On the Degrees of the Vertices of a Directed Graph

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ABSTRACT: *In a previous paper the realizability of a finite set of positive integers as the degrees of the vertices of a linear graph was discussed. Here we are concerned with the realizability of a finite set of pairs of non-negative integers* $\{(d_i^+, d_i^-): i = 1, 2, \cdots, n\}$ *as the degrees of the vertices of a directed graph. The directed graphs considered in this paper are allowed to have parallel elements but it is assumed to contain no self-loop elements. The integers* d_i^+ *and* d_i^- *specify the number of arrowheads directed toward and away from vertex* v_i*, respectively. Other related problems such as; realizability of a given set of non-negative integer pairs as a connected directed graph, strongly connected directed graph, and cycleless directed graph are discussed. The problem of orienting a given graph and the Runyon problem are also considered.*

Introduction

Directed graphs have been used as a model for sequential machines, transportation networks, and signal flow graphs (a graphical representation of a set of linear algebraic equations) (**1**, **2**). Although a number of very interesting papers on the theory of directed graphs have been published (**3**, **4**, **5**), it seems that more work of a theoretical nature is needed before we are able to attack (with certain efficiency) some of the problems encountered in the applied areas.

This paper presents an extension of results on the degrees of the vertices of a (nonoriented) linear graph (**6**, **7**) to the case of directed (oriented) graphs.[1] The statements and proofs of Theorems 1 and 2 in this paper, although more complicated than the corresponding theorems for the nonoriented case (**6**), follow the same general pattern.

Let **G** be a directed (oriented) graph with n vertices (nodes), *i.e.*, let **G** be an n-vertex linear graph G with an arrowhead placed upon each of its elements (branches, arcs).[2] Let d_i ($i = 1, 2, \cdots, n$) represent the number of elements (branches) incident at (connected to) vertex v_i in **G**. The integer d_i is called the degree of vertex v_i in **G**. Let $d_i = d_i^+ + d_i^-$, where d_i^+ represents the number of arrowheads directed toward vertex v_i and d_i^- represents the number of arrowheads directed away from vertex v_i. The non-negative integer pair (d_i^+, d_i^-) is called the degree pair of vertex v_i. Given a set of non-negative

[1] A paper by J. K. Senior (**7**), which was brought to the attention of the author by J. W. Moon, contains results which are very similar with the author's results on the degrees of the vertices of (undirected) graphs (**6**, **8**). The development in this paper, however, follows more closely the reasoning that was used in the author's papers on this subject.

[2] Directed graphs and directed subgraphs are represented by boldface letters throughout the paper.

integer pairs (d_1^+, d_1^-), (d_2^+, d_2^-), $\cdots$, (d_n^+, d_n^-) represented by $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$; how can we tell whether or not there exists a directed graph **G** whose vertices $v_1, v_2, \cdots, v_n$ have degree pairs (d_1^+, d_1^-), (d_2^+, d_2^-), $\cdots$, (d_n^+, d_n^-)? If such a graph **G** exists, we say the set $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ is realizable, or graph **G** realizes the set $\{(d_i^+, d_i^-), i = 1, 2, \cdots, n\}$.

The realizability of a set of integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ as the degree pairs of a directed graph, as degree pairs of a connected directed graph, as degree pairs of a "strongly connected" directed graph, and as a degree pairs of a cycleless directed graph (a directed graph without directed circuits) is discussed. Related problems such as how to orient a given (non-oriented) graph to satisfy a given set of degree pair specifications, and the problem of finding a minimal set of branches whose removal from a directed graph leaves the graph cycleless, which is referred to as the Runyon problem, are considered.

Throughout this paper it is assumed that every given set of integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ is ordered such that $d_i^+ + d_i^- \leq d_{i+1}^+ + d_{i+1}^-$ for $i = 1, 2, \cdots, n - 1$, and also it is assumed that $d_i^+ + d_i^- > 0$. Definitions of some of the terms used in this paper are found in the Appendix. Definitions of all other terms may be found in (**1**).

Realizability

In this section, we will state and prove the necessary and sufficient conditions for a given set of non-negative integer pairs $\{(d_i^+, d_i^-), i = 1, 2, \cdots, n\}$ to be realizable as the degree pairs of the vertices of a directed graph. Using Gale's results (**3**), it is possible to arrive at a solution to the above problem for a different class of directed graphs. (Such a solution is explicitly stated in (**2**), Chap. 9.) In the solution presented in (**2**), it is assumed that a directed graph does not contain parallel elements (a pair of elements $e_1(v_i, v_j)$ and $e_2(v_i, v_j)$ connected between the same pair of vertices with their arrowheads toward v_j), and also a directed graph is allowed to have self-loop elements, *i.e.*, elements of the type $e(v_i, v_i)$ are allowed. In the case presented here, a directed graph is allowed to have parallel elements, but a directed graph is assumed to contain no self-loop elements. It will be seen that the result derived here is in a considerably simpler form, and can be tested much more rapidly for realizability. The above problem was also attacked by Ore (**4**), Theorem 2.2.1. However, the result of Theorem 1 is in a much more convenient form and the proof presented here, although a bit lengthy, is quite elementary. The proof of Theorem 1 also suggests a simple procedure for constructing a directed graph from the given set of integer pairs.

Lemma 1: Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be any two sets of real numbers such that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{1}$$

Then $\sum_{i=1}^{n-1} x_i \geq y_n$ if, and only if $\sum_{i=1}^{n-1} (x_i + y_i) \geq x_n + y_n$.

Proof: Let us assume $\sum_{i=1}^{n-1} x_i \geq y_n$, then

$$\sum_{i=1}^{n} x_i \geq y_n + x_n. \tag{2}$$

Using Eq. 1, we can also write

$$\sum_{i=1}^{n} y_i \geq y_n + x_n. \tag{3}$$

Adding the inequalities of Eqs. 2 and 3, we obtain $\sum_{i=1}^{n} (x_i + y_i) \geq 2(y_n + x_n)$ which implies the desired inequality.

Let us now assume that $\sum_{i=1}^{n-1} (x_i + y_i) \geq x_n + y_n$ which may be written as

$$\sum_{i=1}^{n-1} x_i + \sum_{i=1}^{n} y_i - y_n \geq x_n + y_n. \tag{4}$$

Making use of Eq. 1, the inequality of Eq. 4 can be written as $\sum_{i=1}^{n-1} x_i + \sum_{i=1}^{n} x_i - y_n \geq x_n + y_n$, which implies the desired inequality $\sum_{i=1}^{n-1} x_i \geq y_n$.

Lemma 2: A sufficient set of conditions for the three integer pairs (d_1^+, d_1^-), (d_2^+, d_2^-), (d_3^+, d_3^-) to be realizable as the degree pairs of the vertices of a three vertex directed graph is[3]:

$$\text{(a)} \quad \sum_{i=1}^{3} d_i^+ = \sum_{i=1}^{3} d_i^-$$

$$\text{(b)} \quad \sum_{i=1}^{2} (d_i^+ + d_i^-) \geq d_3^+ + d_3^-.$$

Proof: Let G be a three-vertex directed graph. Let $n_{ij} \geq 0$ ($i, j = 1, 2, 3$ and $i \neq j$) be the number of elements which are connected between vertices v_i and v_j and which have arrowheads toward vertex v_j. We would like to show that given any set of three non-negative integer pairs that satisfies conditions (a) and (b) of the hypothesis, we can find a graph G (by calculating the values of n_{ij}) which realizes the given set of integer pairs.

If G is to realize the given set of integer pairs as its degree pairs, then the following set of equations must be satisfied:

[3] The conditions (a) and (b) of Lemma 2 are also necessary for realizability. This fact will become clear in the proof of Theorem 1.

$$\begin{aligned} n_{21} + n_{31} &= d_1^+ \\ n_{12} + n_{13} &= d_1^- \\ n_{12} + n_{32} &= d_2^+ \\ n_{21} + n_{23} &= d_2^- \\ n_{13} + n_{23} &= d_3^+ \\ n_{31} + n_{32} &= d_3^-. \end{aligned}$$

The above set of equations are not linearly independent; therefore, in general, there is no unique solution. Solving for the n_{ij}'s in terms of n_{21}, we obtain

$$\begin{aligned} n_{31} &= d_1^+ - n_{21} \\ n_{23} &= d_2^- - n_{21} \\ n_{13} &= d_3^+ - n_{23} = d_3^+ - d_2^- + n_{21} \\ n_{32} &= d_3^- - n_{31} = d_3^- - d_1^+ + n_{21} \\ n_{12} &= d_2^+ - n_{32} = d_2^+ - d_3^- + d_1^+ - n_{21}. \end{aligned}$$

Since the only acceptable solution is one in which $n_{ij} \geq 0$ for $i, j = 1, 2, 3$, the following inequalities must be satisfied:

$$0 \leq n_{21} \leq \min (d_1^+, d_2^-, d_1^+ + d_2^+ - d_3^-) \tag{5}$$

and

$$n_{21} \geq \max (d_2^- - d_3^+, d_1^+ - d_3^-). \tag{6}$$

From Lemma 1 and the fact that $d_i^+, d_i^- \geq 0$, we can see that there always exists an integer n_{21} which would satisfy condition of Eq. 5; therefore, the question is can n_{21} be picked such that the inequality of Eq. 6 is also satisfied? To show that such an n_{21} exists, we must show that

$$\max (d_2^- - d_3^+, d_1^+ - d_3^-) \leq \min (d_1^+, d_2^-, d_1^+ + d_2^+ - d_3^-). \tag{7}$$

We will consider two cases: (i) $d_2^- - d_3^+ \geq d_1^+ - d_3^-$, and (ii) $d_2^- - d_3^+ < d_1^+ - d_3^-$. In the first case, the inequality of Eq. 7 is reduced to

$$d_2^- - d_3^+ \leq \min (d_1^+, d_2^-, d_1^+ + d_2^+ - d_3^-)$$

which is always true, because $d_2^- - d_3^+ \leq d_1^+$ (due to Lemma 1)[4], $d_2^- - d_3^+ \leq d_2^-$, and $d_2^- - d_3^+ \leq d_1^+ + d_2^+ - d_3^-$ (due to condition (a) of the hypothesis). In the second case, the inequality of Eq. 7 is reduced to

$$d_1^+ - d_3^- \leq \min (d_1^+, d_2^-, d_1^+ + d_2^+ - d_3^-)$$

which is true, because $d_1^+ - d_3^- \leq d_1^+$, $d_1^+ - d_3^- \leq d_2^-$ (due to Lemma 1), and $d_1^+ - d_3^- \leq d_1^+ + d_2^+ - d_3^-$. This proves that there always exists an integer n_{21} such that

$$\max (0, d_2^- - d_3^+, d_1^+ - d_3^-) \leq n_{21} \leq \min (d_1^+, d_2^-, d_1^+ + d_2^+ - d_3^-)$$

which in turn proves the existence of the desired graph.

[4] It should be noted that $(d_1^+ + d_1^-) + (d_3^+ + d_3^-) \geq d_2^+ + d_2^-$ since $d_3^+ + d_3^- \geq d_2^+ + d_2^-$ and $d_1^+ + d_1^- > 0$.

Theorem 1: Given a set of pairs of non-negative integers $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$, $(n \geq 2)$, the set is realizable as the degree pairs of the vertices of an n-vertex directed graph if, and only if

$$\text{(a)} \quad \sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^-, \text{ and}$$

$$\text{(b)} \quad \sum_{i=1}^{n-1} (d_i^+ + d_i^-) \geq d_n^+ + d_n^-.$$

Proof: Given a n-vertex directed graph **G**, it is clear that $\sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^- = N(\mathbf{G})$, where $N(\mathbf{G})$ is equal to the number of elements (branches) in **G**. To prove the necessity of (b) assume otherwise, that is, there exists a graph **G** in which $\sum_{i=1}^{n-1} (d_i^+ + d_i^-) < d_n^+ + d_n^-$. Let vertex v_n in **G** correspond to the integer $d_n^+ + d_n^-$. Then, the inequality $\sum_{i=1}^{n-1} (d_i^+ + d_i^-) < d_n^+ + d_n^-$ implies that there exists in **G** at least one element which is incident at v_n which is not incident at any other vertex. This is impossible, hence

$$\sum_{i=1}^{n-1} (d_i^+ + d_i^-) \geq d_n^+ + d_n^-.$$

The sufficiency is proved by induction. If $n = 2$, then condition (a) requires that $d_1^+ + d_2^+ = d_1^- + d_2^-$, and condition (b) and the fact that the integers are given in a nondecreasing order requires that $d_1^+ + d_1^- = d_2^+ + d_2^-$. From these equations we conclude that $d_1^+ = d_2^-$ and $d_1^- = d_2^+$. We can now see that a two vertex graph with $d_1^+ + d_1^-$ parallel elements connected between vertex v_1 and vertex v_2 and with d_1^+ of these elements having arrowheads toward vertex v_1 and with the remaining d_2^+ elements having arrowheads toward vertex v_2 will be the realization of the two pairs of integers (d_1^+, d_1^-), (d_2^+, d_2^-). If $n = 3$, we have already shown in Lemma 2, the sufficiency of conditions (a) and (b). To complete the induction, we assume that the assertion is true for $n < k$, $(k > 3)$, then we will show that it is also true for $n = k$. Let (d_1^+, d_1^-), (d_2^+, d_2^-), $\cdots$, (d_k^+, d_k^-) be a set of k non-negative integer pairs which satisfies conditions (a) and (b) of the hypothesis. Consider the following three cases separately: (i) $d_1^+ \leq d_k^-$ and $d_1^- \leq d_k^+$, (ii) $d_1^+ \leq d_k^-$ and $d_1^- > d_k^+$, and (iii) $d_1^+ > d_k^-$ and $d_1^- \leq d_k^+$. (Note that the fourth combination $d_1^+ > d_k^-$ and $d_1^- > d_k^+$ cannot occur, for $d_1^+ + d_1^- \leq d_k^+ + d_k^-$.)

Case (i): Consider the set of $k - 1$ non-negative integer pairs

$$(d_2^+, d_2^-), (d_3^+, d_3^-), \cdots, (d_{k-1}^+, d_{k-1}^-), (d_k^+ - d_1^-, d_k^- - d_1^+). \tag{8}$$

This set of integer pairs (obtained from the original set) clearly satisfies condition (b). If $(d_k^+ - d_1^-) + (d_k^- - d_1^+) \geq d_{k-1}^+ + d_{k-1}^-$, *i.e.*, if the integer pairs are in a proper order, then to prove condition (b) is satisfied, we must show that

$$\sum_{i=2}^{k-1} (d_i^+ + d_i^-) \geq (d_k^+ - d_1^-) + (d_k^- - d_1^+);$$

but this is a consequence of the hypothesized inequality $\sum_{i=1}^{k-1} (d_i^+ + d_i^-) \geq d_k^+ + d_k^-$. If $(d_k^+ - d_1^-) + (d_k^- - d_1^+) < d_{k-1}^+ + d_{k-1}^-$, then to prove that the set of integer pairs given in Eq. 8 satisfies condition (b), we must show that

$$\sum_{i=2}^{k-2} (d_i^+ + d_i^-) + (d_k^+ - d_1^- + d_k^- - d_1^+) \geq d_{k-1}^+ + d_{k-1}^-. \tag{9}$$

Since $k > 3$, we may write the inequality of Eq. 9 as

$$(d_2^+ + d_2^-) - (d_1^+ + d_1^-) + (d_k^+ + d_k^-) + \sum_{i=3}^{k-2} (d_i^+ + d_i^-) \geq d_{k-1}^+ + d_{k-1}^-. \quad (10)^{[5]}$$

We known $(d_2^+ + d_2^-) \geq (d_1^+ + d_1^-)$ and $(d_k^+ + d_k^-) \geq (d_{k-1}^+ + d_{k-1}^-)$; therefore, the inequality of Eq. 10 is satisfied for $k > 3$. This proves that if $d_1^+ \leq d_k^-$ and $d_1^- \leq d_k^+$, then set of $k - 1$ integer pairs given in Eq. 8 satisfies conditions (a) and (b) of the hypothesis, hence, according to the induction hypothesis, is realizable as a $(k - 1)$-vertex directed graph $\mathbf{G}_1$. To realize the original set of k integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, k\}$, we add a vertex v_1 to the directed graph $\mathbf{G}_1$. Between vertex v_1 and the vertex in $\mathbf{G}_1$ corresponding to the integer pair $(d_k^+ - d_1^-, d_k^- - d_1^+)$ we connect $d_1^+ + d_1^-$ parallel elements. On these elements we place d_1^+ arrowheads directed toward vertex v_1 and d_1^- arrowheads directed away from v_1. The resulting graph **G** realizes the original set of integer pairs. This ends the inductive proof of the first case.

Case (ii): If $d_1^+ \leq d_k^-$ and $d_1^- > d_k^+$, the technique used in the previous case is not applicable, the integer pairs given by Eq. 8 will not be non-negative, *i.e.*, $d_k^+ - d_1^- < 0$. In other words, since $d_1^- > d_k^+$, all elements incident at vertex v_1 cannot be incident at vertex v_k. More specifically, in a possible realization at most $d_1^+ + d_k^+$ elements are connected between vertices v_1 and v_k. The remaining $d_1^- - d_k^+$ elements which are incident at v_1 must be incident at other vertices of **G**. Keeping in mind the above introductory remarks, let us consider the set of integer pairs (d_1^+, d_1^-), (d_2^+, d_2^-), $\cdots$, (d_k^+, d_k^-) which is assumed to satisfy condition (a) and (b). From the above set, we will construct a new set of $k - 1$ integer pairs that hopefully will satisfy conditions (a) and (b) and which, due to the induction hypothesis, is realizable as a directed graph $\mathbf{G}_1$. Now, consider the following set of integer pairs

$$(d_2^{+\prime}, d_2^-), (d_3^{+\prime}, d_3^-), \cdots, (d_{k-1}^{+\prime}, d_{k-1}^-), (d_k^{+\prime}, d_k^- - d_1^+), \tag{11}$$

[5] In Eq. 10, $\sum_{i=3}^{k-2} (d_i^+ + d_i^-)$ is assumed to be equal to zero when $k = 4$.

where $d_i^{+\prime}$'s are computed recursively as follows: Let us start with integers d_1^- and d_k^+. Subtract from both integers a number x_1 equal to the minimum of the two, *i.e.*, $x_1 = \min(d_1^-, d_k^+)$, and set $d_k^{+\prime} = d_k^+ - \min(d_1^-, d_k^+) = 0$. Consider the integers $d_1^- - x_1$ and d_{k-1}^+; again subtract from both $x_2 = \min(d_1^- - x_1, d_{k-1}^+)$ and set $d_{k-1}^{+\prime} = d_{k-1}^+ - x_2$. Then consider integers $d_1^- - x_1 - x_2$ and d_{k-2}^+ and subtract from both $x_3 = \min(d_1^- - x_1 - x_2, d_{k-2}^+)$ and set $d_{k-2}^{+\prime} = d_{k-2}^+ - x_3$. Continue this process until the values of $d_{k-i}^{+\prime}$ are found for $i = 0, 1, \cdots, k-1$. We would like to show that the set of integer-pairs given by Eq. 11 satisfy conditions (a) and (b) of the hypothesis. To prove condition (a) is satisfied, we must show that

$$\sum_{i=2}^{k} d_i^{+\prime} = \sum_{i=2}^{k-1} d_i^- + (d_k^- - d_1^+). \qquad (12)$$

From the definition of $d_i^{+\prime}$, we can see that $\sum_{i=2}^{k} d_i^{+\prime} = \sum_{i=2}^{k} d_k^+ - d_1^-$; hence Eq. 12 may be written as $\sum_{i=2}^{k} d_i^+ - d_1^- = \sum_{i=2}^{k} d_i^- - d_1^+$ which is true according to the hypothesis. To show that the set of integer pairs given by Eq. 11 also satisfies condition (b), let

$$d_i' = \begin{cases} d_i^{+\prime} + d_i^-, & 2 \le i \le k-1 \\ d_i^{+\prime} + d_i^- - d_1^+, & i = k \end{cases}. \qquad (13)$$

Let $d_p' = \max(d_2', d_3', \cdots, d_k')$. Then, in terms of d_i''s condition (b) may be written as

$$\sum_{i=2}^{k} d_i' - d_p' \ge d_p'. \qquad (14)$$

Let $d_i = d_i^+ + d_i^-$ for $i = 1, 2, \cdots, k$. We know that $d_i \ge d_i'$ for $i = 2, 3, \cdots, k$. From Eq. 13 and the definition of $d_i^{+\prime}$, we can conclude that $\sum_{i=2}^{k} d_i' = \sum_{i=2}^{k} d_i - d_1$. Making this substitution in the inequality of Eq. 14, we obtain the inequality

$$\sum_{i=2}^{k} d_i - d_1 - d_p' \ge d_p'. \qquad (15)$$

To prove the inequality of Eq. 15, we consider two cases: $d_p' = d_k'$, and $d_p' > d_k'$. If $d_p' = d_k'$, then Eq. 15 is proved by showing that

$$\sum_{i=2}^{k} d_i - d_1 - d_k' \ge d_k'$$

which may be written as

$$\sum_{i=2}^{k} (d_i^+ + d_i^-) - (d_1^+ + d_1^-) - (d_k^- - d_1^+) \ge d_k^- - d_1^+$$

which is the same as

$$\sum_{i=1}^{k} d_i^+ \ge d_k^- + d_1^- - \sum_{i=2}^{k-1} d_i^-. \qquad (16)$$

However, the inequality of Eq. 16 is always true due to condition (a) of the hypothesis. If $d_p' > d_k'$, then the inequality of Eq. 15 is easily established by remembering that $k > 3$, $d_k \ge d_p'$, $d_{k-1} \ge d_p'$, and $d_2 \ge d_1$. We have now shown that the set of $k-1$ integer pairs given by Eq. 11 satisfies conditions (a) and (b), hence it is realizable as a $(k-1)$-vertex directed graph $\mathbf{G}_1$. Let the vertex of $\mathbf{G}_1$ corresponding to the integer pair $(d_i^{+\prime}, d_i^-)$ be labeled v_i, for $i = 2, 3, \cdots, k-1$, and the vertex of $\mathbf{G}_1$ corresponding to the integer pair $(d_k^{+\prime}, d_k^- - d_1^+)$ be labeled v_k. To graph $\mathbf{G}_1$ we add a vertex v_1. Between v_1 and v_k we connect d_1^+ elements with arrowheads toward v_1 and d_k^+ elements with arrowheads toward v_k. Then, we connect $d_i^+ - d_i^{+\prime}$ elements between vertex v_1 and vertices v_i for $i = 2, 3, \cdots, k-1$ with all of these elements having arrowheads away from v_1. The resulting graph **G** will realize the original set of integer pairs; this completes the induction for case (ii).

Case (iii): The proof of this case is identical to case (ii) and, therefore, omitted.

The following Corollary is an immediate consequence of Theorem 1 and Lemma 1.

Corollary: Necessary and sufficient conditions for a set of non-negative integer pairs $\{(d_i^+, d_i^-), i = 1, 2, \cdots, k\}$ $(k \ge 2$ and $d_i^+ + d_i^- \le d_{i+1}^+ + d_{i+1}^-)$ to be realizable as the degree pairs of the vertices of a directed graph are:

$$\text{(a)} \quad \sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^-,$$

and

$$\text{(b)} \quad \sum_{i=1}^{n-1} d_i^+ \ge d_n^-.$$

Realizability as a Connected Directed Graph

A directed graph **G** is said to be connected if the nonoriented graph G, obtained from **G** by removing the arrowhead on the elements of **G** is connected (1). Two directed graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ which realize the same set of integer pairs are called *d*-invariant directed graphs. In other words, if there exists a one to one correspondence between vertices of $\mathbf{G}_1$ and $\mathbf{G}_2$ such that corresponding vertices have the same degree pairs, then directed graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ are *d*-invariant. Consider a pair of elements $e(v_i, v_j)$ and $e(v_k, v_o)$ in a directed graph $\mathbf{G}_1$. Assume (as in the Appendix) that the arrowhead in each element is toward the second vertex, *i.e.*, for example the arrowhead on element $e(v_i, v_j)$ is toward vertex v_j. Assume also that vertices v_i, v_j, v_k, and v_o are all distinct. Remove the pair of elements $\{e(v_i, v_j); e(v_k, v_o)\}$ from G_1 and replace them by the pair of elements $\{e(v_i, v_o); e(v_k, v_j)\}$, the resulting graph G_2 will clearly be d-invariant from G_1. The operation of replacement of the pair of elements $\{e(v_i, v_j); e(v_k, v_o)\}$ by $\{e(v_i, v_o); e(v_k, v_j)\}$ is called an elementary d-invariant transformation (6).

Consider a directed graph G, the subgraphs $g_1, g_2, \cdots, g_r$ $(r > 1)$, are called the components (maximally connected subgraphs) of G, if g_i for $i = 1, 2, \cdots, r$ is connected, there is no path (not necessarily a directed path) from any vertex in g_i to any vertex in g_j (and vice versa), and every element of G is in exactly one of these subgraphs (1, 6). The proofs of Lemma 3 and Theorem 2 being similar to the proofs of Lemma 1 and Theorem 2 of the previous paper (6) are, therefore, omitted.

Lemma 3: If G contains $r > 1$ components and if one of the components of G contains a circuit (not necessarily directed), then there exists a directed graph G_1 which is d-invariant from G but has $r - 1$ components.

Theorem 2: Necessary and sufficient conditions for a set of non-negative integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ to be realizable as the degree pairs of the vertices of a connected directed graph are:

$$\text{(a)}\quad \sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^-$$

$$\text{(b)}\quad \sum_{i=1}^{n-1} (d_i^+ + d_i^-) \geq d_n^+ + d_n^-$$

$$\text{(c)}\quad \sum_{i=1}^{n} d_i^+ \geq (n - 1).$$

The following Corollary can easily be established as a consequence of Theorem 2.

Corollary: A necessary and sufficient condition for a set of integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ to be realizable as the degree pairs of the vertices of a connected circuitless directed graph (a tree) is

$$\sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^- = n - 1.$$

Realizability as a Strongly Connected Graph and Cycleless Graphs

In this section, we are concerned with the following questions: Under what circumstances is a set of non-negative integer pairs realizable as a strongly connected directed graph, and as a cycleless directed graph? A directed graph G is said to be strongly connected if for every pair of vertices v_i and v_j in G there is a directed path from v_i to v_j and a directed path from v_j to v_i.

Lemma 4: A directed graph G is strongly connected if, and only if, G is connected and every element of G is in at least one cycle in G.

Proof: The necessity is self-evident; to prove sufficiency, consider a directed graph G which is connected and every element of G is in a cycle. If G contains two vertices, clearly G is strongly connected. Assume that the assertion is correct if G contains $k - 1$ vertices. Let G contain k vertices. Let $e(v_i, v_j)$ be an element of G. Let G_1 be a graph constructed from G by adding an element $e(v_j, v_i)$ between vertices v_j and v_i of G. If G_1 is strongly connected then G is strongly connected; the addition of element $e(v_j, v_i)$ to G did not introduce any new paths in G. Let directed graph G_2 be constructed from G_1 by shorting (coalescing) vertices v_i and v_j and removing all of the resulting self-loop elements. The directed graph G_2 is clearly connected and every element in G_2 is in some cycle. Therefore G_2 is strongly connected. Clearly reversing the operation, forming G_1 from G_2, we obtain a strongly connected graph G_1. We also know that if G_1 is strongly connected so is G, hence the Lemma.

A vertex v_i in a directed graph G is said to be *compact* if the degree pair of this vertex (d_i^+, d_i^-) has the property that $\min(d_i^+, d_i^-) = 0$.

Theorem 3: Necessary and sufficient conditions for a set of non-negative integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ to be realizable as a (the degree pairs of the vertices of a) strongly connected directed graph are:

(a) The set $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ is realizable (satisfies conditions of Theorem 1).

(b) $\min(d_i^+, d_i^-) > 0$, for $i = 1, 2, \cdots, n$.

Proof: The necessity of condition (a) is known. The necessity of (b) is established by noting that if in a directed graph G for some vertex v_j the $\min(d_j^+, d_j^-) = 0$, then v_j is a compact vertex and an element incident at v_j cannot possibly be in a cycle in G; therefore, it will not be strongly connected. We must now show the sufficiency of conditions (a) and (b) for realizability as a strongly connected directed graph. From Lemma 4, if we prove that the given set of integers $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ is realizable as a connected graph in which every element is in some cycle, the theorem is proved. We first note that the set $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ is realizable as a connected graph, for the set of integer pairs satisfies conditions (a) and (b) of Theorem 2 and, since $d_i^+ \geq 1$ for all i, $\sum_{i=1}^{n} d_i^+ \geq n$. Let G be a connected realization of the set $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$. If every element in G is in some cycle, then we have no problem. Suppose there exists at least one element in G which is not in any cycle. Let this element be $e(v_i, v_j)$. Since vertex v_j is not compact, there exists an element $e(v_j, v_k)$, and similarly there must exist an element $e(v_k, v_1)$. Continuing this process, we obtain a directed edge train or a directed chain $E_1 = e(v_j, v_k)e(v_k, v_1)e(v_1, v_m) \ldots$ Since the graph G is finite, this chain will either eventually reach v_i (which leads to a contradiction, for then $e(v_i, v_j)$ would be in a cycle) or the chain will contain a cycle. Using similar arguments we can establish the existence of another chain $E_2 = \cdots e(v_q, v_p)e(v_p, v_r)e(v_r, v_i)$. If the chains E_1 and E_2 intersect each other at any vertex, then again $e(v_i, v_j)$ will be in some cycle. Therefore, we have two chains E_1 and E_2 each containing a cycle and which do not have common vertices. Take an element of the cycle in E_1, say $e(v_x, v_y)$, and an element of the cycle in E_2, say $e(v_s, v_t)$. Performing an elementary d-invariant transformation involving these two elements, *i.e.*, replacing the pair of elements $\{e(v_x, v_y); e(v_s, v_t)\}$, in G by $\{e(v_x, v_t); e(v_s, v_y)\}$, we obtain a graph G_1 which is d-invariant from G and in which element $e(v_i, v_j)$ is in a cycle. Furthermore, if we examine G_1, we can see that every element that was in some cycle in G is also in some cycle in G_1. Clearly we can continue this process until we obtain a directed graph G_k which is d-invariant from G and in which every element is in some cycle.

Corollary 1: A directed graph G contains a cycle if G contains at most one compact vertex.

Proof: We have already shown (see the proof of Theorem 3) that if G has no compact vertices, then G contains a cycle. What remains to be shown is that if G contains one compact vertex, then G still contains a cycle. Let v_i be the compact vertex of G. Let $I(v_i)$ represent the subgraph of G consisting of those elements of G which are incident at v_i. Let vertex v_{i_1} be a vertex of G which is adjacent to v_i, *i.e.*, there is an element in $I(v_i)$ which is connected between v_i and v_{i_1}. Let us assume all elements of $I(v_i)$ have arrowheads away from v_i. Since v_{i_1} is not compact, there exists an element of $e(v_{i_1}, v_j)$; and since v_j is not compact, there is an element $e(v_j, v_k)$. Continuation of this argument provides the existence of a chain (directed edge-train) E. Since chain E will always encounter noncompact vertices, chain E must eventually (*i.e.*, if extended to sufficient length) contain a cycle. If all elements of $I(v_i)$ have arrowheads toward v_i, then using a similar proof we can show the existence of a cycle, in the directed graph G with one compact vertex. One way of proving this second case is by reversing the orientation of all elements in G which results in a new directed graph G′. Clearly G′ is cycleless if, and only if, G is. We can show that G′ must contain a cycle (by the technique used in the first part of this proof), therefore G contains a cycle. The following Corollary is an obvious consequence of Corollary 1.

Corollary 2: A necessary condition for a realizable set of non-negative integer pairs $\{(d_i^+, d_i^-); i = 1, 2, \cdots, n\}$ to be realizable as the degree pair of the vertices of a cycleless graph is that there must exist at least two integers i and $j (1 \leq i, j \leq n)$ such that $\min(d_i^+, d_i^-) = \min(d_j^+, d_j^-) = 0$.

Unfortunately the condition described in Corollary 2 is not sufficient for cycleless realizability. For example, the set of integer pairs (2, 0), (0, 2), (2, 2), (3, 3) is realizable and satisfies the condition of the Corollary, but there exists no cycleless directed graph that realizes the above set of integer pairs.

Although the problem of cycleless realizability of a set of non-negative integer pairs is not satisfactorily solved, Lemma 5 will suggest a possible step-by-step method for arriving at a realization.

Orientability

A comparison of Theorem 1 and corresponding Theorem for the nonoriented case (6) may lead us to believe that a nonoriented graph G whose vertices have degrees $d_1, d_2, \cdots, d_n$ can be oriented (*i.e.*, arrowheads can be placed on the elements of G) such that the vertices of G will have degree pairs (d_1^+, d_1^-), $(d_2^+, d_2^-), \cdots, (d_n^+, d_n^-)$, if

$$\text{(a)} \quad \min(d_i^+, d_i^-) \geq 0 \quad \text{for} \quad i = 1, 2, \cdots, n$$

$$\text{(b)} \quad (d_i^+ + d_i^-) = d_i \quad \text{for} \quad i = 1, 2, \cdots, n,$$

and

$$\text{(c)} \quad \sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^-.$$

To illustrate that this is not the case, consider the graph of Fig. 1.[6] The graph of Fig. 1 has the degrees 2, 3, 4, 5, 6. We endeavor to show that the graph of Fig. 1 cannot be oriented such that it will realize the set of degree pairs (2, 0), (2, 1), (3, 1), (1, 4), (2, 4). To see this, consider the subgraph of the graph of Fig. 1 consisting of the parallel elements connected between vertices v_4 and v_5.

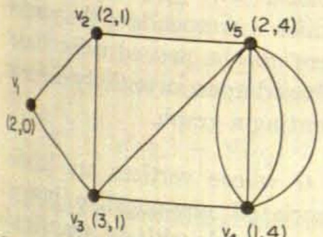

Fig. 1. A nonoriented graph with an unrealizable orientation specification.

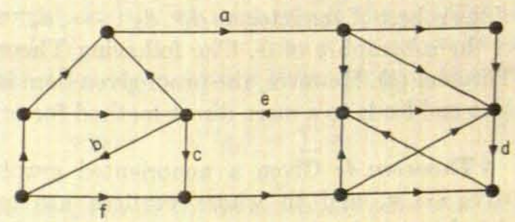

Fig. 2. An example of a non-minimal chord-set.

The desired number of arrowheads directed toward vertex v_4 is one and toward v_5 is two. Since there are four elements directly connected between v_4 and v_5, the sum of the number of arrowheads directed toward vertex v_4 and v_5 must be at least four, hence, the orientation specification is not realizable. Reexamining Theorem 1 and the corresponding theorem for the nonoriented case, we see that Theorem 1 merely proves that there exists a graph G' which has the same vertex degrees as the graph of Fig. 1 and which can be oriented to realize the orientation specification: (2, 0), (2, 1), (3, 1), (1, 4), (2, 4).

With the above introduction we re-define the problem as follows: Let G be a nonoriented graph. Let the vertices of G be labeled $v_1, v_2, \cdots, v_n$ and let the

[6] This graph was chosen so that the vertex degrees identify the vertices, *i.e.*, there are no two vertices with the same degrees.

degrees of these vertices be $d_1, d_2, \cdots, d_n$. Let there be associated to each vertex v_i of G of a non-negative integer d_i^+ for $i = 1, 2, \cdots, n$. (The non-negative integers $d_1^+, d_2^+, \cdots, d_n^+$ will be referred to as the orientation specification.) The problem: Is it possible to put arrowheads on the elements of G such that the number of arrowheads directed toward vertex v_i is d_i^+ for $i = 1, 2, \cdots, n$? Clearly the orientation specification $d_1^+, d_2^+, \cdots, d_n^+$ must satisfy the following two conditions:

$$\text{(a)}\quad 0 \le d_i^+ \le d_i \qquad \text{for} \qquad i = 1, 2, \cdots, n$$

$$\text{(b)}\quad 2 \sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i.$$

Figure 1 demonstrated that the above two conditions are not sufficient conditions for orientability of a given graph with the orientation specification $d_1^+, d_2^+, \cdots, d_n^+$. This problem will be discussed in this section.

If g_1 and g_2 are two subgraphs of G, then by the "ring sum" of g_1 and g_2, denoted by $g_1 \oplus g_2$, we mean a subgraph consisting of those elements of G which are either in g_1 or in g_2 but not in both, by the "union" of g_1 and g_2, denoted by $g_1 \cup g_2$, we mean a subgraph consisting of those elements of G which are either in g_1 or in g_2 (or in both), and finally by the "intersection" of g_1 and g_2, denoted by $g_1 \cap g_2$, we mean a subgraph consisting of those elements of G which are in g_1 and g_2 (*i.e.*, which are in both). Let $N(g)$ be the number of elements in a subgraph g of G, and let $d(g) = \sum_{i \epsilon g} d_i^+$ be the sum of the subset of the integers $d_1^+, d_2^+, \cdots, d_n^+$ which corresponds to the vertices of the subgraph g of G. The following Theorem can be proved using Gale's Theorem (3). However, the proof given here is based upon an entirely different idea and leads to a more direct method for orienting a graph.

Theorem 4: Given a nonoriented graph G whose vertices are labeled $v_1, v_2, \cdots, v_n$ and to whose vertices are associated non-negative integers $d_1^+, d_2^+, \cdots, d_n^+$, respectively; then, graph G is orientable with d_i^+ arrowheads directed toward vertex v_i (for $i = 1, 2, \cdots, n$) if, and only if, for every subgraph g_k of G

$$\text{(a)}\quad d(g_k) - N(g_k) \ge 0,$$

and

$$\text{(b)}\quad d(G) = N(G).$$

Proof: The necessity of condition (b) has already been discussed. To prove the necessity of condition (a), assume otherwise, that is, for some subgraph g_j of G, $d(g_j) - N(g_j) < 0$. This inequality implies that there are more elements in g_j than there are arrowheads directed toward the vertices in g_j which is an impossibility, hence the necessity of condition (a).

We will prove the sufficiency by induction on the number of elements in G. If G has one or two elements (regardless of how the two elements are connected) the sufficiency of conditions (a) and (b) can easily be established. Assume that if G contains $k - 1$ elements and the orientation specification of G satisfies

conditions (a) and (b), then G can be oriented as desired. Now, consider a graph G with k elements (and, say, n vertices). Let the orientation specification $d_1^+, d_2^+, \cdots, d_n^+$ be such that the conditions (a) and (b) are satisfied. Consider an element e in G. Without the loss of generality, let element e be connected between vertices v_1 and v_2. Let $G' = G \oplus e$, where vertices of G' are labeled as in G. If G' and one of the following two sets of orientation specification

$$\text{(i)}^7\quad d_1^+ - 1, d_2^+, d_3^+, \cdots, d_{n-1}^+, d_n^+ = d_1^{+\prime}, d_2^{+\prime}, \cdots, d_n^{+\prime}$$
$$\text{(ii)}^7\quad d_1^+, d_2^+ - 1, d_3^+, \cdots, d_{n-1}^+, d_n^+ = d_1^{+\prime\prime}, d_2^{+\prime\prime}, \cdots, d_n^{+\prime\prime}$$

satisfy conditions (a) and (b) of the hypothesis, then, since G' contains $k - 1$ elements, G' can be oriented such that it would have the orientation specification given either by (i) or (ii)[8]. Let G′ be such a directed graph. If G′ realizes the orientation specification given by (i), then we can construct the directed graph G by adding element e to G′ and putting an arrowhead on element e directed toward vertex v_1. If G′ realizes the orientation specification given by (ii), then G is constructed by adding element e to G′ with the arrowhead on e being directed toward vertex v_2. Clearly in either case, the resulting graph G will have the desired orientation specification. Therefore, our main task is to show that if G and the orientation specification $d_1^+, d_2^+, \cdots, d_n^+$ satisfy conditions (a) and (b), then G' and one of the two orientation specifications given by (i) and (ii) will satisfy conditions (a) and (b).

We know that $N(G') = d(G) - 1$, therefore, condition (b) is satisfied regardless which of the two sets of orientation specifications are used. To prove that condition (a) will be satisfied by at least one of the orientation specifications, suppose otherwise. If G' and the orientation specification given by (i) do not satisfy condition (a), then there exists a subgraph g_p in G' such that

$$d'(g_p) - N(g_p) < 0, \qquad \text{where} \qquad d'(g_p) = \sum_{i \epsilon g_p} d_i^{+\prime}. \tag{17}$$

If G' and the orientation specification given by (ii) do not satisfy condition (a), then there exists a subgraph g_q in G' such that

$$d''(g_q) - N(g_q) < 0, \qquad \text{where} \qquad d''(g_q) = \sum_{i \epsilon g_q} d_i^{+\prime\prime}. \tag{18}$$

It will be shown that the inequalities of Eqs. 17 and 18 cannot be simultaneously satisfied. To do this, we will examine subgraphs g_p and g_q.

If subgraph g_p contains vertices v_1 and v_2, then consider subgraph $g_p \cup e$, which is a subgraph of graph G. From the hypothesis we have

$$d(g_p \cup e) - N(g_p \cup e) \ge 0.$$

Since g_p is assumed to contain v_1 and v_2, $d(g_p \cup e) = d'(g_p) + 1$ and we know $N(g_p \cup e) = N(g_p) + 1$, therefore, if g_p contains v_1 and v_2, the inequality of

[7] By the equality sign in these equations, we mean that $d_1^+ - 1 = d_1^{+\prime}, d_2^+ = d_2^{+\prime}, \cdots, d_n^+ = d_n'$ and also $d_1^+ = d_1^{+\prime\prime}, d_2^+ - 1 = d_2^{+\prime\prime}, \cdots, d_n^+ = d_n^{+\prime\prime}$.

[8] It is possible that G' is orientable regardless of which of the two sets of orientation specifications are picked.

Eq. 17 cannot be satisfied. Suppose g_p contains neither v_1 nor v_2, then, since $g_p \in G$, we have $d(g_p) - N(g_p) \geq 0$ and $d(g_p) = d'(g_p)$, hence the inequality of Eq. 17 cannot be satisfied. Similarly if g_p contains v_2 but not v_1, then again the inequality of Eq. 17 cannot be satisfied. The only remaining possibility is that g_p contains v_1 but not v_2. By similar reasoning, we arrive at the conclusion that the only case that the inequality of Eq. 18 could be satisfied is when g_q contains v_2 but not v_1. In any case subgraphs g_p and g_q are in G, and from the hypothesis, we have

$$d(g_p) - N(g_p) \geq 0 \tag{19}$$

and

$$d(g_q) - N(g_q) \geq 0. \tag{20}$$

Comparing Eqs. 17 and 19 and remembering that g_p contains v_1 but not v_2, that is $d(g_p) = d'(g_p) + 1$, we conclude that

$$d(g_p) - N(g_p) = 0 \tag{21}$$

and similarly we can show that

$$d(g_q) - N(g_q) = 0. \tag{22}$$

Now, we will show that the simultaneous assumption of Eqs. 21 and 22 will lead into a contradiction.

Let $g = g_p \cup g_q \cup e$, then, from the hypothesis, we know

$$d(g) - N(g) \geq 0. \tag{23}$$

We will examine the inequality of Eq. 23 in the light of Eqs. 21 and 22. Let $g_p \cap g_q = 0$ (*i.e.*, let $g_p \cap g_q$ be a null subgraph),[9] then

$$d(g) - N(g) \leqq d(g_p) + d(g_q) - [N(g_p) + N(g_q) + N(e)]$$

which, using Eqs. 21 and 22, becomes

$$d(g) - N(g) \leqq -N(e) = -1$$

with contradicts Eq. 23. If $g_p \cap g_q = g_r$, then

$$d(g) - N(g) \leqq d(g_p) + d(g_q) - d(g_r) - [N(g_p) + N(g_q) - N(g_r) + N(e)]$$

which may be written as

$$d(g) - N(g) \leqq -d(g_r) + N(g_r) - N(e).$$

However, we know $d(g_r) - N(g_r) \geq 0$; therefore, we have

$$d(g) - N(g) \leq -N(e) = -1$$

which again contradicts Eq. 23. This concludes the proof of the Theorem.

Although the proof of Theorem 4 suggests a procedure for orienting a given graph G (to satisfy a given orientation specification), it is not a practical procedure because of the enormous amount of time that must be spent to decide the orientation of an element. The importance of Theorem 4 lies in the fact that it characterizes the difficulties that may arise in the problem of orienting a given graph.

[9] It must be noted that g_p and g_q could have common vertices.

A question that deserves attention is: How many "different" ways can a given graph G be oriented to realize a given orientation specification? The following Theorem sheds some light on this problem.

Theorem 5: Let G be a graph whose vertices are labeled $v_1, v_2, \cdots, v_n$ and whose elements are labeled $e_1, \cdots, e_p$. Let **G** be a possible way of orienting G such that there are d_i^+ arrowheads directed toward vertex v_i for $i = 1, 2, \cdots, n$, then **G** is unique (*i.e.*, there is no other way of orienting G to realize the given orientation specification $d_1^+, d_2^+, \cdots, d_n^+$) if, and only if, **G** is cycleless.

Proof: If **G** contains a cycle C, then we can construct from **G** a different way of orienting G by reversing the arrowheads on the elements of cycle C in **G**. It remains to show that if **G** is cycleless, then there is no other way of orienting G. The proof for this part of the theorem is left out. The reader can easily construct a proof after reading the proof of Lemma 5 in the next section.

Cycleless Directed Graphs and the Runyon Problem

Suppose given a directed graph **G**, we would like to find a minimal subgraph whose removal from **G** breaks all cycles in **G**. However, the first aim in this section is to describe a process by which we can test a directed graph to see whether or not it contains a cycle.

A directed graph **G** has *successively compact vertices* if **G** has a compact vertex v_i and $\mathbf{G}_1 = \mathbf{G} \oplus I(v_i)$ (where, as before, $I(v_i)$ is the set of elements in **G** which are incident at v_i) has a compact vertex v_j and $\mathbf{G}_2 = \mathbf{G}_1 \oplus I(v_j)$ has a compact vertex v_k, and so forth.

Lemma 5: A directed graph **G** is cycleless if, and only if, **G** has successively compact vertices.

Proof: If **G** is cycleless then **G** has a compact vertex v_i (due to Corollary 1, Theorem 3). Since $\mathbf{G}_1 = \mathbf{G} \oplus I(v_i)$ is also cycleless, $\mathbf{G}_1$ must have a compact vertex v_j and so forth. This proves that if **G** is cycleless, then **G** has successively compact vertices. Suppose **G** has successively compact vertices, we would like to show that **G** is cycleless. Let v_i be a compact vertex of **G**. Since none of the elements incident at v_i can be in any cycles, **G** and $\mathbf{G}_1 = \mathbf{G} \oplus I(v_i)$ must contain the same cycles. However, we know $\mathbf{G}_1$ contains a compact vertex v_j, therefore $\mathbf{G}_1$ and $\mathbf{G}_2 = \mathbf{G}_1 \oplus I(v_j)$ contain the same cycles. Since this process can be continued until every element of the original graph is removed, this proves that **G** has as many cycles as a null graph, which has no cycles; therefore the Lemma.

Lemma 5 suggests a method for attacking the Runyon problem.[10] The problem is: Given a directed graph G, how do we find a maximal cycleless subgraph of G (or how do we find a subgraph of G containing the maximum number of elements and no cycles)? In a nonoriented graph a maximal circuitless subgraph is called a tree and the complement of a tree is called a chord-set (1). Analogously, we will define a set of elements $\mathbf{g}_c$ of G (a subgraph of $\mathbf{g}_c$ of G) to be a chord-set of G if $G \oplus \mathbf{g}_c$ is cycleless. If there exists no chord-set $\mathbf{g}_c'$ such that $N(\mathbf{g}_c') < N(\mathbf{g}_c)$, then $\mathbf{g}_c$ is called a minimal chord-set. In such terms, the problem is to find a minimal chord-set of a directed graph G. In general, there is more than one minimal chord-set in a directed graph. Lemma 5 suggests an efficient procedure for finding a chord-set of G which may (or may not) be a minimal chord-set. The procedure may be outlined as follows. Let G be a given directed graph. If there are any compact vertices in G remove all elements in G which are incident at these vertices. Continue this process successively until the resulting graph G_1 has no compact vertices. Let $p_i = d_i/\min(d_i^+, d_i^-)$ be called the parity index of the vertex v_i in G_1. Let p_j be the maximum of the parity indices of the vertices of G_1. Let the corresponding vertex in G_1 be v_j. Let $I(v_j) = I^+(v_j) \cup I^-(v_j)$, $(I^+(v_j) \cap I^-(v_j) = 0)$, where $I^+(v_j)$ is the subset of $I(v_j)$ which contains elements with arrowheads toward vertex v_j and $I^-(v_j)$ is the subset of $I(v_j)$ which contains elements with arrowheads away from v_j. Let subgraph $I^*(v_j)$ be defined as follows:

$$I^*(v_j) = \begin{cases} I^+(v_j), & \text{if} \quad N[I^+(v_j)] \le N[I^-(v_j)] \\ I^-(v_j), & \text{if} \quad N[I^+(v_j)] > N[I^-(v_j)]. \end{cases}$$

Remove $I^*(v_j)$ from G_1. The subgraph $I^*(v_j)$ is part of the desired chord-set. In the remaining graph $G_1 \oplus I^*(v_j)$, vertex v_j is compact and all elements incident at v_j are removed. If there are any other compact vertices, remove all elements incident at these vertices. Finally, there is found a graph G_2 which has no compact vertices. Search for a vertex v_k in G_2 with maximum parity, then $I^*(v_k)$ (defined as $I^*(v_j)$) is the second part of the desired chord-set. Continue this process until the graph is reduced to a null graph. The union, $I^*(v_j) \cup I^*(v_k) \cup \cdots = \mathbf{g}_c$ is a chord-set, because the vertices of $G \oplus \mathbf{g}_c$ are successfully compact, and therefore, $G \oplus \mathbf{g}_c$ is cycleless. Unfortunately, the above process gives a chord-set which is not necessarily a minimal chord-set. A sufficient condition for a chord-set $\mathbf{g}_c$ to be a minimal chord-set of G is that there is a set of $N(\mathbf{g}_c)$ element disjoint cycles (*i.e.*, no two cycles have an element in common) such that each element in $\mathbf{g}_c$ is in one of these cycles. The above condition is not necessary, that is, it is possible that g_c is a minimal chord-set of G but the number of disjoint cycles in G is less than $N(\mathbf{g}_c)$.

A cut-set $\mathbf{S}_i$ of a directed graph G is a minimal set of elements of G which when removed from G increases the number of components of G (1). An incident set, $I(v_i)$, is also considered to be a cut-set, in such a case the isolated vertex is considered a component of G. Let $S_i = S_i^+ \cup S_i^- = S_i^+ \oplus S_i^-$ (*i.e.*, $S_i^+ \cap S_i^- = 0$), where S_i^+ and S_i^- are two subsets of cut-set S_i. The two subsets S_i^+ and S_i^- are distinguished from each other by picking an arbitrary orientation for the cut-set and then the subset of elements that have the same orientations as the cut-set are denoted by S_i^+ and the remaining elements (if any) in S_i are denoted by S_i^-. Let the subgraph S_i^* be defined as

$$S_i^* = \begin{cases} S_i^+, & \text{if} \quad N(S_i^+) \le N(S_i^-) \\ S_i^-, & \text{if} \quad N(S_i^+) > N(S_i^-). \end{cases}$$

Theorem 6: A necessary condition[11] for a chord-set $\mathbf{g}_c$ of directed graph G to be a minimal chord-set is that for every cut-set $\mathbf{S}_i$ in G

$$N(\mathbf{S}_i \cap \mathbf{g}_c) \le N(S_i^*).$$

Proof: Assume that $\mathbf{g}_c$ is a minimal chord-set but there exists a cut-set $\mathbf{S}_i$ in G such that

$$N(\mathbf{S}_i \cap \mathbf{g}_c) > N(S_i^*).$$

Since $\mathbf{S}_i$ is a cut-set, every cycle in G that contains an element of $\mathbf{S}_i$ must also contain an element of S_i^*. Then, every cycle in G that contains an element of $\mathbf{S}_i \cap \mathbf{g}_c$ must also contain an element of S_i^*. Therefore, replacing the subgraph $\mathbf{S}_i \cap \mathbf{g}_c$ in g_c by S_i^* must result in another chord-set $\mathbf{g}_c'$, *i.e.*, $\mathbf{g}_c' = \mathbf{g}_c \oplus \mathbf{S}_i \cap \mathbf{g}_c \oplus S_i^*$ is also a chord-set. However, since we assumed that $N(\mathbf{S}_i \cap \mathbf{g}_c) > N(S_i^*)$, $N(\mathbf{g}_c') < N(g_c)$ which proves that $\mathbf{g}_c$ could not be a minimal chord-set, hence the theorem.

The following example will illustrate how the result of Theorem 6 may be used to reduce the number of elements in a chord-set. Consider the directed graph of Fig. 2. The elements a, b, c, d together form a chord-set $\mathbf{g}_c$ of the directed graph of Fig. 2. Chord-set $\mathbf{g}_c$ could have been obtained by the process described in this section. Consider the cut-set $\mathbf{S}_i$ consisting of elements a, e, c, f of the directed graph of Fig. 2. Since $N(\mathbf{g}_c \cap \mathbf{S}_i) = 2 > N(S_i^*) = 1$, according to Theorem 6, $\mathbf{g}_c$ is not minimal and $\mathbf{g}_c$ may be reduced to $\mathbf{g}_c' = \mathbf{g}_c \oplus \mathbf{g}_c \cap \mathbf{S}_i \cap S_i^* = bed$ which is a minimal chord-set; because there exists in the directed graph of Fig. 2 three element disjoint cycles.

Conclusions and Further Problems

The author hopes that this paper has demonstrated that a number of interesting problems arise in the study of the degree pairs of the vertices of directed graphs, and that a few of these problems are of some physical significance. A number of unsolved problems were suggested in the body of the paper. A problem of some physical significance (which was suggested by Herz (1, 9) in connection with axiomatics) is: Given a directed graph G, find a minimal subgraph $\mathbf{g}$ of G such that for every pair of distinct vertices v_i and v_j if there is a directed path from v_i to v_j in G, there is also a directed path from v_i to v_j in $\mathbf{g}$. Herz has shown that if G is cycleless, then $\mathbf{g}$ is unique. Some theoretical problems that may deserve attention are: Suppose given a directed graph G and a minimal chord-set $\mathbf{g}_c$, how can we find other minimal chord-sets

[10] This problem was originally suggested by J. P. Runyon of the Bell Telephone Laboratories—see Seshu and Reed (1), page 299.

[11] Unfortunately, it can be shown that the condition of Theorem 6 is not sufficient for a chord-set to be a minimal chord-set.

in **G**? Under what circumstances is the minimal chord-set unique (*i.e.*, **G** has only one minimal chord-set)? The problem of unique realizability (which has been considered for the non-oriented case in (8)) of a set of non-negative integer pairs as a directed graph is also an interesting and unsolved problem.

Appendix

A graph is a collection of two types of entities, *elements* (branches, arcs) and *vertices* (nodes, end-points). Each element $e(v_i, v_j)$ is connected between (incident at) a pair vertices v_i and v_j $(v_i \neq v_j)$. A graph g is said to be a *subgraph* of G if elements (branches) of g are in G. Every subgraph contains all of the vertices which are associated with its elements. The *complement of a subgraph* g is a subgraph $\bar{g}$ which contains all of the elements of G which are not in g together with every vertex associated with these elements. A *directed* (or an *oriented*) *graph* **G** is a graph G in which every element has an assigned direction, *i.e.*, given a graph G, if we place an arrowhead on each element of G, we obtain a directed graph **G**. In a directed graph an element $e(v_i, v_j)$ is connected between vertices v_i and v_j and has an arrowhead toward v_j. A *directed edge-train* (or a *chain*) is a set of elements (a subgraph of **G**) that can be ordered in the form $e(v_i, v_j), e(v_j, v_k), \cdots, e(v_r, v_t)\, e(v_t, v_s)$. If the vertices $v_i, v_j, v_k, \cdots, v_r, v_t, v_s$ are all distinct the chain is called a *directed path* from v_i to v_s. If the vertices $v_j, v_k, \cdots, v_r, v_t$ are distinct but $v_i = v_s$ then the directed chain is called a *cycle* (or a *directed circuit*).

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Book Reviews

OCEANOGRAPHICAL ENGINEERING, by R. L. Weigel. 532 pages, diagrams and illustrations, $8\frac{1}{2} \times 11$ in. Englewood Cliffs, N. J., Prentice-Hall, Inc. 1964. Price, $18.00

This book is a welcome addition to a very rapidly expanding branch of engineering in which, until now, no standard reference work has appeared. Professor Weigel has drawn upon his extensive experience as an industrial consultant, as well as teacher of courses in hydraulics, coastal engineering, and design of hydraulic structures, to compile this extensive volume of facts and figures, theory and practice relating to engineering in an ocean environment. Much of the material presented could otherwise only have been obtained at great effort by plowing through the professional literature in widely disparate disciplines. An important feature of this volume is the author's comments based on his experience which serves to unify this mass of heterogeneous material into a comprehensive text and reference work.

Of nineteen chapters, eleven deal specifically with waves and wave action, while others treat the general characteristics of the ocean environment and the physical and chemical processes which control it. Somewhat curiously, only two chapters deal with engineering *per se*—one on functional design and the other on the mooring and anchoring of floating objects. This lack of engineering emphasis stems most probably from the limitations in the state-of-the-art in modern practice. Only within the past ten or fifteen years has technology been confronted with the problem of designing and maintaining complex engineering structures in the sea, and the successful techniques which have so far been developed are still largely empirical and often confined to proprietary practice.

Professor Weigel has arranged his material in a straightforward and perspicuous manner. Profusely illustrated with diagrams comparing the results of theory and experiment, he has included many reference tables and graphs making it easy to interpolate values for a particular problem. This work also includes abundant references.

The author's intention to present material which is a compromise between a textbook and engineering reference may, however, lead to some confusion in its use for either purpose. For instance, the sections dealing with the mathematical theory of waves contain a great deal of material of questionable use to the practicing engineer, while the serious student of wave theory would more profitably refer to the original papers for comprehensive developments. Moreover, it is often unclear which of the many formulas presented apply best to a given situation for lack of qualification of their limits of validity. Nevertheless, the author has accomplished a monumental job in bringing together a large amount of factual and theoretical information that will undoubtedly find its widest application among those students of the subject who already possess a sound working knowledge of wave mechanics. They will find in this encyclopedic treatment the specific formulation or piece of information they seek.

All in all, *Oceanographical Engineering* is an impressive undertaking likely to exist as a principal reference work for many years.

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TENSORS IN ELECTRICAL ENGINEERING, by J. W. Lynn. 216 pages, diagrams, illustrations, 6×9 in. New York, St. Martin's Press, Inc., 1964. Price, $10.50.

According to the Preface, the present book is written to give graduate students and research workers in electrical machine dynamics a survey of Gabriel Kron's application of tensors, and to describe the methods in which circuits, fields, and electric machinery are being united in one mathematical discipline.

There are seven chapters and four appendices. Chapter 1 deals very briefly with "Determinants and Matrices." The reader will have to consult other textbooks for a more complete knowledge on these subjects. Chapter 2, "Kron's Network Analysis," then follows by introducing the "primitive network," "orthogonal networks," "transformations," and "tensors." Le Corbeiller's book (Ref. 10) may be studied along with this