$\underline{x} = \underline{x}_1 + \cdots + \underline{x}_s$, with $\psi_{\underline{x}}(\lambda) = \phi(\lambda)$. Then $0 = \Psi_i(T)\phi(T)\underline{x} = \phi(T)\Psi_i(T)\underline{x} = \phi(T)\Psi_i(T)\underline{x}_i$. Thus $p_i^{m_i} | \phi(\lambda)\Psi_i(\lambda) \Rightarrow p_i^{m_i} | \phi(\lambda) \Rightarrow \psi | \phi$. The converse is clear, ensuring that $\underline{x}$ belongs to ψ_T.

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A Simple Proof of the Gale-Ryser Theorem

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We consider matrices A of zeros and ones and shall be interested in the sum $c(A)$ of their column vectors and the sum $r(A)$ of their row vectors. Given *compositions* $(p_1, \ldots, p_k), (q_1, \ldots, q_l)$ of a positive integer n, i.e., non-negative integers $p_1, \ldots, p_k, q_1, \ldots, q_l$ such that $p_1 + \cdots + p_k = n = q_1 + \cdots + q_l$, does there exist a $k \times l$ matrix A of zeros and ones such that $c(A) = (p_1, \ldots, p_k), r(A) = (q_1, \ldots, q_l)$? An elegant answer to this combinatorial question is given by the important criterion by Gale and Ryser [**5**, chapter 6, theorem 1.1], which plays a prominent role in various mathematical areas and is commonly viewed as a fairly intricate result. For example, in [**2**, 1.4] it is obtained as a consequence of substantial parts of representation theory, in [**1**, 6.2.4] it is derived (in a generalized form) by means of graph theoretical methods, while [**4**, I.6, Example 2] refers to the proof in [**5**]. The object of this note is to propose a straightforward line of reasoning for the Gale-Ryser criterion.

It is easily seen that it suffices to consider the case that $p = (p_1, \ldots, p_k), q = (q_1, \ldots, q_l)$ are *partitions* of a positive integer n, i.e., that

$$p_1 \geq \cdots \geq p_k > 0, q_1 \geq \cdots \geq q_l > 0, \text{ and } p_1 + \cdots + p_k = n = q_1 + \cdots + q_l.$$

For example, $(3, 2, 2, 2, 1)$ and $(3, 3, 3, 1)$ are partitions of 10.

Any partition $p = (p_1, \ldots, p_k)$ may be visualized by a $k \times p_1$ matrix $A = (a_{i,j})$ of zeros and ones called the *Ferrers matrix* for p and defined by $c(A) = p$ and the following property: If $a_{i,j} = 0$, then $a_{i,k} = 0$ for all $k \geq j$. By transposing A we obtain again a Ferrers matrix A^*, and $p^* := c(A^*)$ is called the *conjugate* partition of p. For example,

$$(3,2,2,2,1) \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (3,2,2,2,1)^* \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

hence $(3,2,2,2,1)^* = (5,4,1)$. More formally, $p^* = (p_1^*, \dots, p_{p_1}^*)$ where

$$p_i^* := |\{j\colon 1 \le j \le k,\ p_j \ge i\}| \quad \text{for all } i \le p_1.$$

Given arbitrary compositions $p = (p_1, \dots, p_k)$, $q = (q_1, \dots, q_l)$ of a positive integer n, we say that q is *dominated* by p if $\sum_{i=1}^m q_i \le \sum_{i=1}^m p_i$ for all positive integers m, where $q_i := 0$ for all $i > l$ and $p_i := 0$ for all $i > k$. In this case we write $p \trianglerighteq q$. For example $(3,3,3,1)$ is dominated by $(5,4,1)$, but there is no dominance relationship between $(3,1,3,3)$ and $(3,2,2,2,1)$.

Theorem (Gale-Ryser). *Let p, q be partitions of a positive integer. Then there exists a $(0,1)$-matrix A such that $c(A) = p$, $r(A) = q$ if and only if q is dominated by p^*.*

The **necessity** of the dominance condition is commonly considered as the trivial part of the theorem [**3**, 4.3.19]. For the reader's convenience we recall a line of reasoning for it where we assume, more generally, that q is a composition while p is a partition. Let A be a $k \times l$ matrix of zeros and ones such that $c(A) = p$ and $r(A) = q$. If A does not contain *gaps*, i.e., if there are no $i \le k$, $j < h \le l$ such that $a_{i,j} = 0$ and $a_{i,h} = 1$, then A is a Ferrers matrix and $p^* = q$. Now let (i, j) be a gap of A and let $h > j$ be maximal such that $a_{i,h} = 1$. Swapping $a_{i,j}$ and $a_{i,h}$ we obtain a matrix $\tilde{A}$ such that $c(\tilde{A}) = p$, and the number of gaps of $\tilde{A}$ is lower than the number of gaps of A. As $r(\tilde{A}) \trianglerighteq r(A)$, it follows by induction on the number of gaps that $p^* \trianglerighteq q$, as asserted.

In the sequel we prove the **sufficiency** of the dominance condition for the existence of a matrix A such that $c(A) = p$, $r(A) = q$. Let $p = (p_1, \dots, p_k)$, $q = (q_1, \dots, q_l)$ be partitions of a positive integer n such that $p^* \trianglerighteq q$. First we observe that there is a $k \times l$ matrix B of zeros and ones such that $c(B) = p$ and $r(B) \trianglerighteq q$, namely, the Ferrers matrix for p with $l - p_1$ columns of zeros adjoined. Thus, it will suffice to prove the following

Claim. Given a $k \times l$ matrix A of zeros and ones such that $c(A) = p$, $r(A) \trianglerighteq q$ and $r(A) \ne q$, we can find a $k \times l$ matrix A' of zeros and ones such that $c(A') = p$, $r(A') \trianglerighteq q$, and $\|r(A') - q\| < \|r(A) - q\|$ (where $\|\ \|$ is the ordinary Euclidean norm).

Since $\|r(A) - q\|^2$ is an integer, after a finite number of steps we are done.

Proof: Let $r(A) = (r_1, \dots, r_l)$. Let i be minimal such that $r_i > q_i$, and let j be minimal such that $r_j < q_j$. Then $i < j$, since $r(A) \trianglerighteq q$. Clearly, for the vector

$$r' := (r_1', \dots, r_l') := (r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_{j-1}, r_j + 1, r_{j+1}, \dots, r_l)$$

we have $\|r' - q\| < \|r - q\|$. Moreover, q is dominated by r', as $r'_s \geq q_s$ for all $s < j$ and $r'_1 + \cdots + r'_s = r_1 + \cdots + r_s \geq q_1 + \cdots + q_s$ for all $s \geq j$. Since

$$r_i > q_i \geq q_j > r_j,$$

we can find a row index h (in fact there are at least two choices for h) such that $a_{h,i} = 1$ and $a_{h,j} = 0$. For any such h, the matrix $A' = (a'_{s,t})$ defined by swapping $a_{h,i}$ and $a_{h,j}$, i.e.,

$$a'_{s,t} := \begin{cases} 1 & \text{if } (s,t) = (h,j) \\ 0 & \text{if } (s,t) = (h,i) \\ a_{s,t} & \text{otherwise} \end{cases},$$

has row vector sum r' and column vector sum p. ■

For example, by the theorem there must exist a matrix A such that $c(A) = (3,2,2,2,1)$, $r(A) = (3,3,3,1)$ because $(3,2,2,2,1)^* = (5,4,1) \trianglerighteq (3,3,3,1)$. Starting with the extended Ferrers matrix B for $(3,2,2,2,1)$, the procedure obtained from our proof leads to a solution for A as follows:

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A.$$

By modifying A appropriately, (for example, by permuting the first three columns of A), the reader will find many other possibilities in this case. Note that the steps of the procedure are not uniquely determined. In particular the proof shows that in the case of $p^* \neq q$ we can always find at least two matrices A such that $c(A) = p$ and $r(A) = q$. Finally, we remark that the generalized theorem [**1**, 6.2.4] may easily be obtained along the same lines.

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