# ESTIMATES FOR SPEED OF COMPUTERS WITH INTERLEAVED MEMORY SYSTEMS

By

A. IVÁNYI and I. KÁTAI

Department of Numerical Methods and Computer Science of the L. Eötvös University, Budapest

*(Received December 15, 1975)*

## 1. Introduction

The main parameter of computers used for scientific research and other areas is the speed given as the number of executed operations in a time unit. Since the speed is limited by the slowest element of processing chain (peripheries, memory, central processor), therefore the parallelisation is a useful idea to increase the power.

From the point of view of hardware realisation one of the possibilities is to construct multiprocessor systems. For instance computer system SYMBOL contains 8 processors of different purposes, and the realised quadrant of ILLIAC – IV consists of 64 processors for identical purpose.

An other idea is realised in the computers BESM – 6 and CDC – 7600. Core memory of BESM – 6 consists of 8 modules (each module has 4096 words and the word's length is 48 bit, memory cycle time for every module is 2 $\mu$s) allowing access to 1 – 1 word in every module in the same time. The physically consecutive adresses fall under different (cyclically neighbouring) modules [1].

In their papers HELLERMAN [2], VULIHMAN [3], BURNETT and COFFMAN [4] proposed a lot of mathematical models for the functioning of computers with interleaved memory system, and determined the speed approximately by analytical method and by simulation too.

The aim of our paper is to continue these investigations by improving the known speed estimates for a special mathematical model in case of the request sequence distributed uniformly.

## 2. The formulation of problem

We shall consider the following scheme of computers with interleaved memory system due to VULIHMAN [3], which represents a generalization of HELLERMAN's model [2].

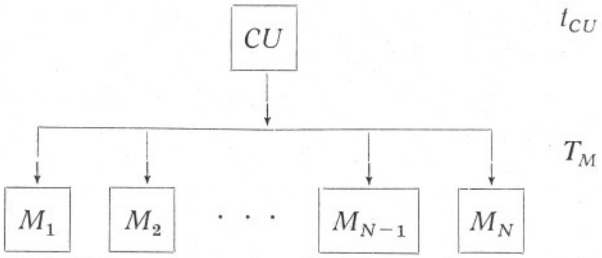


*Fig. 1.* Scheme of computer with interleaved memory system

The sequence of requests $\omega = \alpha_1, \alpha_2, \ldots$ to the moduls $M_1, M_2, \ldots, M_N$ is produced by central unit ($CU$). The number of elements in $\omega$ is called the length of $\omega$ and it is denoted by $|\omega|$. The time between the origins of consecutive requests is denoted by $t_{CU}$ ($t_{CU} \geq 0$). Every request turns to own module (to one of $M_1, M_2, \ldots, M_N$) and reserves it for a time $T_M$ ($T_M \geq t_{CU}, T_M > 0$). For the simplicity we choose $T_M$ to be the unit of time, and put

$$t = \frac{t_{CU}}{T_M}.$$

This time other request cannot return to the same module. For the simplicity let $M_i = i$ ($i = 1, 2, \ldots, N$). If the next request in the sequence finds his own module busy then $CU$ waits while this module will free, and this time $CU$'s functioning is interrupted.

By the work AHO's DENNING's and ULLMAN's [5] we introduce the following notions. The processing cost $C(N, t, \omega)$ of $\omega$ is defined as the processing time of $\omega$. The set of all possible $\omega$ of length $k$ ($k = 1, 2, \ldots$) is denoted by $V^k$. Suppose that $\omega \in V^k$ is a random variable with the distribution $Pr_k(\omega)$. The average processing cost of one request for all $\omega$ in $V^k$ by definition is

$$C(N, t, k, Pr) = \frac{1}{k} \sum_{\omega \in V^k} Pr_k(\omega) \cdot C(N, t, \omega). \tag{2.1}$$

We call $\omega$ to be homogeneous if $\alpha_1 = \alpha_2 = \ldots = \alpha_{|\omega|}$ and to be cyclical if $\alpha_{t+1} = \alpha_t + 1 \pmod N$ ($t = 1, 2, \ldots, |\omega| - 1$). We call $Pr = \{Pr_1(\omega), Pr_2(\omega), \ldots\}$ to be homogeneous, if $Pr_k(\omega) = 0$ for all nonhomogeneous $\omega$ and to be cyclical, if $Pr(\omega) = 0$ for all noncylical $\omega$. According to these we use the notation $HOM$ and $CYCL$ for these special probability distributions.

For fixed $N$, $t$ and $Pr$ the speed of interleaved memory is defined as

$$V(N, t, Pr) = \liminf_{k \to \infty} \frac{1}{C(N, t, k, Pr)}. \tag{2.2}$$

It is easy to see that this limit inferior always exists and the following assertions hold:

a) for every $N, t, Pr$ we have

$$1 = V(N, t, HOM) \leq V(N, t, Pr) \leq V(N, t, CYCL) = \min\left(N, \frac{1}{t}\right). \tag{2.3}$$

b) If $t' > t''$, then

$$V(N, t', Pr) \leq V(N, t'', Pr). \tag{2.4}$$

c) If $N' > N''$, then

$$V(N', t, Pr) \geq V(N'', t, Pr). \tag{2.5}$$

A lot of program behaviour models ($Pr$'s) was investigated and published. One of the first models is proposed by L. A. BELADY [5] and was called $RAN$ (random).

In this model it is supposed that the request $\alpha_1, \alpha_2, \ldots$ are independent random variables and distributed uniformly, i. e.

$$P\{\alpha_i = j\} = \frac{1}{N} \quad (i > 0, \quad j = 1, \ldots, N). \tag{2.6}$$

In this case we have for every $\omega \in V^k$ $(k = 1, 2, \ldots)$

$$RAN(\omega) = \frac{1}{N^k}. \tag{2.7}$$

In his book HELLERMAN [2] proved, that

$$V(N, O, RAN) = \sum_{k=1}^{N} \frac{k^2(N-1)!}{N^k(N-k)!} \tag{2.8}$$

and asserted that

$$0{,}96 \cdot N^{0{,}56} \leq V(N, O, RAN) \leq 1{,}04 \cdot N^{0{,}56}, \quad 1 \leq N \leq 45. \tag{2.9}$$

In his paper [3] V. E. VULIHMAN proved that

$$V(N, O, RAN) = \frac{e^N}{N^N}\left[N! - N(N-1)\int_0^N t^{N-2} e^{-t}\, dt\right] + 1. \tag{2.10}$$

Neglecting the integral and by using the Stirling-formula he deduced that

$$V(N, O, RAN) < \sqrt{2\pi N} \cdot e^{\frac{1}{12N}} + 1. \tag{2.11}$$

Recently L. N. KOROLEV published in his book [1] the inequality

$$V(N, O, RAN) \leq \sqrt{2\pi N} \tag{2.12}$$

and in [7] it was proved that

$$V(N, O, RAN) \leq \frac{\sqrt{2\pi N}}{2} e^{\frac{1}{12N}}. \tag{2.13}$$

No we prove that (2.13) is very sharp.

## 3. The estimates of speed

For the brevity let $V(N) = V(N, O, RAN)$.

THEOREM. *For every integer $N \geq 1$ the inequality*

$$\left| V(N) - \sqrt{\frac{\pi N}{2}} \right| < 1 \tag{3.1}$$

*holds.*

We use the following lemmas.

LEMMA 1. *For every $N \geq 1$*

$$V(N) = \frac{N!}{N^N} \sum_{k=0}^{N-1} \frac{N^k}{k!}. \tag{3.2}$$

LEMMA 2. *For every $N \geq 1$ we have*

$$\sum_{k=0}^{N-1} \frac{N^k}{k!} < \frac{1}{2} e^N. \tag{3.3}$$

LEMMA 3. *For every $N \geq 1$ we have*

$$\sum_{k=0}^{N} \frac{N^k}{k!} > \frac{1}{2} e^N. \tag{3.4}$$

## 4. Proof of Lemmas and Theorem

PROOF OF LEMMA 1. Using partial integration to evaluate the integral in (2.10) after some arithmetical operations we shall become (3.2).

We remark that this lemma represents a special case of formula (3) in [4].

PROOF OF LEMMA 2. Let $a_k = \frac{N^k}{k!}$. We shall show that

$$\sum_{k=0}^{N-1} a_k < \sum_{k=N}^{2N-1} a_k, \tag{4.1}$$

whence by observing that

$$e^N = \sum_{k=0}^{\infty} a_k,$$

the assertion immediately follows.

It is obvious that $a_{N-1} = a_N$. By an easy calculation we get

$$\frac{a_{N+k}}{a_{N-k-1}} = \frac{N^{2k}}{(N^2-1^2)(N^2-2^2)\dots(N^2-k^2)} \quad (k = 1, 2, \dots, N-1),$$

whence $a_{N+k} > a_{N-k-1}$ $(k = 1, 2, \dots, N-1)$, and after summation we have the inequality (4.1).

Proof of Lemma 3. By partial integration it is easy to show that

$$\sum_{k=0}^{p} \frac{p^k}{k!} = e^p \left[ 1 - \frac{p^{p+1}}{p!} \int_0^1 x^p e^{-px} dx \right] (p = 1, 2, \dots). \tag{4.2}$$

Let the integral denoted by $H(p)$ and use the substitution $y = 1-h$ in it:

$$H(p) = \int_0^1 y^p e^{-py} dy = e^{-p} \int_0^1 (1-h)^p e^{hp} dh. \tag{4.3}$$

For $0 \leq h < 1$ the inequalities

$$(1-h)e^h = e^{h+\ln(1-h)} = \exp\left(-\left[\frac{h^2}{2} + \frac{h^3}{3} + \dots\right]\right) \leq \exp\left(-\frac{h^2}{2}\right) \tag{4.4}$$

hold. Hence we get

$$H(p) \leq e^{-p} \int_0^1 \exp\left(-\frac{h^2}{2} p\right) dh. \tag{4.5}$$

After the substitution $s = \frac{h^2}{2} p$

$$H(p) \leq \frac{e^{-p}}{\sqrt{2p}} \int_0^{\frac{p}{2}} \exp(-s) s^{-1/2} ds < \frac{e^{-p}}{\sqrt{2p}} \int_0^{\infty} \exp(-s) s^{-1/2} ds =$$

$$= \frac{e^{-p}}{\sqrt{2p}} \Gamma(0{,}5) = \frac{e^{-p}}{\sqrt{2p}} \sqrt{\pi}. \tag{4.6}$$

Then by (4.2), (4.6) and the Stirling's formula we get

$$\sum_{k=0}^{p} \frac{p^k}{k!} \geq e^p \left(1 - \frac{1}{2e^{\Theta_p/12p}}\right) \quad (\Theta_p > 0), \tag{4.7}$$

and so (3.4) holds for every natural number.

Proof of Theorem. From (3.2) and (3.3) by the Stirling's formula we have

$$V(N) < \frac{N!}{2N^N} e^N = \frac{\sqrt{2N\pi}}{2} e^{\Theta_N/12N} < \sqrt{\frac{\pi N}{2}} + 1.$$

Similarly, from (3.2) and (3.4) we have

$$V(N) = \frac{N!}{N^N}\left(\sum_{k=0}^{N} \frac{N^k}{k!} - \frac{N^N}{N!}\right) > \frac{N!\,e^N}{2N^N} - 1 =$$

$$= \sqrt{\frac{\pi N}{2}}\, e^{\Theta_N/12N} - 1 > \sqrt{\frac{\pi N}{2}} - 1\,,$$

whence (3.1) immediately follows.

## 5. Summary

In this paper we investigated the speed of computers with interleaved memory. We have a general definition of speed, and a lower and upper estimates of speed with a little difference for the Vulihman's model in the case of random request distribution.

We used a simple direct proof. We remark that using a result due to G. SZEGŐ [8] we can prove the following assertion:

$$V(N) = \sqrt{\frac{\pi N}{2}} - \frac{1}{3} + \tau_N\,, \tag{5.1}$$

where $\tau_N$ tends monotonically to zero, and

$$\tau_1 = \frac{4}{3} - \frac{\pi}{2} \approx 0{,}08 \quad \text{and} \quad \tau_2 = \frac{11}{6} - \sqrt{\pi} \approx 0{,}06\,. \tag{5.2}$$

ACKNOWLEDGEMENT. We are indebted to L. N. KOROLEV, Professor of Moscow State University, who proposed the problem.

### References

[1] Л. Н. КОРОЛЕВ, *Структура ЭВМ и их математическое обеспечение*, Наука, Москва, 1974.
[2] H. HELLERMAN, *Digital computer system principles*, Mc Graw Hill, New York, 1967.
[3] В. Е. ВУЛИХМАН, Исследование методов организации оперативной памяти и потока информации в высокопроизводительных электронных машинах, Кандидатская диссертация, Москва, Университет им. М. В. Ломоносова, 1972.
[4] G. J. BURNETT, E. G. COFFMAN, Combinatorial problem related to interleaved memory systems, *Journal of ACM*, **20** (1973), 39 – 45.
[5] A. V. AHO, P. J. DENNING, ULLMAN J. D., Principles of optimal page replacement, *Journal of ACM*, **18** (1971), 80 – 93.
[6] L. A. BELADY, A study of replacement algorithms for a virtual storage computer, *IBM Systems Journal*, **5** (1966), 282 – 288.
[7] А. М. ИВАНИ, Верхняя оценка средней скорости ЭВМ с блочной памятью (в печати).
[8] G. SZEGŐ, Über enige von S. Ramanujan gestellte Aufgaben, *Journal of the London Math. Society*, **3** (1928), 225 – 232.