Acta Univ. Sapientiae, Informatica, **4,** 1 (2012) xx–yy

Imbalance sequences of football tournaments (February 6, 2012)

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Abstract. An open problem is the complexity to decide whether a sequence of nonnegative integer numbers is the final score sequence of a football tournament [9, 18, ?, ?]. In this paper we propose a polynomial algorithm to decide whether a sequence of nonnegative integers is the imbalance sequence of a football tournament.

1 Introduction

Let $a,\ b$ and n be nonnegative integers ($b \geq a \geq 0,\ n \geq 1$), $\mathcal{T}(a, b.n)$ be the set of directed multigraphs $T = (V, E)$, where $|V| = n$, and elements of each pair of different vertices $u,\ v \in V$ are connected with at least a and at most b arcs [15]. $T \in \mathcal{T}(a, b, n)$ is called (a, b, n)-*tournament.* $(1, 1, n)$-tournaments are the usual tournaments, and $(0, 1, n)$-tournaments are also called oriented graphs or simple directed graphs [10]. The set $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcup_{b\geq 1,\ n\geq 1} \mathcal{T}(0, b, n).$$

An (a, b, n)-tournament is called *complete,* if the set of permitted results is $\{0 : c, 1 : c-1, \ldots, c : 0\}$ for all possible c $(a \leq c \leq b)$. Is some of these results are prohibited, then the tournament is called *incomplete.*

Computing Classification System 1998: G.2.2. [Graph Theory]: Subtopic - Network problems.
Mathematics Subject Classification 2010: 05C85, 68R10
Key words and phrases: tournament, imbalance sequence, polynomial algorithm, complexity

The first result on the out-degree sequences of complete $(1, 1, n)$-tournaments belongs to Landau, who in 1953 published the following necessary and sufficient condition.

Theorem 1 *If* $n \geq 1$ *then a sequence* $(s_1, \ldots, s_n)$ *of integers satisfying* $0 \leq s_1 \leq \ldots \leq s_n$ *is the score sequence of some* $(1, 1, n)$*-tournament* T *if and only if*

$$\sum_{i=1}^{k} s_i \geq B_k, \quad 1 \leq k \leq n, \tag{1}$$

with equality when k = n.

Further results connected with complete tournaments can be found e.g. in [15, 16, 19, ?].

For example football is an incomplete $(2, 3, n)$-tournament since the permitted results are $0 : 3$ and $1 : 1$ while $0 : 2$ and $1 : 2$ are not permitted.

According to this definition $\mathcal{T}$ is the set of the finite directed loopless multigraphs.

For any vertex $v \in V$ let $d(v)^+$ and $d(v)^-$ denote the out-degree and in-degree of x, respectively. Define $f(v) = d(v)^+ - d(v)^-$ as the imbalance of the vertex v. The imbalance sequence of $T \in \mathcal{T}$ is formed by listing the vertex imbalances of the vertices in nonincreasing or nondecreasing order.

The following result due to Avery [2] and Mubayi, Will and West [?] provides a necessary and sufficient condition for a nonincreasing sequence F of integers to be the imbalance sequence of a tournament $T \in \mathcal{T}(0, 1, n)$.

Theorem 2 (Avery [2], Mubayi, Will, West [25]) *A nonincreasing sequence of integers* $F = (f_1, \ldots, f_n)$ *is an imbalance sequence of a* $(0, 1, n)$ *tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq k(n - k),$$

for $1 \leq k < n$ *with equality when* $k = n$.

Proof. See [2, 25]. □

2 Imbalance sequences of complete tournaments

In 1991 Avery [2]

In 2001 Mubay, Will and West [25]

3 Incomplete tournaments

The first results on incomplete tournaments were published by Reid and Zhang in 1998.

Theorem 3 (Reid, Zhang [28]) *If* $n \geq 1$ *then the nondecreasing sequence of nonnegatíve integer numbers* $s = (s_1, \ldots, s_n)$ *is the outdegree sequence of a semicompleted tournamant* T *if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \quad \textit{and} \quad s_k \leq n-1 \tag{2}$$

for all $1 \leq k \leq n$.

Theorem 4 (Reid, Zhang [28]) *If* $n \geq 1$ *and* $s = (s_1, \ldots, s_n)$ *is a nondecreasing sequence of nonnegatíve integers, then there exists a tournament* T *with out-degree sequence* $t = t_1, \ldots, t_n$ *such, that* $t_i \leq s_i$ *for* $1 \leq i \leq n$.*the outdegree sequence of a semicompleted tournamant* T *if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \quad \textit{and} \quad s_k \leq n-1 \tag{3}$$

for all $1 \leq k \leq n$.

Theorem 5 (Reid, Zhang [28]) *Theorem 2, Theorem 3 and Theorem 4 are equivalent.*

In 2002 Iványi [14] solved the following problem posed by Antal Bege [3]: How many wins are necessary and sufficient in a $(1, 1, n)$ tournament to guarantee that the teams have different number of points.

Let $x \geq 0$ be a real number and define *the real football tournament* with the permitted results $0 : 0$ and $0 : 0$ and $-1 : 1 + x$. Let $f(x, n)$ denote the above described number of wins in a real football tournament of n teams.

Theorem 6 *If* $n \geq 1$ *then*

$$f(n, 1) = (3/2 - \sqrt{2})n^2 + O(n), \tag{4}$$

if x *is zero or an integer greater or equal 2, then*

$$f(n, x) = \frac{\lfloor n/2 \rfloor (\lfloor n/2 \rfloor + 1)}{2} = \frac{n^2}{8} + \rho(n), \tag{5}$$

where $\rho(n) = 0$, *if* n *is even and* $\rho(n) = 1/8$, *if* $n \geq 3$ *is odd, and if* x *is irrational, then*

$$f(n, x) = O(n^{3/2}). \tag{6}$$

Proof. See [14]. □

4 Imbalance sequences of football tournaments

Let the permitted results of the *simplified football tournament* T_n $0:0$ and $0:1$.

The following assertion gives a necessary and sufficient condition for a nonincreasing sequence $s = (s_1, \ldots, s_n)$ to be the imbalance sequence of a football tournament.

Theorem 7 *A nonincreasing sequence of integers* $S = (s_1, \ldots, s_n)$ *is an imbalance sequence of a football tournament if and only if* s_i $(i = 1, \ldots, n)$ *is a multiple of 3 and* $s' = (s_1/3, \ldots, s_n/3)$ *is the imbalance sequence of a complete* $(0, 1, n)$*-tournament.*

Proof. The result $1:1$ between players P_i and P_j do not change the imbalance of P_i and P_j. The other possible result $3:0$ adds 3 to the imbalance of P_i and subtracts 3 from the imbalance of P_j. Therefore if s is an imbalance sequence of a football tournament, then all elements of s are the multiple of 3.

s is the imbalance sequence of a football tournament if and only if s' is the imbalance sequence of a simplified football tournament. s' is the imbalance sequence of a simplified football tournament if and only if it is the imbalance sequence of a complete $(0, 1, n)$-tournament. □

5 Optimization

6 Summary

The referenced papers can be found at the homepage of the author at http://compalg.inf.elte.hu/~tony/Kutatas/EGHH/.

Acknowledgements. The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP 4.2.1/B-09/1/KMR-2010-0003.

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