Degree sequences of multigraphs

Antal Iványi
(Budapest, Hungary)

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Definitions

Definition

Let a, b and n be integers, $n \geq 1$ and $b \geq a \geq 0$. (a, b, n)*-graphs* are loopless graphs on n vertices, in which different vertices are connected with at least a and at most b edges.

Definition

Let l, m and u be nonnegative integers, further $l \leq u$ and $m \geq 1$. The sequence $s = (s_1, \ldots, s_m)$ of integers is (l, u, m)*-bounded*, if $l \leq s_i \leq u$ hold for all $1 \leq i \leq m$ indices. An $s = (s_1, \ldots, s_m)$ (l, u, m)-bounded sequence is called (l, u, m)*-regular*, if $u \geq s_1 \geq \cdots \geq s_m \geq l$. An (l, u, m)-regular sequence is called (l, u, m)*-graphic*, if there exists a (l, u, m)-graph having s as its degree sequence.

The investigated problem

Our aim is to investigate the conditions and algorithms which decide whether a regular sequence $s = (s_1, \ldots, s_n)$ of integers is the degree sequence of an (a, b, n)-graph.

Erdős-Gallai algorithm (EG)

```
Erdős-Gallai(n, s, L)
01 L = 0                          // line 01: setting of the probable value
02 H_1 = s_1                      // line 02: computing of H_1
03 for i = 2 to n                 // lines 02–03: computing of the further H_i's
04       H_i = H_{i-1} + s_i
05 if H_n is odd                  // lines 04–07: test of the parity
06    return L
07 for i = 1 to n - 1             // line 07–15: test of s
08          R = 0                 // line 08: initialization
09          for k = j + 1 to n    // lines 09–10: tail capacity
10              R = R + min(j, s_k)
11          if H_j - j(j - 1) > R // line 11: test of s
12             return L           // line 12: s is non graphical
13 L = 1                          // lines 13–14: s is graphical
14 return L
```

n	$R(0,1,n)$	$G(0,1,n)$	$R(0,2,n)$	$G(0,2,n)$	$R(2,3,n)$	$G(2,3,n)$
1	1	1	1	1	1	1
2	3	2	6	3	10	4
3	10	4	35	10	84	23
4	35	11	210	52	715	189
5	126	31	1287	283	6188	1582
6	462	102	8008	1706	54264	13583
7	1716	342	50388	10436	480700	122345
8	6435	1213	319770	65370	4292145	1092573
9	24310	4361	2042975	413111	38567100	9816598
10	92378	16016	13123110	2633537	348330136	88680716
11	352716	59348	84672315	16882153	3159461968	804480107

Table 1: The number of (a, b, n)-regular and (a, b, n)-graphical sequences for $a = 0$ and $b = 1$, $a = 0$ and $b = 2$, $a = 2$ and $b = 3$ and for $n = 1, \ldots, 11$ vertices.

Havel-Hakimi algorithm (HH)

Havel-Hakimi(n, s, L)

01 $L = 0$
02 **for** $i = 1$ **to** $n - 1$ // line 02–07: test of the elements of s
03   **if** $s_{s_i+i} == 0$ // lines 02–03: s is not graphical
04   **return** L
05   **for** $j = i + 1$ **to** $s_i + i$
06         $s_j = s_j - 1$
07   sort $(s_{i+1}, \ldots, s_n)$ in decreasing order
08 $L = 1$ // lines 08–09: s is graphical
09 **return** L

n/i	1	2	3	4	5	6	7	8	9	10	11
1	0										
2	1	0									
3	6	0	0								
4	22	2	0	0							
5	85	8	2	0	0						
6	311	35	12	2	0	0					
7	1169	128	58	17	2	0	0				
8	4369	488	239	100	24	2	0	0			
9	16524	1805	942	471	173	32	2	0	0		
10	62650	6800	3601	2021	956	289	43	2	0	0	
11	239008	25571	13677	8147	4561	1877	470	55	2	0	0

Table 2: The number of the filtered non $(0, 1, n)$-graphical sequences in the i-th round of HH for $n = 1, \ldots, 11$ vertices and $i = 1, \ldots, 10$.

n/i	1	2	3	4	5	6	7	8	9	10	11
1	0										
2	1	0									
3	1	2	0								
4	1	8	1	0							
5	1	16	12	1	0						
6	1	29	48	22	1	0					
7	1	47	130	127	35	1	0				
8	1	72	306	488	290	54	1	0			
9	1	104	618	1492	1475	591	78	1	0		
10	1	145	1158	3863	5757	3868	1112	110	1	0	
11	1	195	1998	8890	18440	18662	9053	1958	149	1	0

Table 3: The number of the filtered $(0, 1, n)$-graphical sequences in the i-th round of HH for $n = 1, \ldots, 11$ vertices and $i = 1, \ldots, 10$.

Let $n_i(a, b, n, \mathrm{A})$, resp. $m_i(a, b, n, \mathrm{A})$ denote the number of filtered by algorithm A non (a, b, n)-graphical, resp. (a, b, n)-graphical sequences in the ith round of the testing of all (a, b, n)-regular sequences, further let

$$N = \sum_{i=1}^{n-1} n_i \quad \text{és} \quad M = \sum_{i=1}^{n-1} m_i, \tag{1}$$

$$X(a, b, n, \mathrm{A}) = \frac{\sum_{i=1}^{n-1} in_i}{N}, \tag{2}$$

$$Y(a, b, n, \mathrm{A}) = \frac{\sum_{i=1}^{n-1} im_i}{M}, \tag{3}$$

$$Z(a, b, n, \mathrm{A}) = \frac{\sum_{i=1}^{n-1} i(m_i + n_i)}{N + M}, \tag{4}$$

$$X'(a, b, n, \mathrm{A}) = \frac{\sum_{i=1}^{n-1} in_i}{N(n-1)}, \tag{5}$$

$$Y'(a, b, n, \mathrm{A}) = \frac{\sum_{i=1}^{n-1} im_i}{M(n-1)}, \tag{6}$$

$$Z'(a, b, n, \mathrm{A}) = \frac{\sum_{i=1}^{n-1} i(m_i + n_i)}{(N + M)(n-1)}. \tag{7}$$

Table 4 characterizes the efficiency of algorithm HHL during the testing of $(0,1,n)$-regular sequences for $n = 1, \ldots, 11$ vertices. In line 11 of Table 4 we see $X'(0,1,11) = 0.136887459$ and $Y'(0,1,11) = 0.615705668$. According to these data in the case of 11 vertices the filtering of *all* nongraphical sequences needs in average the 14 % of the rounds, while the filtering of the graphical sequences requires 62 % of the rounds implying that the complete filtering requires in average 22 % of the rounds.

n / measure	X	Y	Z	X'	Y'	Z'
2	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
3	1.000000000	1.750000000	1.300000000	0.500000000	0.875000000	0.650000000
4	1.083333333	2.454545455	1.514285714	0.361111111	0.818181818	0.504761905
5	1.126315789	3.032258065	1.595238095	0.281578947	0.758064516	0.398809524
6	1.180555556	3.588235294	1.712121212	0.236111111	0.717647059	0.342424242
7	1.220524017	4.111111111	1.796620047	0.203420670	0.685185185	0.299436674
8	1.262734584	4.629843364	1.897435897	0.180390655	0.661406195	0.271062271
9	1.299062610	5.140793396	1.988235294	0.162382826	0.642599175	0.248529412
10	1.335323852	5.650162338	2.083407305	0.148369317	0.627795815	0.231489701
11	1.368874588	6.157056683	2.174534186	0.136887459	0.615705668	0.217453419

Table 4: Efficiency of HH for the testing of all $(0,1,n)$-regular sequences for $n = 2, \ldots, 11$ vertices.

Havel-Hakimi linear testing algorithm (HHL)

The original Havel-Hakimi algorithm in worst case requires quadratic time to test the $(0, 1, n)$-regular sequences. Using the new concepts weight point (w_i) and reserve (r_i) we reduced the worst running time to $O(n)$.
If $s_1 < bi$, then let $w_i = 0$, otherwise let w_i the largest k $(1 \leq k \leq n)$ having the property $s_k \geq bi$. But if $s_1 < bi$, then let $w_i = 0$. In the case $b = 1$ the new definition is the same as the old one.
In HHL the weight point w_i determines the increment of the tail capacity when we switch to the investigation of the next element of s.
The remainder r_i belonging to s_i is defined as the unused part of the actual tail capacity and can be computed by the formulas

$$r_i = w_1 - 1 - s_1 \tag{8}$$

and

$$r_i = w_i - r_{i-1} - s_i \quad \text{for} \quad 1 \leq i \leq n - 1. \tag{9}$$

HAVEL-HAKIMI-LINEAR(n, s, L)

```
01 L = 0                                   // lines 01: set the probable value
02 if s_1 == 0                             // lines 02–04: test of the sequence of zeros
03     L = 1
04     return L
05 if s_{s_1+1} == 0                       // lines 05–07: test of s_1 in constant time
06     return L
07 H_1 = s_1                               // line 07: initialization of H
08 for i = 2 to n                          // lines 09–09: further H_i's
09       H_i = H_{i-1} + s_i
10 if H_n is odd                           // lines 10–11: test of the parity
11     return L
12 w_1 = n                                 // lines 12–15: computation of weight point and reserve
13 while s_{w_1} < 1                       // lines 13–24: testing of s
14           w_1 = w_1 - 1
15 r_1 = w_1 - 1 - s_1
16 s_{n+1} = 0
17 for i = 2 to n - 1
18      if s_i ≤ i or s_{i+1} = 0
19          L = 1
```

20   **return** L
21    $w_i = w_{i-1}$
22   **while** $s_{w_i} < i$ and $w_i > 0$
23          $w_i = w_i - 1$
24   **if** $s_i > w_i - 1 + r_{i-1}$ // line 24: Is s graphical?
25   **return** L // line 25: s is not graphical
26    $r_i = w_i + r_{i-1} - s_i$ // line 26: updating of the reserve
27 $L = 1$ // lines 27–28: s is graphical
28 **return**

Theorem

The running time of Havel-Hakimi-Linear *is in best case* $\Theta(1)$, *and in worst case* $\Theta(n)$.

Example 1.

Let our first example $s = (3^3, 1)$. According to lines 01–15 $r_1 = 0$. For $i = 2$ we get $w_i = 3$ and the condition of line 22 is not satisfied, therefore s is *not* $(0, 1, 4)$*-graphical.*

Example 2.

Let our next example $s = (5, 3^2, 2, 1^3)$. In lines 01–15 we get $w_1 = 7$ and $r_1 = 1$. For $i = 2$ according to lines $w_i = 3$, the condition of line 22 does not hold and according to line 25 $r_2 = 1$. When $i = 3$, then $s_i \geq i$ and so according to line 16 s is $(0, 1, 7)$*-graphical.*

Example 3.

Now let $s = (5, 4, 1^5)$. At first get $r_1 = 1$, then for $i = 2$ we have $w_i = 2$, therefore the conditions in line 22 holds, so s is *not* $(0, 1, 7)$*-graphical.*

Theorem of Chungphaisan I

Theorem

(Chungphaisan [1]) Let $n \geq 1$. *An* $s = (s_1, \ldots, s_n)$ $(0, b, n)$*-regular sequence is* $(0, b, n)$*-graphical if and only if*

$$\sum_{i=1}^{n} s_i \quad \text{is even} \tag{10}$$

and

$$\sum_{i=1}^{j} s_i - bj(j-1) \leq \sum_{k=j+1}^{n} \min(jb, s_k) \quad (j = 1, \ldots, n-1). \tag{11}$$

In worst case the algorithm based on this theorem requires quadratic time, but the following assertion allows to test the sequences in linear time.

Theorem of Chungphaisan II

Theorem

If $n \geq 1$, then an $s = (s_1, \ldots, s_n)$ $(0, b, n)$-regular sequence is $(0, b, n)$-graphical if and only if

$$\sum_{i=1}^{n} s_i \quad \textit{is even} \tag{12}$$

and

$$H_i > bi(y_i - 1) + H_n - H_y \quad (i = 1, \ldots, n-1), \tag{13}$$

where

$$y_i = \max(i, w_i) \quad (i = 1, \ldots, n-1). \tag{14}$$

Chungphaisan-Erdős-Gallai algorithm

CHUNGPHAISAN-ERDŐS-GALLAI-LINEAR(n, s, b, L)

```
01 H_1 = s_1                                  // line 01: initialization of H_1
02 for i = 2 to n − 1                         // line 02–03: computation of the elements of H
03     H_i = H_{i−1} + s_i
04 if H_n is odd                              // line 04–05: test of the parity
05    return
06 w = n                                      // lines 06: initialization of the first weight point
07 for i = 1 to n − 1                         // lines 07–14: test of s
08     while s_w < ib and w > 0
09         w = w − 1
10     y = max(i, w)
11     if H_i > bi(y − 1) + H_n − H_y
12        return L                            // lines 12: acceptance of s
13 L = 1                                      // lines 13–14: acceptance of b
14 return L
```

Theorem

The running time of CHUNGPHAISAN-ERDŐS-GALLAI-LINEAR *is* $\Theta(n)$ *in all cases.*

Examples

Example 1

Let $b = 3$ and $s' = (13, 10, 5, 5, 4, 1)$. $H_6 = 38$ is even. If $i = 1$, then $w_i = y = 5$ and the condition in line 18 is not satisfied $(13 \leq 3 \cdot 1 \cdot (5 - 1))$. If $i = 2$, then $w_i = y = 2$ and the condition in line 18 holds $(23 > 3 \cdot 2 \cdot (2 - 1)) + 5 + 5 + 4 + 1$, therefore s is *not* $(0, 3, 6)$*-graphical.*

Example 2

Let b remain 3, but change s to $s' = (13, 10, 5, 5, 4, 3)$. The first difference comparing with the previous example comes when $i = 2$. Now $23 \leq 3 \cdot 2 \cdot (2 - 1)) + 5 + 5 + 4 + 3$, and the condition in line 18 holds for $i = 3,\ 4$ and 5 too, therefore s' is $(0, 3, 6)$-graphical.

n/i	1	2	3	4	5	6	7	8	9	10
1	0									
2	3	0								
3	22	3	0							
4	132	26	2	0						
5	824	164	31	4	0					
6	5084	1026	276	75	3	0				
7	31902	6288	2018	829	111	50				
8	201366	39090	13282	7231	1837	203	4	0		
9	1281918	244833	84340	53594	20681	4259	298	6	0	
10	8207232	1548774	529578	365461	183262	59726	8709	470	5	0
11	52819163	9866545	3331910	2385963	1404590	632058	155070	17213	660	7

Table 5: The number of the excluded in the ith ($i = 1, \ldots, 10$) by ChEGL non $(0, 2, n)$-graphical sequences for $a = 0$, $b = 2$ and $n = 1, \ldots, 11$ vertices.

n/i	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	0								
3	1	9	0							
4	1	7	42	0						
5	1	10	29	224	0					
6	1	14	49	183	1297	0				
7	1	18	70	345	1143	7658	0			
8	1	23	97	559	2326	7262	46489	0		
9	1	28	125	846	4038	15927	46074	286007	0	
10	1	34	159	1191	6520	29629	107724	295609	1779026	0
11	1	40	193	1624	9668	50663	213399	728610	1900061	11154877

Table 6: The number of the filtered $(0, 2, n)$-graphical sequences in the ith ($i = 1, \ldots, 10$) round of ChEGL for $n = 1, \ \ldots, \ 11$ vertices.

n / measure	X	Y	Z	X'	Y'	Z'
2	1, 000000000	1, 000000000	1, 000000000	1, 000000000	1, 000000000	1, 000000000
3	1, 120000000	1, 900000000	1, 342857143	0, 560000000	0, 950000000	0, 671428571
4	1, 187500000	2, 820000000	1, 576190476	0, 395833333	0, 940000000	0, 525396825
5	1, 232649071	3, 803030303	1, 759906760	0, 308162268	0, 950757576	0, 439976690
6	1, 280785891	4, 788212435	1, 957042957	0, 256157178	0, 957642487	0, 391408591
7	1, 322698224	5, 770438549	2, 137870128	0, 220449704	0, 961739758	0, 356311688
8	1, 363989613	6, 751572493	2, 320248929	0, 194855659	0, 964510356	0, 331464133
9	1, 402468979	7, 733105601	2, 496464714	0, 175308622	0, 966638200	0, 312058089
10	1, 439464334	8, 714770487	2, 670148311	0, 159940482	0, 968307832	0, 296683146
11	1, 474743645	9, 697001722	2, 839981439	0, 147474365	0, 969700172	0, 283998144

Table 7: The efficiency of ChEGL during the testing of $(0, 2, n)$-regular sequences for $n = 1, \ldots, 11$ vertices.

Degree sequences of (a, b, n)-graphs

Theorem 2 due to Chungphaisan has the following straightforward consequence.

Corollary

Let $n \geq 2$. An $s = (s_1, \ldots, s_n)$ (a, b, n)-reg.ular sequence is (a, b, n)-graphical if and only if the sequence $s' = (s_1 - a(n-1), \ldots, s_n - a(n-1))$ is $(0, b-a, n)$-graphical.

Proof.

In an (a, b, n)-graph the elements of every pair of vertices is connected with at least a arcs. Therefore if we remove a arcs, then we get a $(0, b-a, n)$-graph. □

Using Corollary 5 it is easy to test an (a, b, n)-regular sequence: we use ChEGL with input sequence $s' = (s_1 - a(n-1), \ldots, s_n - a(n-1))$.

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