *be nondecreasing sequences of integers satisfying* (4.1)*, further either* $f_p > 0$ $f_p \leq bq$, $g_q \leq bp$ *or* $g_p > 0$, $g_q \leq bp$, $f_p \leq bq$ *Let* F' *be obtained from* F *by deleting* f_p*, and* G' *be obtained as follows. Choose* h, $1 \leq h \leq b$*, such that* $(h-1)q < f_p \leq hq$ *and increase* $f_p - (h-1)q$ *smallest elements of* G *by* h *each, and* $q - (f_p - (h-1)q) = hq - f_p$ *remaining elements by* $(h-1)$ *each. Then* F *and* G *are imbalance sequences of some* $(0, b, p, q)$*-tournament if and only if* F' *and* G' *are imbalance sequences of some* $(0, b, p, q)$*-tournament.*

Proof. Due to the symmetry it is sufficient to prove the theorem for the case when $f_p > 0$.

Let F' and G' be the imbalance sequences of some $(0, b, p, q)$-tournament D' with parts U' and V'. Then a $(0, b, p, q)$-tournament D with imbalance sequences F and G can be obtained by adding a transmitter u_p to U' such that $u_p(h-0)v_i$ for those vertices v_i in V' whose imbalances were increased by h and $u_p((h-1)-0)v_j$ for those vertices v_j in V' whose imbalances were increased by $h-1$ in going from F and G to F' and G'.

Conversely, suppose F and G are the imbalance sequences of a $(0, b, p, q)$-tournament D with parts U and V. By Lemma 3.1, assume D to be of β-type. Then there is a vertex u_p in U with imbalance f_p (or a vertex v_q in V with imbalance g_q, or both u_p and v_q) which is a transmitter. Let the vertex u_p in U with imbalance f_p be a transmitter. Clearly, $f_p > 0$ so $d^+_{u_p} > 0$ and $d^-_{u_p} = 0$.

Let V_1 be the set of $f_p - (h-1)q$ vertices of smallest imbalances in V, and let $V_2 = V - V_1$.. Construct D such that $u_p(h-0)v_i$ for all $v_i \in V_1$ and $u_p((h-1)-0)v_j$ for all vertices $v_j \in V_2$. This construction is possible since if there there are less than h arcs say $h-t$ arcs from u_p to any vertex in V_1, then these t arcs from u_p will be directed towards vertices in V_2, and by transformations will be made directed to v_i in V_1. Clearly $D - \{u_p\}$ realizes F' and G'. $\square$

As a consequence of Theorem 4.3, we have the following recursive and constructive criteria for $(0, 1, p, q)$-tournaments.

Corollary 4.2. *Let $F = [f_1, \cdots, f_p]$ and $G = [g_1, \cdots, g_q]$ be non-decreasing sequences of integers $f_p > 0$, $f_p \leq q$ and $g_q \leq p$. Let F' be obtained from F by deleting one element f_p, and G' be obtained from G by increasing g_p smallest elements of G by 1 each. Then F and G are the imbalance sequences of some $(0, 1, p, q)$-tournament if and only if F' and G' are imbalance sequences.*

4.4. Examples

Example 1. The first example illustrates the application of Theorem 4.3. Let $p = 4$, $F_1 = [-2, -2, 3, 4]$, $q = 3$, and $G_1 = [-5, -1, 3]$. Then according to Lemma we have

$$2 \leq b_{min} \leq 5.$$

Theorem 4.2 results $b_{min} = 2$.

The steps of the reconstruction using Theorem 4.3 are as follows. In $f_p > 0$ and $h = 2$, implying $F_2 = [-2, -2, 3]$ and $G_2 = [-3, 0, 4]$, and the constructed arcs are $u_4(2 - -0)v_1$, $u_4(1 - -0)v_2$, and $u_4(1 - -0)v_3$. Now $f_3 > 0$ and $g_3 > 0$, therefore we can choose. Let us choose g_3, then $h = 2$, implying $G_3 = [-3, 0]$ and after sorting $F_3 = [-1, 0, 4]$ and the constructed arcs are $v_3(2 - -0)u_1$, $v_3(1 - -0)u_2$ and $v_3(1 - -0)u_3$. Now $f_3 > 0$ and $h = 2$, implying $F_4 = [-1, 0]$ and after sorting $G_4 = [-1, 2]$ and the constructed arcs are $u_3(2 - -0)v_1$ and $u_3(2 - -0)v_2$. Now $g_2 > 0$ and $h = 1$ implying $G_5 = [-1]$ and $F_5 = [0, 1]$ and the constructed arcs are $v_2(1 - -0)u_1$ and $v_2(1 - -0)u_2$. Now $f_1 > 0$, so $h = 1$, implying $F_6 = [0]$ and $G_6 = [0]$. The constructed arcs are $u_1(1 - -0)v_1$.

The constructed $(0, 2, 4, 3)$-undigraph having imbalance sequences $F = (-2, -2, 3, 4)$ and $(-5, -1, 3)$ is shown in Figure 3.

Example 2. The second example illustrates the application of Theorem 4.2. Let $p = 4$ and $q = 5$, $F_1 = [-3, 1, 2, 2]$ and $G_1 = [-3, -1, 0, 1, 1]$. In this case Lemma 4.4 results

$$1 \leq b_{min} \leq 3.$$

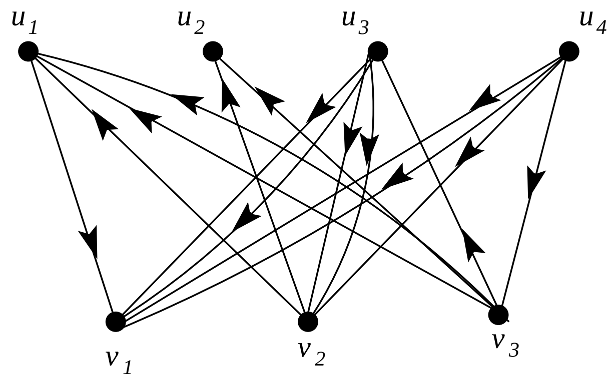

Figure 3. Result of the reconstruction of the imbalance sequences in Example 1.

Theorem 4.2 gives the precise value $b_{min} = 1$.

The steps of the recursive reconstruction using Theorem 4.2 are as follows.

We choose $f_4 > 0$, implying $F_2 = [-3, 1, 2]$, $G_2 = [-2, 0, 0, 1, 1]$ and $u_4(1 - -0)v_1$, $u_4(1 - -0)v_2$. Now $f_3 > 0$, implying $F_3 = [-3, 1]$, after sorting $G_3 = [-1, 0, 1, 1, 1]$ and $u - 3(1 - -0)v_1$, $u_3(1 - -0)v_2$. Now $f_2 > 0$, implying $F_4 = [-3]$, $G_4 = [0, 0, 1, 1, 1]$ and $u_2(1 - -0)v_1$. Now $g_4 > 1$, implying $G_5 = [0, 0, 1, 1]$, $F_5 = [-2]$ and $v_5(1 - -0)u_1$. Now $g_4 > 0$, implying $G_6 = [0, 0, 1]$, $F_6 = [-1]$ and $v_4(1 - -0)u_1$. Now $g_6 = 1$, implying $G_7 = [0, 0]$, $F_7 = [0]$, and $v_2(1 - -0)u_1$.

The constructed $(0, 1, 4, 5)$-undigraph having imbalance sequences $F = (-3, 1, 2, 2)$ and $G = [-3, -1, 0, 1, 1)$ is shown in Figure 4.

5. Imbalance sets in bipartite multidigraphs

K. B. Reid in 1978 [29] introduced the concept of the score set of tournaments as the set of different scores (out-degrees) of a tournament. At the same time he formulated the conjecture that for any set of non-negative integers S there exists a tournament T having S as its score set. At the same time he proved the conjecture for sets containing 1, 2,

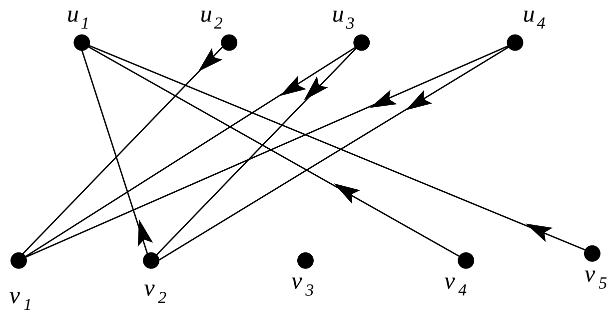

Figure 4. Result of the reconstruction of the imbalance sequences in Example 2.

or 3 elements. Hager in 1986 [6] proved the conjecture for $|S| = 4$ and $|S| = 5$ and Yao [32] published a proof of the conjecture

In an analogous manner we define the imbalance set of a bipartite multigraph $B = (U \cup V, E)$ as the union of the sets of different imbalances of the vertices in U and V.

5.1. Existence of a $(0, 1, p, q)$-tournament with prescribed imbalance sets

First we show the existence of a $(0, 1, p, q)$-tournament with given set of integers as imbalance sets.

Theorem 5.1. *Let* $F = [f_1, \ldots, f_p]$ *and* $G = [-g_1, \ldots, -g_p]$, *where* $f_1, \ldots, f_p, g_1, \ldots, g_p$ *are positive integers with* $f_1 < \cdots < f_p$ *and* $g_1 < \cdots < g_p$. *Then there exists a connected* $(0, 1, p, p)$*-tournament with imbalance set* $F \cup G$.

Proof. Construct a $(0, 1, p, q)$-tournament $B(U \cup V, E)$ as follows. Let $U = U_1 \cup \cdots \cup U_p$, $V = V_1 \cup \cdots \cup V_p$ with $U_i \cap U_j = \emptyset$ $(i \neq j)$, $V_i \cap V_j = \emptyset$ $(i \neq j)$, $|U_i| = b_i$ for all i, $1 \leq i \leq p$ and $|V_j| = a_j$ for all j, $1 \leq j \leq p$. Let there be an arc from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq p$, so that we obtain the $(0, 1, p, q)$-tournament $B(U \cup V, E)$ with the given imbalance sets of vertices as follows.

For $1 \leq i, j \leq p$, $f_{u_i} = |V_i| - 0 = f_i$, for all $u_i \in U_i$ and $g_{v_j} = 0 - |U_j| = -g_j$, for all $v_i \in V_i$.

Therefore, the imbalance set of $B(U \cup V, E)$ is $F \cup G$.

The oriented bipartite graph constructed above is not connected. In order to see the existence of oriented bipartite graph, whose underlying graph is connected, we proceed as follows.

Taking $U_i = \{u_1, \ldots, u_{b_i}\}$ and $V_j = \{v_1, \ldots, v_{a_j}\}$, and let there be an arc from each vertex of U_i to every vertex of V_j except the arcs between u_{g_i} and v_{f_j}, that is $u_{b_i}(0-0)g_{a_j}$, $1 \le i \le p$ and $1 \le j \le p$. We take $u_{g_1}(0-0)g_{f_2}$, $u_{g_2}(0-0)v_{f_3}$, and so on $u_{g_{(n-1)}}(0-0)v_{f_n}$, $u_{g_n}(0-0)v_{f_1}$. The underlying graph of this $(0,1,p,p)$-tournament is connected.

5.2. Existence of a $(0, b, p, q)$-tournament with prescribed imbalances

Finally, we prove the existence of a $(0, b, p, q)$-tournament with prescribed sets of positive integers as its imbalance set.

Let $(f_1, \ldots, f_p, g_1, \ldots, g_q)$ denote the greatest common divisor of $f_1, \ldots, f_p, g_1, \ldots, g_q$.

Theorem 5.2. *Let $b \ge 1$ a positive integer, $F = [f_1, \ldots, f_p]$ and $Q = [-g_1, \ldots, -g_q]$, where $f_1, \ldots, f_p, g_1, \ldots, g_q$ are positive integers with $f_1 < \cdots < f_p$, $g_1 < \cdots < g_q$ and $(f_1, \ldots, f_p, g_1, \ldots, g_q) = t \le b_{min}$. Then there exists a connected $(0, b, p, q)$-tournament with imbalance set $P \cup Q$.*

Proof. Since $(f_1, \ldots, f_p, g_1, \ldots, g_q) = t$, where $1 \le t \le b$, there exist positive integers $x_1, \ldots, x_p, y_1, \ldots, y_q$ with $x_1 < \cdots < x_p$, $y_1 < \cdots < y_q$ such that $f_i = tx_i$ for $1 \le i \le p$ and $g_j = ty_j$ for $1 \le j \le q$.

Construct a $(0, b, p, q)$-tournament $B(U \cup V, E)$ as follows. Let $U = U_1 \cup \cdots \cup U_p \cup U^1 \cup \cdots \cup U^p$, $V = V_1 \cup \cdots \cup V_p \cup V^1 \cup \cdots \cup V^p$ with $U_i \cap U_j = \emptyset$, $U_i \cap U^j = \emptyset$, $U^i \cap U^j = \emptyset$, $V_i \cap V_j = \emptyset)$, $V_i \cap V^j = \emptyset$, $V^i \cap V^j = \emptyset$, $i \ne j$, $|U_i| = x_i$ for all i, $1 \le i \le p$ and $|U^i| = g_i$ for all i, $1 \le i \le p$, $|V_i| = x_i$ for all i, $1 \le i \le p$ and $|V^i| = g_i$ for all i, $1 \le i \le q$. Let there be t arcs directed from every vertex of U_i to each vertex of V_i for all i, $1 \le i \le p$ and let there be t arcs directed from every vertex of U^i to each vertex of V^i for all i, $1 \le i \le q$, so that we obtain the $(0, b, p, q)$-tournament $B(U \cup V, E)$ with the imbalances of vertices as follows.

For $1 \le i \le p$,

$$f_{u_i} = t|V_i| - 0 = tx_i = f_i, \text{ for all } u_i \in U_i,$$

$$g_{v_i} = 0 - t|U_i| = -ty_1 = -g_1, \text{for all } v_i \in V_i,$$

for $1 \le i \le q$,

$$f_{u^i} = t|V^i| - 0 = tf_1 = g_1, \text{ for all } u^i \in U^i,$$

$$g_{v^i} = 0 - t|U^i| = -ty_i = -g_i, \text{ for all } v^i \in V^i.$$

Therefore the imbalance set of $B(U \cup V, E)$ is $P \cup Q$.

The $(0, b, p, q)$-tournament constructed above is not connected. In order to see the existence of a $(0, b, p, q)$-tournament, whose underlying graph is connected, we proceed as follows.

Let $U_i = \{u_1, \ldots, u_{g_i}\}$ and $V_j = \{v_1, \ldots, v_{f_j}\}$, and let there be an arc from each vertex of U_i to every vertex of V_j except the arcs between u_{g_i} and v_{f_j}, that is $u_{g_i}(0-0)v_{f_j}$, $1 \le i \le q$ and $1 \le j \le q$. We take $u_{g_1}(0-0)v_{f_2}$, $u_{b_2}(0-0)v_{a_3}$, and so on $u_{b_{(n-1)}}(0-0)v_{a_n}$, $u_{b_n}(0-0)v_{a_1}$. The underlying graph of this $(0, b, p, q)$-tournament is connected.

An overview of the results on score sets can be found in [21].

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Shariefuddin Pirzada
University of Kashmir, Department of Mathematics
Srinagar
India
sdpirzada@yahoo.co.in

Antal Iványi
Eötvös Loránd University, Department of Computer Algebra
Budapest
Hungary
tony@compalg.inf.elte.hu

Namir A. Shah
University of Kashmir, Department of Mathematics
Srinagar
India
wmshah@rediffmail.com.in