Contents

25. Comparison Based Ranking

In practice often appears the problem, how to rank different objects. Researchers of these problems frequently mention different applications, e.g. in biology Landau [72], in chemistry Hakimi [39], in networks Kim, Toroczkai, Miklós, Erdős, and Székely [63], Newman and Barabási [88], in comparison based decision making Bozóki, Fülöp, Kéri, Poesz, Rónyai and [14, 15, 71], in sports Iványi, Lucz, Pirzada, Sótér and Zhou [49, 51, 55, 53, 54, 74, 98, 106, 125].

A popular method is the comparison of two—and sometimes more—objects in all possible manner and distribution some amount of points among the compared objects.

In this chapter we introduce a general model for such ranking and study some connected problems.

25.1. Introduction to supertournaments

Let n, m be positive integers, $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ and $\mathbf{k} = (k_1, k_2, \ldots, k_m)$ vectors of nonnegative integers with $a_i \leq b_i$ $(i = 1, 2, \ldots, m)$ and $0 < k_1 < k_2 < \cdots < k_m$.

An ***$(\mathbf{a}, \mathbf{b}, \mathbf{k}, m, n)$-supertournament*** is an $x \times n$ sized matrix $\mathcal{M}$, whose columns correspond to the players of the tournament (they represent the rankable objects) and the rows correspond to the comparisons of the objects. The permitted elements of $\mathcal{M}$ belong to the set $\{0, 1, 2, \ldots, b_{max}\} \cup \{*\}$, where $m_{ij} = *$ means, that the player P_j is not a participants of the match corresponding to the i-th line, while $m_{ij} = k$ means, that P_j received k points in the match corresponding to the i-th line, and $b_{max} = \max_{1 \leq i \leq n} b_i$.

The sum (dots are taken in the count as zeros) of the elements of the i-th column of $\mathcal{M}$ is denoted by d_i and is called ***the score*** of the ith player P_i :

$$d_i = \sum_{j=1}^{x} m_{ij} \quad (i = 1, \ldots, x). \tag{25.1}$$

The sequence $\mathbf{d} = (d_1, \ldots, d_n)$ is called the ***score vector*** of the tournament. The increasingly ordered sequence of the scores is called the ***score sequence*** of the

match/player	P_1	P_2	P_3	P_4
P_1-P_2	1	1	*	*
P_1-P_3	0	*	2	*
P_1-P_4	0	*	*	2
P_2-P_3	*	0	2	*
P_2-P_4	*	0	*	2
P_3-P_4	*	*	1	1
P_1-P_2-P_3	1	1	0	*
P_1-P_2-P_4	1	0	*	2
P_1-P_3-P_4	1	*	1	0
P_2-P_3-P_4	*	0	0	2
P_1-P_2-P_3-P_4	3	1	1	1
Total score	7	3	8	10

Figure 25.1 Point matrix of a chess+last trick-bridge tournament with $n = 4$ players.

tournament and is denoted by $\mathbf{s} = (s_1, \ldots, s_n)$.

Using the terminology of the sports a supertournament can combine the matches of different sports. For example in Hungary there are popular chess-bridge, chess-tennis and tennis-bridge tournaments.

A sport is characterized by the set of the permitted results. For example in tennis the set of permitted results is $S_{\text{tennis}} = \{0 : 1\}$, for chess is the set $S_{\text{chess}} = \{0 : 2, 1 : 1\}$, for football is the set $S_{\text{football}} = \{0 : 3, 1 : 1\}$ and in the Hungarian card game last trick is $S_{\text{last trick}} = \{(0, 1, 1), (0, 0, 2)$. There are different possible rules for an individual bridge tournament, e.g. $S_{\text{bridge}} = \{(0, 2, 2, 2), (1, 1, 1, 3)\}$.

The number of participants in a match of a given sport S_i is denoted by k_i, the minimal number of the distributed points in a match is denoted by a_i, and the maximal number of points is denoted by b_i.

If a supertournament consists of only the matches of one sport, then we use a, b and k instead of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{k}$ and omit the parameter m. When the number of the players is not important, then the parameter n is also omitted.

If the points can be divided into arbitrary integer partitions, then the given sport is called ***complete***, otherwise it is called ***incomplete***. According to this definitions chess is a complete (2,2)-sport, while football is an incomplete (2,3)-sport.

Since a set containing n elements has $\binom{n}{k}$ k-element subsets, an (a, b, k, n)-tournament consists of $\binom{n}{k}$ matches. If all matches are played, then the tournament is ***finished,*** otherwise it is ***partial***.

In this chapter we deal only with finished tournaments and mostly with complete tournaments (exception is only the section on football).

Figure 25.1 contains the results of a full and complete chess+last trick+bridge supertournament. In this example $n = 4$, $\mathbf{a} = \mathbf{b} = (2, 2, 6), \mathbf{k} = (2, 3, 4)$, and $x = \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 11$. In this example the score vector of the given supertournament is $(7, 3, 8, 10)$, and its score sequence is $(3, 7, 8, 10)$.

In this chapter we investigate the problems connected with the existence and construction of different types of supertournaments having prescribed score sequences.

At first we give an introduction to (a, b)-tournaments (Section 25.2), then summarize the results on (1,1)-tournaments (Section 25.3), then for (a, a)-tournaments (Section 25.4) and for general (a, b)-tournaments (Section 25.5.

In Section 25.6 we deal with imbalance sequences, and in Section 25.7 with supertournaments. In Section 25.8 we investigate special incomplete tournaments (football tournaments) and finally in Section 25.9 we consider examples of the reconstruction of football tournaments.

Exercises

25.1-1 Describe known and possible multitournaments.

25.1-2 Estimate the number of given types of multitournaments.

25.2. Introduction to (a, b)-tournaments

Let a, b $(a \leq b)$ and n $(2 \leq n)$ be nonnegative integers and let $\mathcal{T}(a, b, n)$ be the set of such generalized tournaments, in which every pair of distinct players is connected by at least a, and at most b arcs. The elements of $\mathcal{T}(a, b, n)$ are called ***(a, b, n)-tournaments.*** The vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of the outdegrees of $T \in \mathcal{T}(a, b, n)$ is called ***the score vector*** of T. If the elements of $\mathbf{d}$ are in nondecreasing order, then $\mathbf{d}$ is called the ***score sequence*** of T.

An arbitrary vector $\mathbf{q} = (q_1, \ldots, q_n)$ of nonnegative integers is called ***multigraphic vector,*** (or degree vector) if there exists a loopless multigraph whose degree vector is $\mathbf{q}$, and $\mathbf{d}$ is called ***dimultigraphic vector*** (or *score vector*) iff there exists a loopless directed multigraph whose outdegree vector is $\mathbf{d}$.

A nondecreasingly ordered multigraphic vector is called ***multigraphic sequence,*** and a nondecreasingly ordered dimultigraphic vector is called ***dimultigraphic sequence*** (or *score sequence*).

In there exists a simple graph, resp. a simple digraph with degree/out-degree sequence $\mathbf{d}$, then $\mathbf{d}$ is called simply ***graphic,*** resp. ***digraphic.***

The number of arcs of T going from player P_i to player P_j is denoted by m_{ij} $(1 \leq i, j \leq n)$, and the matrix $\mathcal{M} = [1..n, 1..n]$ is called the ***point matrix*** or *tournament* of the T.

In the last sixty years many efforts have been devoted to the study of both types of vectors, resp. sequences. E.g. in the papers [11, 24, 31, 36, 39, 42, 40, 44, 54, 55, 53, 60, 93, 117, 118, 125, 126, 130, 135] the multigraphic sequences, while in the papers [1, 3, 4, 11, 19, 25, 33, 34, 35, 38, 42, 45, 64, 65, 72, 75, 81, 82, 83, 86, 87, 89, 90, 91, 94, 108, 111, 113, 132, 136, 140] the dimultigraphic sequences have been discussed.

Even in the last two years many authors investigated the conditions when $\mathbf{q}$ is multigraphical (e.g. [7, 13, 18, 21, 27, 28, 32, 29, 47, 48, 58, 63, 66, 67, 76, 79, 95, 97, 112, 128, 133, 134, 137, 138, 142]) or dimultigraphical (e.g. [8, 43, 49, 62, 68, 73, 85, 92, 103, 102, 104, 105, 114, 116, 127, 143]).

It is worth to mention another interesting direction of the study of different kinds of tournament, the score sets [98]

In this chapter we deal first of all with directed graphs and usually follow the terminology used by K. B. Reid [109, 111]. If in the given context a, b and n are fixed or non important, then we speak simply on *tournaments* instead of generalized or (a, b, n)-tournaments.

The first question is: how one can characterize the set of the score sequences of the (a, b)-tournaments? Or, with other words, for which sequences $\mathbf{q}$ of nonnegative integers does exist an (a, b)-tournament whose outdegree sequence is $\mathbf{q}$. The answer is given in Section 25.5.

If T is an (a, b)-tournament with point matrix $\mathcal{M} = [1..n, 1..n]$, then let $E(T)$, $F(T)$ and $G(T)$ be defined as follows: $E(T) = \max_{1\le i,j\le n} m_{ij}$, $F(T) = \max_{1\le i<j\le n}(m_{ij}+m_{ji})$, and $g(T) = \min_{1\le i<j\le n}(m_{ij}+m_{ji})$. Let $\Delta(\mathbf{q})$ denote the set of all tournaments having $\mathbf{q}$ as outdegree sequence, and let $e(D)$, $f(D)$ and $g(D)$ be defined as follows: $e(D) = \{\min\ E(T) \mid T \in \Delta(\mathbf{q})\}$, $f(\mathbf{q}) = \{\min\ F(T) \mid T \in \Delta(\mathbf{q})\}$, and $g(D) = \{\max\ G(T) \mid T \in \Delta(\mathbf{q})\}$. In the sequel we use the short notations E, F, G, e, f, g, and Δ.

Hulett, Will, and Woeginger [48, 139], Kapoor, Polimeni, and Wall [59], and Tripathi et al. [131, 128] investigated the construction problem of a minimal size graph having a prescribed degree set [107, 141]. In a similar way we follow a minimax approach formulating the following questions: given a sequence $\mathbf{q}$ of nonnegative integers,

- How to compute e and how to construct a tournament $T \in \Delta$ characterized by e? In Subsection 25.5.3 a formula to compute e, and an in 25.5.4 an algorithm to construct a corresponding tournament is presented.
- How to compute f and g? In Subsection 25.5.4 we characterize f and g, and in Subsection 25.5.5 an algorithm to compute f and g is described, while in Subsection 25.5.8 we compute f and g in linear time.
- How to construct a tournament $T \in \Delta$ characterized by f and g? In Subsection 25.5.10 an algorithm to construct a corresponding tournament is presented and analyzed.

We describe the proposed algorithms in words, by examples and by the pseudocode used in [22].

25.3. Existence of $(1, 1)$-tournaments with prescribed score sequence

The simplest supertournament is the classical tournament, in our notation the $(1, 1, n)$-tournament.

Now, we give the characterization of score sequences of tournaments which is due to Landau [72]. This result has attracted quite a bit of attention as nearly a dozen different proofs appear in the literature. Early proofs tested the readers patience with special choices of subscripts, but eventually such gymnastics were replaced by more elegant arguments. Many of the existing proofs are discussed in a survey written by K. Brooks Reid [108]. The proof we give here is due to Thomassen [127]. Further, two new proofs can be found in the paper due to Griggs and Reid [35].

Theorem 25.1 (Landau [72]) *A sequence of nonnegative integers $q = (q_1, \ldots, q_n)$ is the score vector of a $(1,1,n)$-tournament if and only if for each subset $I \subseteq \{1, \ldots, n\}$*

$$\sum_{i \in I} q_i \geq \binom{|I|}{2}, \tag{25.2}$$

with equality, when $|I| = n$.

This theorem, called Landau theorem is a nice necessary and sufficient condition, but its direct application can require the test of exponential number of subsets.

If instead of the nonordered vector we consider a nondecreasingly ordered sequence $q = (q_1, \ldots, q_n)$, then due to the monotonity $q_1 \leq \cdots \leq q_n$ the inequalities (25.2), called Landau inequalities, we get the following consequence.

Corollary 25.2 (Landau [72]) *A nondecreasing sequence of nonnegative integers $\mathbf{q} = (q_1, \ldots, q_n)$ is the score sequence of some $(1,1,n)$-tournament, if and only if*

$$\sum_{i=1}^{k} q_i \geq \binom{k}{2} \tag{25.3}$$

for $i = 1, \ldots, n$, with equality for $k = n$.

Proof **Necessity** If a nondecreasing sequence of nonnegative integers $\mathbf{q}$ is the score sequence of an $(1,1,n)$-tournament T, then the sum of the first k scores in the sequence counts exactly once each arc in the subtournament W induced by $\{v_1, \ldots, v_k\}$ plus each arc from W to $T - W$. Therefore the sum is at least $\frac{k(k-1)}{2}$, the number of arcs in W. Also, since the sum of the scores of the vertices counts each arc of the tournament exactly once, the sum of the scores is the total number of arcs, that is, $\frac{n(n-1)}{2}$.

Sufficiency (Thomassen [127]) Let n be the smallest integer for which there is a nondecreasing sequence $\mathbf{s}$ of nonnegative integers satisfying Landau's conditions (25.3), but for which there is no $(1,1,n)$-tournament with score sequence $\mathbf{s}$. Among all such $\mathbf{s}$, pick one for which $\mathbf{s}$ is as lexicografically small as possible.

First consider the case where for some $k < n$,

$$\sum_{i=1}^{k} s_i = \binom{k}{2}. \tag{25.4}$$

By the minimality of n, the sequence $\mathbf{s}_1 = [s_1, \ldots, s_k]$ is the score sequence of some tournament T_1. Further,

$$\sum_{i=1}^{m} (s_{k+i} - k) = \sum_{i=1}^{m+k} s_i - mk \geq \binom{m+k}{2} - \binom{k}{2} - mk = \binom{m}{2}, \tag{25.5}$$

for each m, $1 \leq m \leq n - k$, with the equality when $m = n - k$. Therefore, by the minimality of n, the sequence $\mathbf{s}_2 = [s_{k+1}-k, s_{k+2-k}, \ldots, s_n-k]$ is the score sequence of some tournament T_2. By forming the disjoint union of T_1 and T_2 and adding all arcs from T_2 to T_1, we obtain a tournament with score sequence $\mathbf{s}$.

Now, consider the case where each inequality in (25.3) is strict when $k < n$ (in particular $q_1 > 0$). Then the sequence $\mathbf{s}_3 = [s_1 - 1, \ldots, s_{n-1}, s_n + 1]$ satisfies (25.3) and by the minimality of q_1, $\mathbf{s}_3$ is the score sequence of some tournament T_3. Let u and v be the vertices with scores $s_n + 1$ and $s_1 - 1$ respectively. Since the score of u is larger than that of v, then according to Lemma 25.5 T_3 has a path P from u to v of length ≤ 2. By reversing the arcs of P, we obtain a tournament with score sequence $\mathbf{s}$, a contradiction. ■

Landau's theorem is the tournament analog of the Erdős-Gallai theorem for graphical sequences [24]. A tournament analog of the Havel-Hakimi theorem [41, 44] for graphical sequences is the following result.,

Theorem 25.3 (Reid, Beineke [110]) *A nondecreasing sequence $(q_1, \ldots, q_n)$ of nonnegative integers, $n \geq 2$, is the score sequence of an $(1, 1, n)$-tournament if and only if the new sequence*

$$(q_1, \ldots, q_{q_n}, q_{q_n+1} - 1, \ldots, q_{n-1} - 1), \tag{25.6}$$

arranged in nondecreasing order, is the score sequence of some $(1, 1, n-1)$-tournament.

Proof See [110]. ■

25.4. Existence of an (a, a)-tournament with prescribed score sequence

For the (a, a)-tournament Moon [82] proved the following extension of Landau's theorem.

Theorem 25.4 (Moon [82], Kemnitz, Dulff [62]) *A nondecreasing sequence of nonnegative integers $q = (q_1, \ldots, q_n)$ is the score sequence of an (a, a, n)-tournament if and only if*

$$\sum_{i=1}^{k} q_i \geq a\binom{k}{2}, \tag{25.7}$$

for $i = 1, \ldots, n$, with equality for $k = n$.

Proof See [62, 82]. ■

Later Kemnitz and Dulff [62] reproved this theorem.

The proof of Kemnitz and Dulff is based on the following lemma, which is an extension of a lemma due to Thomassen [127].

Lemma 25.5 (Thomassen [127]) *Let u be a vertex of maximum score in an (a, a, n)-tournament T. If v is a vertex of T different from u, then there is a directed path P from u to v of length at most 2.*

Proof ([62]) Let $v_1, \ldots, v_l$ be all vertices of T such that $(u, v_i) \in E(T)$, $i = 1, \ldots, l$. If $v \in \{v_1, \ldots, v_l\}$ then $|P| = 1$ for the length $|P|$ of path P. Otherwise if there exists a vertex v_i, $1 \leq i \leq l$, such that $(v_i, v) \in E(T)$ then $|P| = 2$. If for all i, $1 \leq i \leq l$ $(v_i, v) \notin E(T)$ then there are k arcs $(v, v_i) \in T$ which implies $d^+(v) \geq kl + k > kl \geq d^+u$, a contradiction to the assumption that u has maximum score. ■

Proof of Theorem 25.4. The necessity of condition (25.7) is obvious since there are $a\binom{k}{2}$ arcs among any k vertices and there are $a\binom{k}{2}$ arcs among all n vertices.

To prove the sufficiency of (25.7) we assume that the sequence $S_n = (s_1, \ldots, s_n)$ is a counterexample to the theorem with minimum n and smallest s_1 with that choice of n. Suppose first that there exists an integer m, $1 \leq m < n$, such that

$$\sum_{i=1}^{m} s_i = k\binom{k}{2}. \tag{25.8}$$

Because the minimality of n, the sequence $(s - 1, \ldots, s_n)$ is the score sequence of some $(1, 1, n)$-tournament T_1.

Consider the sequence $R_{n-m} = (r - 1, r_2, \ldots, r_{n-m})$ defined by $r_i = s_{m+1} - km$, $i = 1, \ldots, n - m$. because of $\sum_{i=1}^{m+1} s_i \geq k\binom{m+1}{2}$ by assumption it follows that

$$s_{m+1} = \sum_{i=1}^{m+1} s_i - \sum_{i=1}^{m} s_i \geq k\binom{m+1}{2} - k\binom{m}{2} - km$$

which implies $r_i \geq 0$. Since S_n is nondecreasing also R_{n-m} is a nondecreasing sequence of nonnegative integers.

For each l with $1 \leq l \leq n - m$ it holds that

$$\sum_{i=1}^{l} r_i = \sum_{i=1}^{l}(s_{m+1} - km) = \sum_{i=1}^{l+m} s_i - \sum_{i=1}^{m} s_i - lam \geq k\binom{l+m}{2} - k\binom{m}{2} - lam = k\binom{l}{2} \tag{25.9}$$

with equality for $l = n - m$ since by assumption

$$\sum_{i=1}^{l+m} s_i \geq a\binom{l+m}{2}, \quad \sum_{i=1}^{m} s_i = a\binom{m}{2}. \tag{25.10}$$

Therefore the sequence R_{n-m} fulfils condition (25.8), by the minimality of n, R_{n-m} is the score sequence of some $(a, a, n - m)$-tournament T_2. By forming the disjoint union of T_1 and T_2 we obtain a (a, a, n)-tournament T with score sequence S_n in contradiction to the assumption that S_n is counterexample.

Now we consider the case when the inequality in condition (25.8) is strict for each m, $1 \leq m < n$. This implies in particular $s_1 > 0$.

The sequence $\overline{S}_n = (s-1, s_2, s_3, \ldots, s_{n-1}, s_n)$ is a nondecreasing sequence of nonnegative integers which fulfils condition (25.8). Because of the minimality of S_n, $\overline{S}_n$ is the score sequence of some (a, a, n)-tournament T_3. Let u denote a vertex of T_3 with score $s_n + 1$ and v a vertex of T_3 with score $S_1 - 1$. Since u has maximum score in T_3 there is a directed path P from u to v of length at most 2 according to Lemma 25.5. By reversing the arcs of the path P we obtain an (a, a, n)-tournament T with score sequence S_n. This contradiction completes the proof. ■

25.5. Existence of an (a, b)-tournament with prescribed score sequence

In this section we show that for arbitrary prescribed sequence of nondecreasingly ordered nonnegative integers there exists an (a, b)-tournament

25.5.1. Existence of a tournament with arbitrary degree sequence

Since the numbers of points m_{ij} are not limited, it is easy to construct a $(0, q_n, n)$-tournament for any $\mathbf{q}$.

Lemma 25.6 *If $n \geq 2$, then for any vector of nonnegative integers $\mathbf{q} = (q_1, \ldots, q_n)$ there exists a loopless directed multigraph T with outdegree vector $\mathbf{q}$ so, that $E \leq q_n$.*

Proof Let $m_{n1} = d_n$ and $m_{i,i+1} = q_i$ for $i = 1, 2, \ldots, n-1$, and let the remaining m_{ij} values be equal to zero. ■

Using weighted graphs it would be easy to extend the definition of the (a, b, n)-tournaments to allow *arbitrary real values* of a, b, and $\mathbf{q}$. The following algorithm, Naive-Construct works without changes also for input consisting of real numbers.

We remark that Ore in 1956 [89, 90, 91] gave the necessary and sufficient conditions of the existence of a tournament with prescribed indegree and outdegree vectors. Further Ford and Fulkerson [25, Theorem11.1] published in 1962 necessary and sufficient conditions of the existence of a tournament having prescribed lower and upper bounds for the indegree and outdegree of the vertices. Their results also can serve as basis of the existence of a tournament having arbitrary outdegree sequence.

25.5.2. Description of a naive reconstructing algorithm

Sorting of the elements of D is not necessary.

Input. n: the number of players ($n \geq 2$);
$\mathbf{q} = (q_1, \ldots, q_n)$: arbitrary sequence of nonnegative integer numbers.

Output. $\mathcal{M} = [1..n, 1..n]$: the point matrix of the reconstructed tournament.

Working variables. i, j: cycle variables.

Naive-Construct$(n, \mathbf{q})$

```
for i = 1 to n
    for j = 1 to n
        m_ij = 0
m_n1 = q_n
for i = 1 to n − 1
    m_i,i+1 = q_i
return M
```

The running time of this algorithm is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has n^2 elements, this algorithm is asymptotically optimal.

25.5.3. Computation of e

This is also an easy question. From now on we suppose that $\mathbf{q}$ is a nondecreasing sequence of nonnegative integers, that is $0 \le q_1 \le q_2 \le \ldots \le q_n$. Let $h = \lceil q_n/(n-1) \rceil$.

Since $\Delta(\mathbf{q})$ is a finite set for any finite score vector $\mathbf{q}$, $e(\mathbf{q}) = \min\{E(T) | T \in \Delta(\mathbf{q})\}$ exists.

Lemma 25.7 (Iványi [49]) *If $n \ge 2$, then for any sequence $\mathbf{q} = (q_1, \ldots, q_n)$ there exists a $(0, b, n)$-tournament T such that*

$$E \le h \qquad \text{and} \quad b \le 2h, \tag{25.11}$$

and h is the smallest upper bound for e, and $2h$ is the smallest possible upper bound for b.

Proof If all players gather their points in a uniform as possible manner, that is

$$\max_{1 \le j \le n} m_{ij} - \min_{1 \le j \le n,\ i \ne j} m_{ij} \le 1 \quad \text{for } i = 1,\ 2,\ \ldots,\ n, \tag{25.12}$$

then we get $E \le h$, that is the bound is valid. Since player P_n has to gather q_n points, the pigeonhole principle [9, 10, 23] implies $E \ge h$, that is the bound is not improvable. $E \le h$ implies $\max_{1 \le i < j \le n} m_{ij} + m_{ji} \le 2h$. The score sequence $D = (d_1, d_2, \ldots, d_n) = (2n(n-1), 2n(n-1), \ldots, 2n(n-1))$ shows, that the upper bound $b \le 2h$ is not improvable. ■

Corollary 25.8 (Iványi [51]) *If $n \ge 2$, then for any sequence $\mathbf{q} = (q_1, \ldots, q_n)$ holds $e(D) = \lceil q_n/(n-1) \rceil$.*

Proof According to Lemma 25.7 $h = \lceil q_n/(n-1) \rceil$ is the smallest upper bound for e. ■

25.5.4. Description of a construction algorithm

The following algorithm constructs a $(0, 2h, n)$-tournament T having $E \leq h$ for any $\mathbf{q}$.

Input. n: the number of players $(n \geq 2)$;
$\mathbf{q} = (q_1, \ldots, q_n)$: arbitrary sequence of nonnegative integer numbers.

Output. $\mathcal{M} = [1..n, 1..n]$: the point matrix of the tournament.

Working variables. $i,\ j,\ l$: cycle variables;
k: the number of the "larger part" in the uniform distribution of the points.

Pigeonhole-Construct$(n, \mathbf{q})$

```
for i = 1 to n
    m_ii = 0
    k = q_i − (n − 1)⌊q_i/(n − 1)⌋
    for j = 1 to k
        l = i + j (mod n)
        m_il = ⌈q_n/(n − 1)⌉
    for j = k + 1 to n − 1
        l = i + j (mod n)
        m_il = ⌊q_n/(n − 1)⌋
return M
```

The running time of Pigeonhole-Construct is $\Theta(n^2)$ in worst case (in best case too). Since the point matrix $\mathcal{M}$ has n^2 elements, this algorithm is asymptotically optimal.

25.5.5. Computation of f and g

Let S_i $(i = 1,\ 2,\ \ldots,\ n)$ be the sum of the first i elements of $\mathbf{q}$. B_i $(i = 1,\ 2,\ \ldots,\ n)$ be the binomial coefficient $i(i-1)/2$. Then the players together can have S_n points only if $fB_n \geq S_n$. Since the score of player P_n is q_n, the pigeonhole principle implies $f \geq \lceil q_n/(n-1) \rceil$.

These observations result the following lower bound for f:

$$f \geq \max\left(\left\lceil \frac{S_n}{B_n} \right\rceil, \left\lceil \frac{d_n}{n-1} \right\rceil \right). \tag{25.13}$$

If every player gathers his points in a uniform as possible manner then

$$f \leq 2 \left\lceil \frac{q_n}{n-1} \right\rceil. \tag{25.14}$$

These observations imply a useful characterization of f.

Lemma 25.9 (Iványi [49]) *If $n \geq 2$, then for arbitrary sequence $\boldsymbol{q} = (q_1, \ldots, q_n)$*

Player/Player	P_1	P_2	P_3	P_4	P_5	P_5	Score
P_1	—	0	0	0	0	0	0
P_2	0	—	0	0	0	0	0
P_3	0	0	—	0	0	0	0
P_4	10	10	10	—	5	5	40
P_5	10	10	10	5	—	5	40
P_6	10	10	10	5	5	—	40

Figure 25.2 Point matrix of a $(0, 10, 6)$-tournament with $f = 10$ for $\mathbf{q} = (0, 0, 0, 40, 40, 40)$.

there exists a (g, f, n)-tournament having $\boldsymbol{q}$ as its outdegree sequence and the following bounds for f and g:

$$\max\left(\left\lceil \frac{S}{B_n} \right\rceil, \left\lceil \frac{q_n}{n-1} \right\rceil\right) \le f \le 2\left\lceil \frac{q_n}{n-1} \right\rceil, \tag{25.15}$$

$$0 \le g \le f. \tag{25.16}$$

Proof (25.15) follows from (25.13) and (25.14), (25.16) follows from the definition of f. ■

It is worth to remark, that if $q_n/(n-1)$ is integer and the scores are identical, then the lower and upper bounds in (25.15) coincide and so Lemma 25.9 gives the exact value of F.

In connection with this lemma we consider three examples. If $q_i = q_n = 2c(n-1)$ $(c > 0,\ i = 1,\ 2,\ \ldots,\ n-1)$, then $q_n/(n-1) = 2c$ and $S_n/B_n = c$, that is S_n/B_n is twice larger than $q_n/(n-1)$. In the other extremal case, when $q_i = 0$ $(i = 1,\ \ldots, n-1)$ and $q_n = cn(n-1) > 0$, then $q_n/(n-1) = cn$, $S_n/B_n = 2c$, so $q_n/(n-1)$ is $n/2$ times larger, than S_n/B_n.

If $\mathbf{q} = (0, 0, 0, 40, 40, 40)$, then Lemma 25.9 gives the bounds $8 \le f \le 16$. Elementary calculations show that Figure 25.2 contains the solution with minimal f, where $f = 10$.

In 2009 we proved the following assertion.

Theorem 25.10 (Iványi [49] *For $n \ge 2$ a nondecreasing sequence $\boldsymbol{q} = (q_1, \ldots, q_n)$ of nonnegative integers is the score sequence of some (a, b, n)-tournament if and only if*

$$aB_k \le \sum_{i=1}^{k} q_i \le bB_n - L_k - (n-k)q_k \quad (1 \le k \le n), \tag{25.17}$$

where

$$L_0 = 0, \ \text{and}\ L_k = \max\left(L_{k-1},\ bB_k - \sum_{i=1}^{k} q_i\right) \quad (1 \le k \le n). \tag{25.18}$$

The theorem was proved by Moon [82], and later by Kemnitz and Dolff [62] for (a, a, n)-tournaments is the special case $a = b$ of Theorem 25.10. Theorem 3.1.4 of [57] is the special case $a = b = 2$. The theorem of Landau [72] is the special case $a = b = 1$ of Theorem 25.10.

25.5.6. Description of a testing algorithm

The following algorithm Interval-Test decides whether a given $\mathbf{q}$ is a score sequence of an (a, b, n)-tournament or not. This algorithm is based on Theorem 25.10 and returns $W =$ True if $\mathbf{q}$ is a score sequence, and returns $W =$ False otherwise.

Input. a: minimal number of points divided after each match;
b: maximal number of points divided after each match.

Output. W: logical variable ($W =$ True shows that D is an (a, b, n)-tournament.

Local working variable. i: cycle variable;
$L = (L_0, L_1, \ldots, L_n)$: the sequence of the values of the loss function.

Global working variables. n: the number of players ($n \geq 2$);
$\mathbf{q} = (q_1, \ldots, q_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores.

```
Interval-Test(a, b)
for i = 1 to n
    L_i = max(L_{i-1}, bB_n - S_i - (n - i)q_i)
    if S_i < aB_i
       W = False
       return W
       if S_i > bB_n - L_i - (n - i)q_i
          W ← False
          return W
return W
```

In worst case Interval-Test runs in $\Theta(n)$ time even in the general case $0 < a < b$ (in the best case the running time of Interval-Test is $\Theta(n)$). It is worth to mention, that the often referenced Havel–Hakimi algorithm [39, 44] even in the special case $a = b = 1$ decides in $\Theta(n^2)$ time whether a sequence D is digraphical or not.

25.5.7. Description of an algorithm computing f and g

The following algorithm is based on the bounds of f and g given by Lemma 25.9 and the logarithmic search algorithm described by D. E. Knuth [68, page 410].

Input. No special input (global working variables serve as input).

Output. b: f (the minimal F);
a: g (the maximal G).

Local working variables. i: cycle variable;
l: lower bound of the interval of the possible values of F;

u: upper bound of the interval of the possible values of F.

Global working variables. n: the number of players ($n \geq 2$);
$\mathbf{q} = (q_1, \ldots, q_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores;
W: logical variable (its value is TRUE, when the investigated D is a score sequence).

```
MinF-MaxG
01 B_0 = S_0 = L_0 = 0                          ▷ Initialization
02 for i = 1 to n
03     B_i = B_{i-1} + i - 1
04     S_i = S_{i-1} + q_i
05 l = max(⌈S_n/B_n⌉, ⌈q_n/(n - 1)⌉)
06 u = 2 ⌈q_n/(n - 1)⌉
07 W = True                                     ▷ Computation of f
08 Interval-Test(0, l)
09 if W == True
10     b = l
11     go to 21
12 b = ⌈(l + u)/2⌉
13 Interval-Test(0, f)
14 if W == True
15     go to 17
16 l = b
17 if u == l + 1
18     b = u
19     go to 37
20 go to 14
21 l = 0                                        ▷ Computation of g
22 u = f
23 Interval-Test(b, b)
24 if W == True
25     a ← f
26     go to 37
27 a = ⌈(l + u)/2⌉
28 Interval-Test(0, a)
29 if W == True
30     l ← a
31     go to 33
32 u = a
33 if u == l + 1
34     a = l
35     go to 37
36 go to 27
39 return a, b
```

MinF-MaxG determines f and g.

Lemma 25.11 (Iványi [51]) *Algorithm* Ming-MaxG *computes the values f and g for arbitrary sequence $\boldsymbol{q} = (q_1, \ldots, q_n)$ in $O(n \log(q_n/(n))$ time.*

Proof According to Lemma 25.9 F is an element of the interval $[\lceil q_n/(n-1)\rceil, \lceil 2q_n/(n-1)\rceil]$ and g is an element of the interval $[0, f]$. Using Theorem B of [68, page 412] we get that $O(\log(q_n/n))$ calls of Interval-Test is sufficient, so the $O(n)$ run time of Interval-Test implies the required running time of MinF-MaxG. ∎

25.5.8. Computing of f and g in linear time

Analyzing Theorem 25.10 and the work of algorithm MinF-MaxG one can observe that the maximal value of G and the minimal value of F can be computed independently by the following Linear-MinF-MaxG.

Input. No special input (global working variables serve as input).
Output. b: f (the minimal F).
a: g (the maximal G).
Local working variables. i: cycle variable.
Global working variables. n: the number of players ($n \geq 2$);
$\mathbf{q} = (q_1, \ldots, q_n)$: a nondecreasing sequence of nonnegative integers;
$B = (B_0, B_1, \ldots, B_n)$: the sequence of the binomial coefficients;
$S = (S_0, S_1, \ldots, S_n)$: the sequence of the sums of the i smallest scores.

Linear-MinF-MaxG
01 $B_0 = S_0 = L_0 = 0$ ▷ Initialization
02 **for** $i = 1$ **to** n
03 $\quad B_i = B_{i-1} + i - 1$
04 $\quad S_i = S_{i-1} + q_i$
05 $a = 0$
06 $b = \min 2 \lceil q_n/(n-1) \rceil$
07 **for** $i = 1$ **to** n ▷ Computation of g
08 $\quad a_i = \left\lceil (2S_i/(n^2 - n) \right\rceil) < a$
09 $\quad$**if** $a_i > a$
10 $\quad\quad a = a_i$
11 **for** $i = 1$ **to** n ▷ Computation of f
12 $\quad L_i = \max(L_{i-1}, bB_n - S_i - (n-i)q_i$
13 $\quad b_i = (S_i + (n-i)q_i + L_i)/B_i$
14 $\quad$**if** $b_i < b$
15 $\quad\quad b = b_i$
16 **return** a, b

Lemma 25.12 *Algorithm* Linear-MinG-MaxG *computes the values f and g for arbitrary sequence $\mathbf{q} = (q_1, \ldots, q_n)$ in $\Theta(n)$ time.*

Proof Lines 01, 05, 06, and 16 require only constant time, lines 02–06, 07–10, and 11–15 require $\Theta(n)$ time, so the total running time is $\Theta(n)$. ∎

25.5.9. Tournament with f and g

The following reconstruction algorithm Score-Slicing2 is based on balancing between additional points (they are similar to “excess”, introduced by Brauer et al. [16]) and missing points introduced in [49]. The greediness of the algorithm Havel–Hakimi [39, 44] also characterizes this algorithm.

This algorithm is an extended version of the algorithm Score-Slicing proposed in [49].

The work of the slicing program is managed by the following program Mini-Max.

Input. n: the number of players ($n \geq 2$);
$\mathbf{q} = (q_1, \ldots, q_n)$: a nondecreasing sequence of integers satisfying (25.17).

Output. $\mathcal{M} = [1 \mathinner{.\,.} n, 1 \mathinner{.\,.} n]$: the point matrix of the reconstructed tournament.

Local working variables. i, j: cycle variables.

Global working variables. $p = (p_0, p_1, \ldots, p_n)$: provisional score sequence;
$P = (P_0, P_1, \ldots, P_n)$: the partial sums of the provisional scores;
$\mathcal{M}[1 \mathinner{.\,.} n, 1 \mathinner{.\,.} n]$: matrix of the provisional points.

```
Mini-Max(n, q)
MinF-MaxG(n, q)                           ▷ Initialization
p_0 = 0
P_0 = 0
for i = 1 to n
    for j = 1 to i − 1
        M[i, j] = b
        for j = i to n
            M[i, j] = 0
    p_i = q_i
if n ≥ 3                                  ▷ Score slicing for n ≥ 3 players
   for k = n downto 3
       Score-Slicing2(k)
if n == 2                                 ▷ Score slicing for 2 players
   m_{1,2} = p_1
   m_{2,1} = p_2
return M
```

25.5.10. Description of the score slicing algorithm

The key part of the reconstruction is the following algorithm Score-Slicing2 [49].

During the reconstruction process we have to take into account the following bounds:

$$a \leq m_{i,j} + m_{j,i} \leq b \quad (1 \leq i < j \leq n); \tag{25.19}$$

modified scores have to satisfy (25.17); (25.20)

$$m_{i,j} \leq p_i \ (1 \leq i,\ j \leq n, i \neq j); \tag{25.21}$$

$$\text{the monotonicity } p_1 \le p_2 \le \cdots \le p_k \text{ has to be saved } (1 \le k \le n); \tag{25.22}$$

$$m_{ii} = 0 \quad (1 \le i \le n). \tag{25.23}$$

Input. k: the number of the investigated players ($k > 2$);
$\mathbf{p}_k = (p_0, p_1, \ldots, p_k)$ ($k = 3,\ 4,\ \ldots,\ n$): prefix of the provisional score sequence p;
$\mathcal{M}[1 \,.\,.\, n, 1 \,.\,.\, n]$: matrix of provisional points;

Output. M: number of missing points
$\mathbf{p}_k$: prefix of the provisional score sequence.
Local working variables. $A = (A_1, A_2, \ldots, A_n)$ the number of the additional points;
M: missing points: the difference of the number of actual points and the number of maximal possible points of P_k;
d: difference of the maximal decreasable score and the following largest score;
y: number of sliced points per player;
f: frequency of the number of maximal values among the scores p_1, p_2, $\ldots$, p_{k-1};
i, j: cycle variables;
m: maximal amount of sliceable points;
$P = (P_0, P_1, \ldots, P_n)$: the sums of the provisional scores;
x: the maximal index i with $i < k$ and $m_{i,k} < b$.
Global working variables: n: the number of players ($n \ge 2$);
$B = (B_0, B_1, B_2, \ldots, B_n)$: the sequence of the binomial coefficients;
a: minimal number of points distributed after each match;
b: maximal number of points distributed after each match.

```
Score-Slicing2(k, p_k)
P_0 = 0
for i = 1 to k − 1                    ▷ Initialization
     P_i = P_{i−1} + p_i
     A_i = P_i − aB_i
M = (k − 1)b − p_k
while M > 0 and A_{k−1} > 0          ▷ There are missing and additional points
        x = k − 1
        while r_{x,k} == b
                x = x − 1
        f = 1
        while p_{x−f+1} == p_{x−f}
                f = f + 1
        d = p_{x−f+1} − p_{x−f}
        m = min(b, d, ⌈A_x/b⌉, ⌈M/b⌉)
        for i = f downto 1
             y = min(b − r_{x+1−i,k}, m, M, A_{x+1−i}, p_{x+1−i})
             r_{x+1−i,k} = r_{x+1−i,k} + y
             p_{x+1−i} = p_{x+1−i} − y
             r_{k,x+1−i} = b − r_{x+1−i,k}
```

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	1	5	1	1	1	09
P_2	1	—	4	2	0	2	09
P_3	3	3	—	5	4	4	19
P_4	8	2	5	—	2	3	20
P_5	9	9	5	7	—	2	32
P_6	8	7	5	6	8	—	34

Figure 25.3 The point table of a $(2, 10, 6)$-tournament T.

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	1	1	6	1	0	9
P_2	1	—	1	6	1	0	9
P_3	1	1	—	6	8	3	19
P_4	3	3	3	—	8	3	20
P_5	9	9	2	2	—	10	32
P_6	10	10	7	7	0	—	34

Figure 25.4 The point table of T reconstructed by Score-Slicing.

20 $\qquad\qquad M = M - y$
21 $\qquad\quad$ **for** $j = i$ **downto** 1
22 $\qquad\qquad\quad A_{x+1-i} = A_{x+1-i} - y$
23 **while** $M > 0$ $\qquad\qquad\qquad\qquad \triangleright$ No missing points
24 $\qquad i = k - 1$
25 $\qquad y = \max(m_{ki} + m_{ik} - a, m_{ki}, M)$
26 $\qquad r_{ki} = r_{ki} - y$
27 $\qquad M = M - y$
28 $\qquad i = i - 1$
29 **return** p_k, M

Let us consider an example. Figure 25.3 shows the point table of a $(2, 10, 6)$-tournament T. We remark that the termin ***point table*** is used as a synonym of the termin point matrix.

The score sequence of T is $\mathbf{q}$ = (9,9,19,20,32,34). In [49] the algorithm Score-Slicing resulted the point table reprezented in Figure 25.4.

The algorithm Mini-Max starts with the computation of f. MinF-MaxG called in line 01 begins with initialization, including provisional setting of the elements of $\mathcal{M}$ so, that $m_{ij} = b$, if $i > j$, and $m_{ij} = 0$ otherwise. Then MinF-MaxG sets the lower bound $l = \max(9, 7) = 9$ of f in line 07 and tests it in line 10 Interval-Test. The test shows that $l = 9$ is large enough so Mini-Max sets $b = 9$ in line 12 and jumps to line 23 and begins to compute g. Interval-Test called in line 25 shows that $a = 9$ is too large, therefore MinF-MaxG continues with the test of $a = 5$ in line 30. The result is positive, therefore comes the test of $a = 7$, then the test of $a = 8$. Now $u = l + 1$ in line 35, so $a = 8$ is fixed, and the control returns to line 02

Player/Player	P_1	P_2	P_3	P_4	P_5	P_6	Score
P_1	—	4	4	1	0	0	9
P_2	4	—	4	1	0	0	9
P_3	4	4	—	7	4	0	19
P_4	7	7	1	—	5	0	20
P_5	8	8	4	3	—	9	32
P_6	9	9	8	8	0	—	34

Figure 25.5 The point table of T reconstructed by Mini-Max.

of Mini-Max.

Lines 02–09 contain initialization, and Mini-Max begins the reconstruction of a $(8, 9, 6)$-tournament in line 10. The basic idea is that Mini-Max successively determines the won and lost points of P_6, P_5, P_4 and P_3 by repeated calls of Score-Slicing2 in line 12, and finally it computes directly the result of the match between P_2 and P_1.

At first Mini-Max computes the results of P_6 calling calling Score-Slicing2 with parameter $k = 6$. The number of additional points of the first five players is $A_5 = 89 - 8 \cdot 10 = 9$ according to line 03, the number of missing points of P_6 is $M = 5 \cdot 9 - 34 = 11$ according to line 04. Then Score-Slicing2 determines the number of maximal numbers among the provisional scores $p_1, p_2, \ldots, p_5$ ($f = 1$ according to lines 09–14) and computes the difference between p_5 and p_4 ($d = 12$ according to line 12). In line 13 we get, that $m = 9$ points are sliceable, and P_5 gets these points in the match with P_6 in line 16, so the number of missing points of P_6 decreases to $M = 11 - 9 = 2$ (line 19) and the number of additional point decreases to $A = 9 - 9 = 0$. Therefore the computation continues in lines 22–27 and m_{64} and m_{63} will be decreased by 1 resulting $m_{64} = 8$ and $m_{63} = 8$ as the seventh line and seventh column of Figure 25.5 show. The returned score sequence is $p = (9, 9, 19, 20, 23)$.

In the second place Mini-Max calls Score-Slicing2 with parameter $k = 5$, and get $A_4 = 9$ and $M = 13$. At first P_4 gets 1 point, then P_3 and P_4 get both 4 points, reducing M to 4 and A_4 to 0. The computation continues in line 22 and results the further decrease of m_{54}, m_{53}, m_{52}, and m_{51} by 1, resulting $m_{54} = 3$, $m_{53} = 4$, $m_{52} = 8$, and $m_{51} = 8$ as the sixth row of Figure 25.5 shows.

In the third place Mini-Max calls Score-Slicing2 with parameter $k = 4$, and get $A_3 = 11$ and $M = 11$. At first P_3 gets 6 points, then P_3 further 1 point, and P_2 and P_1 also both get 1 point, resulting $m_{34} = 7$, $m_{43} = 2$, $m_{42} = 8$, $m_{24} = 1$, $m_{14} = 1$ and $m_{41} = 8$, further $A_3 = 0$ and $M = 2$. The computation continues in lines 22–27 and results a decrease of m_{43} by 1 point resulting $m_{43} = 1$, $m_{42} = 7$, and $m_{41} = 7$, as the fifth row and fifth column of Figure 25.5 show. The returned score sequence is $p = (9, 9, 15)$.

In the fourth place Mini-Max calls Score-Slicing2 with parameter $k = 3$, and gets $A_2 = 10$ and $M = 9$. At first P_1 and P_2 get 4 points, resulting $m_{13} = 4$, and $m_{23} = 4$, and $M = 2$, and $A_2 = 0$. Then MINI-MAX sets in lines 23–26 $m_{31} = 4$

and $m_{32} = 4$. The returned point vector is $p = (4, 4)$.

Finally Mini-Max sets $m_{12} = 4$ and $m_{21} = 4$ in lines 14–15 and returns the point matrix represented in Figure 25.5.

The comparison of Figures 25.4 and 25.5 shows a large difference between the simple reconstruction of Score-Slicing2 and the minimax reconstruction of Mini-Max: while in the first case the maximal value of $m_{ij} + m_{ji}$ is 10 and the minimal value is 2, in the second case the maximum equals to 9 and the minimum equals to 8, that is the result is more balanced (the given $\mathbf{q}$ does not allow to build a perfectly balanced (k, k, n)-tournament).

25.5.11. Analysis of the minimax reconstruction algorithm

The main result of this paper is the following assertion.

Theorem 25.13 (Iványi [51]) *If $n \geq 2$ is a positive integer and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ is a nondecreasing sequence of nonnegative integers, then there exist positive integers f and g, and a (g, f, n)-tournament T with point matrix $\mathcal{M}$ such that*

$$f = \min(m_{ij} + m_{ji}) \leq b, \tag{25.24}$$

$$g = \max m_{ij} + m_{ji} \geq a \tag{25.25}$$

for any (a, b, n)-tournament, and algorithm Linear-MinF-MaxG *computes f and g in $\Theta(n)$ time, and algorithm* Mini-Max *generates a suitable T in $O(q_n n^2)$ time.*

Proof The correctness of the algorithms Score-Slicing2, Min$F$-Max$G$ implies the correctness of Mini-Max.

Lines 1–46 of Mini-Max require $O(\log(d_n/n))$ uses of MinG-MaxF, and one search needs $O(n)$ steps for the testing, so the computation of f and g can be executed in $O(n \log(q_n/n))$ times.

The reconstruction part (lines 47–55) uses algorithm Score-Slicing2, which runs in $O(bn^3)$ time [49]. Mini-Max calls Score-Slicing2 $n - 2$ times with $f \leq 2\lceil d_n/n \rceil$, so $n^3 q_n/n = q_n n^2$ finishes the proof. ■

The interesting property of f and g is that they can be determined independently (and so there exists a tournament T having both extremal features) is called linking property. One of the earliest occurrences appeared in a paper of Mendelsohn and Dulmage [77]. It was formulated by Ford and Fulkerson [25, page 49] in a theorem on the existence of integral matrices for which the row-sums and the column-sums lie between specified bounds. The concept was investigated in detail in the book written by Mirsky [80]. A. Frank used this property in the analysis of different problems of combinatorial optimization [26, 30].

25.6. Imbalances in $(0, b)$-tournaments

A $(0, b, n)$-tournament is a digraph in which multiarcs multiarcs are permitted, and which has no loops [37].

At first we consider the special case $b = 0$, then the $(0, b, n)$-tournaments.

25.6.1. Imbalances in $(0,1)$-tournaments.

A $(0,1,n)$-tournament is a directed graph (shortly digraph) without loops and without multiple arcs, is also called ***simple digraph*** [37]. The ***imbalance*** of a vertex v_i in a digraph is b_{v_i} (or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The ***imbalance sequence*** of a simple digraph is formed by listing the vertex imbalances in nonincreasing order. A sequence of integers $F = [f_1, f_2, \ldots, f_n]$ with $f_1 \geq f_2 \geq \ldots \geq f_n$ is ***feasible*** if the sum of its elements is zero, and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

The following result provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 25.14 (Mubayi, Will, West [84]) *A sequence is realizable as an imbalance sequence of a $(0,1,n)$-tournament if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, \ldots, b_n]$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ is an imbalance sequence of a $(0,1,n)$-tournament if and only if

$$\sum_{i=1}^{k} b_i \leq k(n-k), \tag{25.26}$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in nondecreasing order, we have the following observation.

Corollary 25.15 *A sequence of integers $B = [b_1, \ldots, b_n]$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of a $(0,1,n)$-tournament if and only if*

$$\sum_{i=1}^{k} b_i \geq k(k-n),$$

for $1 \leq k < n$, with equality when $k = n$.

Various results for imbalances in different tournaments can be found in [49, 51, 99, 100].

25.6.2. Imbalances in $(0,2)$-tournaments

A $(0,b,n)$-tournament is a digraph in which multiarcs are permitted, and which has no loops [37]. If $b \geq 2$ then a $(0,b,n)$-tournament is an orientation of a simple multigraph and contains at most b edges between the elements of any pair of distinct vertices. Let T be a $(0,b,n)$-tournament with vertex set $V = \{v_1, \ldots, v_n\}$, and let

d_v^+ and d_v^- respectively denote the outdegree and indegree of vertex v. Define b_{v_i} (or simply b_i) $= d_{v_i}^+ - d_{u_i}^-$ as imbalance of v_i. Clearly, $-r(n-1) \leq b_{v_i} \leq r(n-1)$. The imbalance sequence of D is formed by listing the vertex imbalances in nondecreasing order.

We remark that $(0, b, n)$-digraphs are special cases of (a, b)-digraphs containing at least a and at most b edges between the elements of any pair of vertices. Degree sequences of $(0, b, n)$-tournaments have been studied by Mubayi, West, Will [84] and Pirzada, Naikoo and Shah [99].

Let u and v be distinct vertices in T. If there are f arcs directed from u to v and g arcs directed from v to u, then we denote this by $u(f-g)v$, where $0 \leq f, g, f+g \leq r$.

A double in T is an induced directed subgraph with two vertices u, and v having the form $u(f_1 - f_2)v$, where $1 \leq f_1, f_2 \leq r$, and $1 \leq f_1 + f_2 \leq r$, and f_1 is the number of arcs directed from u to v, and f_2 is the number of arcs directed from v to u. A triple in D is an induced subgraph with tree vertices u, v, and w having the form $u(f_1 - f_2)v(g_1 - g_2)w(h_1 - h_2)u$, where $1 \leq f_1, f_2, g_1, g_2, h_1, h_2 \leq r$, and $1 \leq f_1 + f_2, g_1 + g_2, h_1 + h_2 \leq b$, and the meaning of f_1, f_2, g_1, g_2, h_1, h_2 is similar to the meaning in the definition of doubles. An oriented triple in D is an induced subdigraph with three vertices. An oriented triple is said to be transitive if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. An r-graph is said to be transitive if all its oriented triples are transitive. In particular, a triple C in an r-graph is transitive if every oriented triple of C is transitive.

The following observation can be easily established and is analogue to Theorem 2.2 of Avery [3].

Lemma 25.16 (Avery 1991 [3]) *If T_1 and T_2 are two $(0, b, n)$-tournaments with same imbalance sequence, then T_1 can be transformed to T_2 by successively transforming (i) appropriate oriented triples in one of the following ways, either (a) by changing the intransitive oriented triple $u(1-0)v(1-0)w(1-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$, which has the same imbalance sequence or vice versa, or (b) by changing the intransitive oriented triple $u(1-0)v(1-0)w(0-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$, which has the same imbalance sequence or vice versa; or (ii) by changing a double $u(1-1)v$ to a double $u(0-0)v$, which has the same imbalance sequence or vice versa.*

The above observations lead to the following result.

Theorem 25.17 (Pirzada, Naikoo, Samee, Iványi 2010 [100]) *Among all $(0, b, n)$-tournaments with given imbalance sequence, those with the fewest arcs are transitive.*

Proof Let $\mathbf{b}$ be an imbalance sequence and let T be a realization of $\mathbf{b}$ that is not transitive. Then T contains an intransitive oriented triple. If it is of the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation *i(a)* of Lemma 25.16 to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$ with the same imbalance sequence and three arcs fewer. If T contains an intransitive oriented triple of the form

$u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation *i(b)* of Lemma 25.16 to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$ same imbalance sequence but one arc fewer. In case T contains both types of intransitive oriented triples, they can be transformed to transitive ones with certainly lesser arcs. If in T there is a double $u(1-1)v$, by operation *(ii)* of Lemma 25.16, it can be transformed to $u(0-0)v$, with same imbalance sequence but two arcs fewer. ■

The next result gives necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of some $(0, b, n)$-tournament.

Theorem 25.18 (Pirzada, Naiko, Samee, Iványi [100]) *A nondecreasing sequence* $\mathbf{b} = (b_1, \ldots, b_n)$ *of integers is an imbalance sequence of a* $(0, b, n)$*-tournament if and only if*

$$\sum_{i=1}^{k} b_i \geq bk(k-n), \tag{25.27}$$

with equality when $k = n$.

Proof **Necessity.** A subtournament induced by k vertices has a sum of imbalances at least $bk(k-n)$.

Sufficiency. Assume that $\mathbf{b} = (b_1, \ldots, b_n)$ is a nonincreasing sequence of integers satisfying conditions (25.27) but is not the imbalance sequence of any $(0, b, n)$-tournament. Let this sequence be chosen in such a way that n is the smallest possible and b_1 is the least with that choice of n. We consider the following two cases.

Case (i). Suppose equality in (25.27) holds for some $k \leq n$, so that

$$\sum_{i=1}^{k} b_i = bk(k-n), \tag{25.28}$$

for $1 \leq k < n$.

By minimality of n, $B_1 = (b_1, \ldots, b_k)$ is the imbalance sequence of some $(0, b, n)$-tournament T_1 with vertex set, say V_1. Let $\mathbf{b}_2 = (b_{k+1}, b_{k+2}, \ldots, b_n)$. Consider

$$\begin{aligned} \sum_{i=1}^{f} b_{k+i} &= \sum_{i=1}^{k+f} b_i - \sum_{i=1}^{k} b_i \\ &\geq b(k+f)[(k+f)-n] - bk(k-n) \\ &= b(k_2 + kf - kn + fk + f_2 - fn - k_2 + kn) \\ &\geq r(f_2 - fn) \\ &= rf(f-n), \end{aligned} \tag{25.29}$$

for $1 \leq f \leq n-k$, with equality when $f = n-k$. Therefore, by the minimality for n, the sequence $\mathbf{b}_2$ forms the imbalance sequence of some $(0, b, n)$-tournament T_2 with vertex set, say V_2. Construct a new $(0, b, n)$-tournament T with vertex set as follows.

Let $V = V_1 \cup V_2$ with, $V_1 \cap V_2 = \emptyset$ and the arc set containing those arcs which are in T_1 and T_2. Then we obtain the $(0, b, n)$-tournament T with the imbalance sequence $\mathbf{b}$, which is a contradiction.

Case (ii). Suppose that the strict inequality holds in (25.27) for some $k < n$, so that

$$\sum_{i=1}^{k} q_i > bk(k-n), \tag{25.30}$$

for $1 \leq k < n$. Let $\mathbf{b}_1 = (q_1 - 1, q_2, \ldots, q_{n-1}, q_n + 1]$, so that $\mathbf{b}_1$ satisfies the conditions (25.27). Thus by the minimality of b_1, the sequences $\mathbf{b}_1$ is the imbalances sequence of some $(0, b, n)$-tournament T_1 with vertex set, say V_1). Let $b_{v_1} = b_1 - 1$ and $b_{v_n} = a_n + 1$. Since $b_{v_n} > b_{v_1} + 1$, there exists a vertex $v_p \in V_1$ such that $v_n(0-0)v_p(1-0)v_1$, or $v_n(1-0)v_p(0-0)v_1$, or $v_n(1-0)v_p(1-0)v_1$, or $v_n(0-0)v_p(0-0)v_1$, and if these are changed to $v_n(0-1)v_p(0-0)v_1$, or $v_n(0-0)v_p(0-1)v_1$, or $v_n(0-0)v_p(0-0)v_1$, or $v_n(0-1)v_p(0-1)v_1$ respectively, the result is a $(0, b, n)$-tournament with imbalances sequence $\mathbf{b}$, which is again a contradiction. This proves the result. ■

Arranging the imbalance sequence in nonincreasing order, we have the following observation.

Corollary 25.19 (Pirzada, Naiko, Samee, Iványi [100]) *A nondecreasing sequence $\mathbf{q} = (q_1, \ldots, q_n)$ of integers is an imbalance sequence of a $(0, b, n)$-tournament if and only if*

$$\sum_{i=1}^{k} q_i \leq bk(n-k),$$

for $1 \leq k \leq n$, with equality when $k = n$.

The ***converse of a $(0, b, n)$-tournament*** T is a $(0, b, n)$-graph T', obtained by reversing orientations of all arcs of T. If $\mathbf{q} = (q_1, \ldots, q_n]$ with $q_1 \leq 2_{\leq} \ldots \leq b_n$ is the imbalance sequence of a $(0, b, n)$-tournament T, then $\mathbf{q}' = (-q_n, -q_{n-1}, \ldots, -q_1]$ is the imbalance sequence of T'.

The next result gives lower and upper bounds for the imbalance b_i of a vertex v_i in a $(0, b, n)$-tournament T.

Theorem 25.20 *If $\mathbf{q} = (q_1, \ldots, b_n)$ is an imbalance sequence of a $(0, b, n)$-tournament T, then for each i*

$$b(i-n) \leq q_i \leq b(i-1). \tag{25.31}$$

Proof Assume to the contrary that $q_i < b(i-n)$, so that for $k < i$,

$$q_k \leq q_i < b(i-n). \tag{25.32}$$

That is,

$$q_1 < b(i-n), q_2 < b(i-n), \ldots, b_i < b(i-n). \tag{25.33}$$

Adding these inequalities, we get

$$\sum_{k=1}^{i} q_k < bi(i-n), \tag{25.34}$$

which contradicts Theorem 25.18.

Therefore, $b(i-n) \le q_i$.

The second inequality is dual to the first. In the converse $(0, b, n)$-tournament with imbalance sequence $\mathbf{q} = (q_1', q_2', \ldots, q_n')$ we have, by the first inequality

$$\begin{aligned} q_{n-i+1}' &\ge b[(n-i+1)-n] \\ &= b(-i+1). \end{aligned} \tag{25.35}$$

Since $b_i = -b_{n-i+1}'$, therefore

$$q_i \le -b(-i+1) = b(i-1).$$

Hence, $q_i \le b(i-1)$. ■

Now we obtain the following inequalities for imbalances in $(0, b, n)$-tournament.

Theorem 25.21 (Pirzada, Naikoo, Samee, Iványi 2010 [100]) *If* $\mathbf{q} = (q_1, \ldots, q_n)$ *is an imbalance sequence of a* $(0, b, n)$*-tournament with* $q_1 \ge q_2 \ge \ldots \ge q_n$*, then*

$$\sum_{i=1}^{k} q_i^2 \le \sum_{i=1}^{k} (2bn - 2bk - q_i)^2, \tag{25.36}$$

for $1 \le k \le n$*, with equality when* $k = n$*.*

Proof By Theorem 25.18, we have for $1 \le k \le n$, with equality when $k = n$

$$bk(n-k) \ge \sum_{i=1}^{k} q_i, \tag{25.37}$$

implying

$$\sum_{i=1}^{k} q_i^2 + 2(2bn-2bk)bk(n-k) \ge \sum_{i=1}^{k} b_i^2 + 2(2bn-2bk)\sum_{i=1}^{k} q_i,$$

from where

$$\sum_{i=1}^{k} q_i^2 + k(2bn-2bk)^2 - 2(2bn-2bk)\sum_{i=1}^{k} q_i \ge \sum_{i=1}^{k} q_i^2, \tag{25.38}$$

and so we get the required

$$\begin{aligned} q_1^2 + q_2^2 + \ldots + q_k^2 + (2bn-2bk)^2 + (2bn-2bk)^2 + \ldots + (2bn-2bk)^2 \\ - 2(2bn-2bk)q_1 - 2(2bn-2bk)b_2 - \ldots - 2(2bn-2bk)b_k \\ \geq \sum_{i=1}^{k} q_i^2, \end{aligned} \tag{25.39}$$

or

$$\sum_{i=1}^{k}(2bn-2bk-q_i)^2 \geq \sum_{i=1}^{k} q_i^2. \tag{25.40}$$

■

25.7. Supertournaments

In Section 25.1 we defined the $(\mathbf{a}, \mathbf{b}, \mathbf{k}, m, n)$-supertournaments.

Now at first we present some results on the special case $m = 1$, that is on the hypertournaments.

25.7.1. Hypertournaments

Hypergraphs are generalizations of graphs [11, 12]. While edges of a graph are pairs of vertices of the graph, edges of a hypergraph are subsets of the vertex set, consisting of at least two vertices. An edge consisting of k vertices is called a k-edge. A k-hypergraph is a hypergraph all of whose edges are k-edges. A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. Instead of scores of vertices in a tournament, Zhou et al. [143] considered scores and losing scores of vertices in a k-hypertournament, and derived a result analogous to Landau's theorem [72]. The score $s(v_i)$ or s_i of a vertex v_i is the number of arcs containing v_i and in which v_i is not the last element, and the losing score $r(v_i)$ or r_i of a vertex v_i is the number of arcs containing v_i and in which v_i is the last element. The score sequence (losing score sequence) is formed by listing the scores (losing scores) in non-decreasing order.

The following characterizations of score sequences and losing score sequences in k-hypertournaments can be found in G. Zhou et al. [144].

Theorem 25.22 (Zhou, Yang, Zhao [144]) *Given two non-negative integers n and k with $n \geq k > 1$, a nondecreasing sequence $\mathbf{q} = [q_1, \ldots, q_n]$ of nonnegative integers is a losing score sequence of some k-hypertournament if and only if for each j,*

$$\sum_{i=1}^{j} q_i \geq \binom{j}{k}, \tag{25.41}$$

with equality when $j = n$.

Proof See [144]. ■

Theorem 25.23 (Zhou, Yang, Zhao [144]) *Given two positive integers* n *and* k *with* $n \geq k > 1$*, a nondecreasing sequence* $\mathbf{q} = (q_1, \ldots, q_n)$ *of nonnegative integers is a score sequence of some* $(0, \infty, k, n)k$*-hypertournament if and only if for each* j*,*

$$\sum_{i=1}^{j} q_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}, \tag{25.42}$$

with equality when $j = n$.

Proof See [144]. ■

Some more results on k-hypertournaments can be found in [17, 69, 96, 97, 143]. The analogous results of Theorem 25.22 and Theorem 25.23 for $[h, k]$-bipartite hypertournaments can be found in [95] and for $[\alpha, \beta, \gamma]$-tripartite hypertournaments can be found in [101].

Throughout this subsection i takes values from 1 to k and j_i takes values from 1 to n_i, unless otherwise is stated.

A ***k-partite hypertournament*** is a generalization of k-partite graphs (and k-partite tournaments). Given non-negative integers n_i and α_i, $(i = 1, 2, \ldots, k)$ with $n_i \geq \alpha_i \geq 1$ for each i, an $[\alpha_1, \alpha_2, \ldots, \alpha_k]$-$k$-partite hypertournament (or briefly k-partite hypertournament) M of order $\sum_1^k n_i$ consists of k vertex sets U_i with $|U_i| = n_i$ for each i, $(1 \leq i \leq k)$ together with an arc set E, a set of $\sum_1^k \alpha_i$ tuples of vertices, with exactly α_i vertices from U_i, called arcs such that any $\sum_1^k \alpha_i$ subset $\cup_1^k U_i'$ of $\cup_1^k U_i$, E contains exactly one of the $\left(\sum_1^k \alpha_i\right)$ $\sum_1^k \alpha_i$-tuples whose α_i entries belong to U_i'.

Let $e = (u_{11}, u_{12}, \ldots, u_{1\alpha_1}, u_{21}, u_{22}, \ldots, u_{2\alpha_2}, \ldots, u_{k1}, u_{k2}, \ldots, u_{k\alpha_k})$, with $u_{ij_i} \in U_i$ for each i, $(1 \leq i \leq k, 1 \leq j_i \leq \alpha_i)$, be an arc in M and let $h < t$, we let $e(u_{1h}, u_{1t})$ denote to be the new arc obtained from e by interchanging u_{1h} and u_{1t} in e. An arc containing α_i vertices from U_i for each i, $(1 \leq i \leq k)$ is called an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-arc.

For a given vertex $u_{ij_i} \in U_i$ for each i, $1 \leq i \leq k$ and $1 \leq j_i \leq \alpha_i$, the score $d_M^+(u_{ij_i})$ (or simply $d^+(u_{ij_i})$) is the number of $\sum_1^k \alpha_i$-arcs containing u_{ij_i} and in which u_{ij_i} is not the last element. The losing score $d_M^-(u_{ij_i})$ (or simply $d^-(u_{ij_i})$) is the number of $\sum_1^k \alpha_i$-arcs containing u_{ij_i} and in which u_{ij_i} is the last element. By arranging the losing scores of each vertex set U_i separately in non-decreasing order, we get k lists called losing score lists of M and these are denoted by $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$ for each i, $(1 \leq i \leq k)$. Similarly, by arranging the score lists of each vertex set U_i

separately in non-decreasing order, we get k lists called score lists of M which are denoted as $S_i = [s_{ij_i}]_{j_i=1}^{n_i}$ for each i $(1 \le i \le k)$.

The following two theorems are the main results of this subsection.

Theorem 25.24 (Pirzada, Zhou, Iványi [106, Theorem 3]) *Given k nonnegative integers n_i and k nonnegative integers α_i with $1 \le \alpha_i \le n_i$ for each i $(1 \le i \le k)$, the k nondecreasing lists $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$ of nonnegative integers are the losing score lists of a k-partite hypertournament if and only if for each p_i $(1 \le i \le k)$ with $p_i \le n_i$,*

$$\sum_{i=1}^{k} \sum_{j_i=1}^{p_i} r_{ij_i} \ge \prod_{i=1}^{k} \binom{p_i}{\alpha_i}, \tag{25.43}$$

with equality when $p_i = n_i$ for each i $(1 \le i \le k)$.

Theorem 25.25 (Pirzada, Zhou, Iványi [Theorem 4][106]) *Given k nonnegative integers n_i and k nonnegative integers α_i with $1 \le \alpha_i \le n_i$ for each i $(1 \le i \le k)$, the k non-decreasing lists $S_i = [s_{ij_i}]_{j_i=1}^{n_i}$ of non-negative integers are the score lists of a k-partite hypertournament if and only if for each p_i, $(1 \le i \le k)$ with $p_i \le n_i$*

$$\sum_{i=1}^{k} \sum_{j_i=1}^{p_i} s_{ij_i} \ge \left(\sum_{i=1}^{k} \frac{\alpha_i p_i}{n_i}\right) \left(\prod_{i=1}^{k} \binom{n_i}{\alpha_i}\right) + \prod_{i=1}^{k} \binom{n_i - p_i}{\alpha_i} - \prod_{i=1}^{k} \binom{n_i}{\alpha_i}, \tag{25.44}$$

with equality, when $p_i = n_i$ for each i $(1 \le i \le k)$.

We note that in a k-partite hypertournament M, there are exactly $\prod_{i=1}^{k} \binom{n_i}{\alpha_i}$ arcs and in each arc only one vertex is at the last entry. Therefore,

$$\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} d_M^-(u_{ij_i}) = \prod_{i=1}^{k} \binom{n_i}{\alpha_i}.$$

In order to prove the above two theorems, we need the following Lemmas.

Lemma 25.26 (Pirzada, Zhou, Iványi [Lemma 5][106]) *If M is a k-partite hypertournament of order $\sum_1^k n_i$ with score lists $S_i = [s_{ij_i}]_{j_i=1}^{n_i}$ for each i $(1 \le i \le k)$, then*

$$\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} s_{ij_i} = \left[\left(\sum_{1=1}^{k} \alpha_i\right) - 1\right] \prod_{i=1}^{k} \binom{n_i}{\alpha_i}. \tag{25.45}$$

Proof We have $n_i \ge \alpha_i$ for each i $(1 \le i \le k)$. If r_{ij_i} is the losing score of $u_{ij_i} \in U_i$, then

$$\sum_{i=1}^{k} \sum_{j_i=1}^{n_i} r_{ij_i} = \prod_{i=1}^{k} \binom{n_i}{\alpha_i}. \tag{25.46}$$

The number of $[\alpha_i]_1^k$ arcs containing $u_{ij_i} \in U_i$ for each i, $(1 \le i \le k)$, and $1 \le j_i \le n_i$ is

$$\frac{\alpha_i}{n_i} \prod_{t=1}^{k} \binom{n_t}{\alpha_t}. \tag{25.47}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j_i=1}^{n_i} s_{ij_i} &= \sum_{i=1}^{k}\sum_{j_i=1}^{n_i} \left(\frac{\alpha_i}{n_i}\right)\prod_{1}^{k}\binom{n_t}{\alpha_t} - \binom{n_i}{\alpha_i} \\ &= \left(\sum_{i=1}^{k}\alpha_i\right)\prod_{1}^{k}\binom{n_t}{\alpha_t} - \prod_{1}^{k}\binom{n_i}{\alpha_i} \\ &= \left[\left(\sum_{1=1}^{k}\alpha_i\right) - 1\right]\prod_{1}^{k}\binom{n_i}{\alpha_i}.\end{aligned} \tag{25.48}$$

■

Lemma 25.27 (Pirzada, Zhou, Iványi [Lemma 6][106]) *If $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$ $(1 \le i \le k)$ are k losing score lists of a k-partite hypertournament M, then there exists some h with*

$$r_{1h} < \frac{\alpha_1}{n_1}\prod_{1}^{k}\binom{n_p}{\alpha_p} \tag{25.49}$$

so that $R_1' = [r_{11}, r_{12}, \ldots, r_{1h} + 1, \ldots, r_{1n_1}]$, $R_s' = [r_{s1}, r_{s2}, \ldots, r_{st} - 1, \ldots, r_{sn_s}]$ $(2 \le s \le k)$ and $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$, $(2 \le i \le k)$, $i \ne s$ are losing score lists of some k-partite hypertournament, t is the largest integer such that $r_{s(t-1)} < r_{st} = \ldots = r_{sn_s}$.

Proof Let $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$ $(1 \le i \le k)$ be losing score lists of a k-partite hypertournament M with vertex sets $U_i = \{u_{i1}, u_{i2}, \ldots, u_{ij_i}\}$ so that $d^-(u_{ij_i}) = r_{ij_i}$ for each i $(1 \le i \le k,\ 1 \le j_i \le n_i)$.

Let h be the smallest integer such that

$$r_{11} = r_{12} = \ldots = r_{1h} < r_{1(h+1)} \le \ldots \le r_{1n_1}$$

and t be the largest integer such that

$$r_{s1} \le r_{s2} \le \ldots \le r_{s(t-1)} < r_{st} = \ldots = r_{sn_s}$$

Now, let

$$R_1' = [r_{11}, r_{12}, \ldots, r_{1h} + 1, \ldots, r_{1n_1}],$$

$$R_s' = [r_{s1}, r_{s2}, \ldots, r_{st} - 1, \ldots, r_{sn_s}$$

$(2 \le s \le k)$, and $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$, $(2 \le i \le k)$, $i \ne s$.

Clearly, R_1' and R_s' are both in non-decreasing order.

Since $r_{1h} < \frac{\alpha_1}{n_1}\prod_1^k \begin{pmatrix} n_p \\ \alpha_p \end{pmatrix}$, there is at least one $[\alpha_i]_1^k$-arc e containing both u_{1h} and u_{st} with u_{st} as the last element in e, let $e' = (u_{1h}, u_{st})$. Clearly, R_1', R_s' and $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$ for each i $(2 \leq i \leq k)$, $i \neq s$ are the k losing score lists of $M' = (M - e) \cup e'$. ■

The next observation follows from Lemma **??**, and the proof can be easily established.

Lemma 25.28 (Pirzada, Zhou, Iványi [Lemma 7][106]) *Let $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$, $(1 \leq i \leq k)$ be k nondecreasing sequences of nonnegative integers satisfying (**??**. If $r_{1n_1} < \frac{\alpha_1}{n_1}\prod_1^k \begin{pmatrix} n_t \\ \alpha_t \end{pmatrix}$, then there exist s and t $(2 \leq s \leq k)$, $1 \leq t \leq n_s$ such that $R_1' = [r_{11}, r_{12}, \ldots, r_{1h}+1, \ldots, r_{1n_1}]$, $R_s' = [r_{s1}, r_{s2}, \ldots, r_{st}-1, \ldots, r_{sn_s}]$ and $R_i = [r_{ij_i}]_{j_i=1}^{n_i}$, $(2 \leq i \leq k)$, $i \neq s$ satisfy (25.43).*

Proof **of Theorem 25.24. Necessity.** Let R_i, $(1 \leq i \leq k)$ be the k losing score lists of a k-partite hypertournament $M(U_i, 1 \leq i \leq k)$. For any p_i with $\alpha_i \leq p_i \leq n_i$, let $U_i' = \{u_{ij_i}\}_{j_i=1}^{p_i} (1 \leq i \leq k)$ be the sets of vertices such that $d^-(u_{ij_i}) = r_{ij_i}$ for each $1 \leq j_i \leq p_i$, $1 \leq i \leq k$. Let M' be the k-partite hypertournament formed by U_i' for each i $(1 \leq i \leq k)$.

Then,

$$\sum_{i=1}^{k}\sum_{j_i=1}^{p_i} r_{ij_i} \geq \sum_{i=1}^{k}\sum_{j_i=1}^{p_i} d_{M'}^-(u_{ij_i}) = \prod_1^k \begin{pmatrix} p_t \\ \alpha_t \end{pmatrix}. \tag{25.50}$$

Sufficiency. We induct on n_1, keeping $n_2, \ldots, n_k$ fixed. For $n_1 = \alpha_1$, the result is obviously true. So, let $n_1 > \alpha_1$, and similarly $n_2 > \alpha_2, \ldots, n_k > \alpha_k$. Now,

$$
\begin{aligned}
r_{1n_1} &= \sum_{i=1}^{k}\sum_{j_i=1}^{n_i} r_{ij_i} - \left(\sum_{j_1=1}^{n_1-1} r_{1j_1} + \sum_{i=2}^{k}\sum_{j_i=1}^{n_i} r_{ij_i}\right) \\
&\leq \prod_1^k \binom{n_t}{\alpha_t} - \binom{n_1-1}{\alpha_1} \prod_2^k \binom{n_t}{\alpha_t} \\
&= \left[\binom{n_1}{\alpha_1} - \binom{n_1-1}{\alpha_1}\right] \prod_2^k \binom{n_t}{\alpha_t} \\
&= \binom{n_1-1}{\alpha_1-1} \prod_2^k \binom{n_t}{\alpha_t} .
\end{aligned}
\tag{25.51}
$$

We consider the following two cases.

Case 1. $r_{1n_1} = \binom{n_1-1}{\alpha_1-1} \prod_2^k \binom{n_t}{\alpha_t}$. Then,

$$
\begin{aligned}
\sum_{j_1=1}^{n_1-1} r_{1j_1} + \sum_{i=2}^{k}\sum_{j_i=1}^{n_i} r_{ij_i} &= \sum_{i=1}^{k}\sum_{j_i=1}^{n_i} r_{ij_i} - r_{1n_1} \\
&= \prod_1^k \binom{n_t}{\alpha_t} - \binom{n_1-1}{\alpha_1-1} \prod_2^k \binom{n_t}{\alpha_t} \\
&= \left[\binom{n_1}{\alpha_1} - \binom{n_1-1}{\alpha_1-1}\right] \prod_2^k \binom{n_t}{\alpha_t} \\
&= \binom{n_1-1}{\alpha_1} \prod_2^k \binom{n_t}{\alpha_t} .
\end{aligned}
\tag{25.52}
$$

By induction hypothesis $[r_{11}, r_{12}, \ldots, r_{1(n_1-1)}]$, $R_2, \ldots, R_k$ are losing score lists of a k-partite hypertournament $M'(U'_1, U_2, \ldots, U_k)$ of order $\left(\sum_{i=1}^k n_i\right) - 1$. Construct a k-partite hypertournament M of order $\sum_{i=1}^k n_i$ as follows. In M', let $U'_1 = \{u_{11}, u_{12}, \ldots, u_{1(n_1-1)}\}, U_i = \{u_{ij_i}\}_{j_i=1}^{n_i}$ for each i, $(2 \leq i \leq k)$. Adding a new vertex u_{1n_1} to U'_1, for each $\left(\sum_{i=1}^k \alpha_i\right)$-tuple containing u_{1n_1}, arrange u_{1n_1} on the last entry. Denote E_1 to be the set of all these $\binom{n_1-1}{\alpha_1-1} \prod_2^k \binom{n_t}{\alpha_t}$ $\left(\sum_{i=1}^k \alpha_i\right)$-tuples. Let $E(M) = E(M') \cup E_1$. Clearly, R_i for each i, $(1 \leq i \leq k)$ are the k losing score lists of M.

Case 2. $r_{1n_1} < \left(\begin{array}{c} n_1 - 1 \\ \alpha_1 - 1 \end{array} \right) \prod_2^k \left(\begin{array}{c} n_t \\ \alpha_t \end{array} \right)$.

Applying Lemma 25.26 repeatedly on R_1 and keeping each R_i, $(2 \leq i \leq k)$ fixed until we get a new non-decreasing list $R_1' = [r'_{11}, r'_{12}, \ldots, r'_{1n_1}]$ in which now $'_{1n_1} = \left(\begin{array}{c} n_1 - 1 \\ \alpha_1 - 1 \end{array} \right) \prod_2^k \left(\begin{array}{c} n_t \\ \alpha_t \end{array} \right)$. By Case 1, R_1', R_i $(2 \leq i \leq k)$ are the losing score lists of a k-partite hypertournament. Now, apply Lemma 25.26 on R_1', R_i $(2 \leq i \leq k)$ repeatedly until we obtain the initial non-decreasing lists R_i for each i $(1 \leq i \leq k)$. Then by Lemma 25.27, R_i for each i $(1 \leq i \leq k)$ are the losing score lists of a k-partite hypertournament. ■

Proof of Theorem 25.25. Let $S_i = [s_{ij_i}]_{j_i=1}^{n_i} (1 \leq i \leq k)$ be the k score lists of a k-partite hypertournament $M(U_i, 1 \leq i \leq k)$, where $U_i = \{u_{ij_i}\}_{j_i=1}^{n_i}$ with $d_M^+(u_{ij_i}) = s_{ij_i}$, for each i, $(1 \leq i \leq k)$.

Clearly,

$d^+(u_{ij_i}) + d^-(u_{ij_i}) = \frac{\alpha_i}{n_i} \prod_1^k \left(\begin{array}{c} n_t \\ \alpha_t \end{array} \right)$, $(1 \leq i \leq k, 1 \leq j_i \leq n_i)$.

Let $r_{i(n_i+1-j_i)} = \mathrm{d}^-(u_{ij_i})$, $(1 \leq i \leq k, 1 \leq j_i \leq n_i)$.

Then $R_i = [r_{ij_i}]_{j_i=1}^{n_i} (i = 1, 2, \ldots, k)$ are the k losing score lists of M. Conversely, if R_i for each i $(1 \leq i \leq k)$ are the losing score lists of M, then S_i for each i, $(1 \leq i \leq k)$ are the score lists of M. Thus, it is enough to show that conditions (25.43) and (25.44) are equivalent provided

$$s_{ij_i} + r_{i(n_i+1-j_i)} = \left(\frac{\alpha_i}{n_i} \right) \prod_1^k \left(\begin{array}{c} n_t \\ \alpha_t \end{array} \right), \qquad (25.53)$$

for each i $(1 \leq i \leq k$ and $1 \leq j_i \leq n_i)$.

First assume (25.44) holds. Then,

$$\begin{aligned}
\sum_{i=1}^{k}\sum_{j_i=1}^{p_i} r_{ij_i} &= \sum_{i=1}^{k}\sum_{j_i=1}^{p_i}\left(\frac{\alpha_i}{n_i}\right)\left(\prod_{1}^{k}\binom{n_t}{\alpha_t}\right) - \sum_{i=1}^{k}\sum_{j_i=1}^{p_i} s_{i(n_i+1-j_i)} \\
&= \sum_{i=1}^{k}\sum_{j_i=1}^{p_i}\left(\frac{\alpha_i}{n_i}\right)\left(\prod_{1}^{k}\binom{n_t}{\alpha_t}\right) - \left[\sum_{i=1}^{k}\sum_{j_i=1}^{n_i} r_{ij_i} - \sum_{i=1}^{k}\sum_{j_i=1}^{n_i-p_i} s_{ij_i}\right] \\
&\geq \left[\sum_{i=1}^{k}\sum_{j_i=1}^{p_i}\left(\frac{\alpha_i}{n_i}\right)\left(\prod_{1}^{k}\binom{n_t}{\alpha_t}\right)\right] \\
&- \left[\left(\left(\sum_{1}^{k}\alpha_i\right)-1\right)\prod_{1}^{k}\binom{n_i}{\alpha_i}\right] \\
&+ \sum_{i=1}^{k}(n_i-p_i)\left(\frac{\alpha_i}{n_i}\right)\prod_{1}^{k}\binom{n_t}{\alpha_t} \\
&+ \prod_{1}^{k}\binom{n_i-(n_i-p_i)}{\alpha_i} - \prod_{1}^{k}\binom{n_i}{\alpha_i} \\
&= \prod_{1}^{k}\binom{n_i}{\alpha_i},
\end{aligned}$$

with equality when $p_i = n_i$ for each i $(1 \leq i \leq k)$.

Thus (1) holds.

Now, when (25.43) holds, using a similar argument as above, we can show that (25.44) holds. This completes the proof. ■

25.7.2. Supertournaments

The majority of the results on hypertournaments can be extended to supertournaments.

The simplest case when all m individual tournaments have own input sequence $\mathbf{q_i} = q_{i,1}, \ldots, q_{i,n_i}$, where $n_i = \binom{n}{k_i}$. Then we can apply the necessary and sufficient conditions and algorithms of the previous sections.

If all m tournaments have a join input sequence $\mathbf{q_1}, \ldots, \mathbf{q_n}$, then all the previous necessary conditions remain valid.

25.8. Football tournaments

The football tournaments are special incomplete $(2, 3, n)$-tournaments, where the set of the permitted results is $S_{\text{football}} = \{0 : 3, 1 : 1\}$.

25.8.1. Testing algorithms

In this section we describe eight properties of football sequences. These properties serve as necessary conditions for a given sequence to be a football sequence.

Definition 25.29 *A **football tournament** F is a directed graph (on $n \geq 2$ vertices) in which the elements of every pair of vertices are connected either with 3 arcs directed in identical manner or with 2 arcs directed in different manner. A nondecreasingly ordered sequence of any F is called **football sequence***

The i-th vertex will be called i-th team and will be denoted by T_i. For the computations we represent a tournament with M, what is an $n \times n$ sized matrix, in which m_{ij} means the number of points received by T_i in the match against T_j. The elements m_{ii}, that is the elements in the main diagonal of M are equal to zero. Let's underline, that the permitted elements are 0, 1 or 3, so $|\mathcal{F}| = 3^{n(n-1)/2}$.

The vector of the outdegrees $d = (d_1, d_2, \ldots, d_n)$ of a tournament F is called score vector. Usually we suppose that the score vector is nondecreasingly sorted. The sorted score vector is called score sequence and is denoted by $f = (f_1, f_2, \ldots, f_n)$. The number of football sequences for n teams is denoted by $\phi(n)$. The values of $\phi(n)$ are known for $i = 1, \ldots, 8$ [70].

In this section at first we describe six algorithms which require $\Theta(n)$ time in worst case, then more complicate algorithms follow.

Linear time testing algorithms

In this subsection we introduce relatively simple algorithms Boundness-Test, Mono- tonity-Test, Intervallum-Test, Loss-Test, Draw-Loss-test, Victory-Test, Strong-Test, and Sport-Test.

Testing of boundness

Since every team T_i plays $n-1$ matches and receives at least 0 and at most 3 points in each match, therefore in a football sequence it holds $0 \leq f_i \leq 3(n-1)$ for $i = 1,\ 2,\ \ldots,\ n$.

Definition 25.30 *A sequence $(q_1, q_2, \ldots, q_n)$ of integers will be called n-**bounded** (shortly: bounded), iff*

$$0 \leq q_i \leq 3(n-1) \quad \text{for} \quad i = 1,\ 2,\ \ldots,\ n. \tag{25.54}$$

Lemma 25.31 (Lucz, Iványi, Sótér [74]) *Every football sequence is a bounded sequence.*

Proof The lemma is a direct consequence of Definition 25.29. ■

The following algorithm executes the corresponding test. Sorting of the elements of q is not necessary. We allow negative numbers in the input since later testing algorithm Decomposition can produce such input for Bounded.

Input. n: the number of teams ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: arbitrary sequence of integer numbers.

Output. W: a logical variable. Its value is True, if the input vector is bounded, and False otherwise.

Working variable. i: cycle variable.

```
Boundness-Test(n, q)
for i = 1 to n
    if q_i < 0 or q_i > 3(n − 1)
        W = False
        return W
W = True
return W
```

In worst case Boundness-Test runs $\Theta(n)$ time, in expected case runs in $\Theta(1)$ time. More precisely the algorithm executes n comparisons in worst case and asymptotically in average 2 comparisons in best case.

Testing of monotonity

Monotonity is also a natural property of football sequences.

Definition 25.32 *A bounded sequence of nonnegative integers $q = (q_1, q_2, \ldots, q_n)$ will be called n-**monotone** (shortly: monotone), if and only if*

$$q_1 \leq q_2 \leq \cdots \leq q_n. \qquad (25.55)$$

Lemma 25.33 (Lucz, Iványi, Sótér [74]) *Every football sequence is a monotone sequence.*

Proof This lemma also is a direct consequence of Definition 25.29. ■

The following algorithm executes the corresponding test. Sorting of the elements of q is not necessary.

Input. n: the number of players ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a bounded sequence of length n.

Output. W: a logical variable. Its value is True, if the input vector is monotone, and False otherwise.

Working variable. i: cycle variable.

```
Monotonity-Test(n, q)
for i = 1 to n − 1
    if q_i < q_{i−1}
        W = False
```

```
        return W
W = TRUE
return W
```

In worst case MONOTONITY-TEST runs $\Theta(n)$ time, in expected case runs in $\Theta(1)$ time. More precisely the algorithm executes n comparisons in worst case.

The following lemma gives the numbers of bounded and monotone sequences. Let $\mathcal{B}(n)$ denote the set of n-bounded, and $\mathcal{M}(n)$ the set of n-monotone sequences, $\beta(n)$ the size of $\mathcal{B}(n)$ and $\mu(n)$ the size of $\mathcal{M}(n)$.

Lemma 25.34 *If $n \geq 1$, then*

$$\beta(n) = (3n-2)^n \tag{25.56}$$

and

$$\mu(n) = \binom{4n-3}{n}. \tag{25.57}$$

Proof (25.56) is implied by the fact that an n-bounded sequence contains n elements and these elements have $3n-2$ different possible values.

To show (25.57) let $m = (m_1, m_2, \ldots, m_n)$ be a monotone sequence and let $m' = (m'_1, m'_2, \ldots, m'_n)$, where $m'_i = m_i + i - 1$. Then $0 \leq m'_1 < m'_2 < \cdots < m'_n < 4n-4$. The mapping $m \to m'$ is a bijection and so $\mu(n)$ equals to the number of ways of choosing n numbers from $4n-3$, resulting (25.57). ■

Testing of the intervallum property

The following definition exploits the basic idea of Landau's theorem [72].

Definition 25.35 *A monotone nonincreasing sequence $q = (q_1, q_2, \ldots, q_n)$ is called* ***intervallum type*** *(shortly: intervallum), if and only if*

$$2\binom{k}{2} \leq \sum_{i=1}^{k} q_i \leq 3\binom{n}{2} - (n-i)q_i \quad (k = 1, 2, \ldots, n). \tag{25.58}$$

Lemma 25.36 *Every football sequence is intervallum sequence.*

Proof The left inequality follows from the fact, that the teams $T_1, T_2, \ldots, T_k$ play $\binom{k}{2}$ matches and they get together at least two points in each matches.

The right inequality follows from the monotonity of m and from the fact, that the teams play $\binom{n}{2}$ matches and get at most 3 points in each match. ■

The following algorithm INTERVALLUM-TEST tests whether a monotone input is intervallum type.

Input. n: the number of teams ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a bounded sequence of length n.

Output. W: a logical variable. Its value is TRUE, if the input vector is intervallum type, and FALSE otherwise.

Working variables. i: cycle variable;

$B_k = \binom{n}{k}$ $(k = 0,\ 1,\ 2,\ \ldots,\ n)$: binomial coefficients;

$S_0 = 0$: initial value for the sum of the input data;

$S_k = \sum_{i=1}^{k} q_i$ $(k = 1,\ 2,\ \ldots,\ n)$: the sum of the smallest k input data.

We consider $B = (B_0, B_1, \ldots, B_n)$ and $S = (S_0, S_1, \ldots, S_n)$ as global variables, and therefore they are used later without new calculations. The number of n-intervallum sequences will be denoted by $\gamma(n)$.

INTERVALLUM-TEST(n, q)

```
B_0 = S_0 = 0
for i = 1 to n
    B_i = B_{i-1} + i - 1
    S_i = S_{i-1} + q_i
    if 2B_i > S_i or S_i > 3B_n - (n - i)q_i
    W = FALSE
        return W
W = TRUE
return W
```

In worst case INTERVALLUM-TEST runs $\Theta(n)$ time. More precisely the algorithm executes $2n$ comparisons, $2n$ additions, $2n$ extractions, n multiplications and 2 assignments in worst case. The number of the n-intervallum sequences will be denoted by $\gamma(n)$.

Testing of the loss property

The following test is based on Theorem 3 of [49, page 86]. The basis idea behind the theorem is the observation that if the sum of the k smallest scores is less than $3\binom{k}{2}$, then the teams $\mathrm{T}_1,\ \mathrm{T}_2,\ \ldots,\ \mathrm{T}_k$ have lost at least $3\binom{k}{2} - S_k$ points in the matches among others. Let $L_0 = 0$ and $L_k = \max(L_{k-1}, 3\binom{n}{2} - S_k)$ $(k = 1,\ 2,\ \ldots,\ n)$.

Definition 25.37 *An intervallum satisfying sequence $q = (q_1, q_2, \ldots, q_n)$ is called **loss satisfying**, iff*

$$\sum_{i=1}^{k} q_i + (n-k)q_k \le 3B_n - L_k \quad (k = 1, 2, \ldots, n). \tag{25.59}$$

Lemma 25.38 (Lucz, Iványi, Sótér [74]) *A football sequence is loss satisfying.*

Proof See the proof of Theorem 3 in [49]. ■

The following algorithm LOSS-TEST exploits Lemma 25.38.

Input. n: the number of teams ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a bounded sequence of length n.

Output. W: a logical variable. Its value is TRUE, if the input vector is Landau type, and FALSE otherwise.

Working variables. i: cycle variable;
$L = (L_0, L_1, \ldots, L_n)$: vector of the loss coefficient;
$S = (S_0, S_1, \ldots, S_n)$: sums of the input values, global variables;
$B = (B_0, B_1, \ldots, B_n)$: binomial coefficients, global variables.

```
LOSS-TEST(n, q)
L_0 = 0
for i = 1 to n
      L_i = max(L_{i-1}, 3B_i - S_i)
      if S_i + (n - i)q_i > 3B_n - L_i
            W = FALSE
            return W
W = TRUE
return W
```

In worst case LOSS-TEST runs in $\Theta(n)$ time, in best case in $\Theta(1)$ time. We remark that $L = (L_0, L_1, \ldots, L_n)$ is in the following a global variable. The number of loss satisfying sequences will be denoted by $\lambda(n)$.

Testing of the draw-loss property

In the previous subsection LOSS-TEST exploited the fact, that small scores signalize draws, allowing the improvement of the upper bound $3B_n$ of the sum of the scores.

Let us consider the loss sequence $(1, 2)$. T_1 made a draw, therefore one point is lost and so $S_2 \leq 2B_2 - 1 = 1$ must hold implying that the sequence $(1, 2)$ is not a football sequence. This example is exploited in the following definition and lemma. Let

$$L'(0) = 0 \quad \text{and} \quad L'_k = \max\left(L'_{i-1}, 3B_k - S_k, \left\lceil \frac{\sum_{i=1}^{k}(q_i - 3\lfloor q_i/3 \rfloor)}{2} \right\rceil\right). \qquad (25.60)$$

Definition 25.39 *A loss satisfying sequence* $q = (q_1, q_2, \ldots, q_n)$ *is called* ***draw loss satisfying,*** *if and only if*

$$\sum_{i=1}^{k} q_k + (n-k)q_k \leq 3B_n - L'_k \quad (k = 1, 2, \ldots, n). \qquad (25.61)$$

Lemma 25.40 (Lucz, Iványi, Sótér [74]) *A football sequence is draw loss satisfying.*

Proof The assertion follows from the fact that small scores and remainders (mod 3) of the scores both signalize lost points and so decrease the upper bound $3B_n$. ■

The following algorithm Draw-Loss-Test exploits Lemma 25.38.

Input. n: the number of teams ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a loss satisfying sequence of length n.

Output. W: a logical variable. Its value is True, if the input vector is Landau type, and False otherwise.

Working variables. i: cycle variable;
L, S: global variables;
$L' = (L'_0, L'_1, \ldots, L'_n)$: modified loss coefficients.

```
Draw-Loss-Test(n, q, W)
L'_0 = 0
for i = 1 to n
    L'_i = max(L_i), ⌈ (Σ_{i=1}^{k} (q_i − 3⌊q_i/3⌋)) / 2 ⌉
    if S_i + (n − i)q_i > 3B_n − L'_i
        W = False
        return W
W = True
return W
```

In worst case Draw-Loss-Test runs in $\Theta(n)$ time, in best case in $\Theta(1)$ time.
We remark that L' is in the following a global variable.
The number of draw loss satisfying sequences will be denoted by $\delta(n)$.

Testing of the victory property

In any football tournament $S_n - 2\binom{n}{2}$ matches end with victory and $3\binom{n}{2} - S_n$ end with draw.

Definition 25.41 *A loss satisfying (shortly: loss) sequence $q = (q_1, q_2, \ldots, q_n)$ is called* ***victory satisfying****, iff*

$$\sum_{i=1}^{n} \left\lfloor \frac{q_i}{3} \right\rfloor \geq S_n - 2\binom{n}{2} \quad (k = 1, 2, \ldots, n). \tag{25.62}$$

Lemma 25.42 (Lucz, Iványi, Sótér [74]) *A football sequence is victory satisfying.*

Proof Team T_i could win at most $\lfloor q_i/3 \rfloor$ times. The left side of (25.62) is an upper bound for the number of possible wins, therefore it has to be greater or equal then the exact number of wins in the tournament. ■

The following algorithm Victory-Test exploits Lemma 25.42.

Input. n: the number of teams ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a loss sequence of length n.

Output. W: a logical variable. Its value is True, if the input vector is Landau type, and False otherwise.

Working variables. i: cycle variable;

$V = (V_0, V_1, V_2, \ldots, V_n)$: where V_i is an upper estimation of the number of possible wins of T_1, T_2, ..., T_i.

S_n, B_n: global variables.

```
VICTORY-TEST(n, q, W)
V_0 = 0
for i = 1 to n
    V_i = V_{i-1} + ⌊q_i/3⌋
if V_n < S_n − 2B_n
    W = FALSE
    return W
W = TRUE
return W
```

VICTORY-TEST runs in $\Theta(n)$ time in all cases. The number of the victory satisfying sequences is denoted by $\nu(n)$.

VICTORY-TEST is successful e.g. for the input sequence $(1, 2)$, but until now we could not find such draw loss sequence, which is not victory sequence. The opposite assertion is also true. Maybe that the sets of victory and draw loss sequences are equivalent?

Testing of the strong draw-loss property

In paragraph "Testing of the draw-loss property" we estimated the loss caused by the draws in a simple way: supposed that every draw implies half point of loss. Especially for short sequences is useful a more precise estimation.

Let us consider the sequence $(2, 3, 3, 7)$. The sum of the remainders (mod 3) is $2 + 1 = 3$, but we have to convert to draws at least three "packs" (3 points), if we wish to pair the necessary draws, and so at least six points are lost, permitting at most $S_n = 12$.

Exploiting this observation we can sharp a bit Lemma 25.40. There are the following useful cases:

1. one small remainder (1 pont) implies the loss of $(1 + 5 \times 3)/2 = 8$ points;
2. one large remainder (2 points) implies the loss of $(2 + 4 \times 3)/2 = 5$ points;
3. one small and one large remainder imply the loss of $(1 + 2 + 3 \times 3)/2 = 6$ points;
4. two large remainders imply the loss of $(2 + 2 + 2 \times 3)/2 = 5$ points;
5. one small and two large remainders imply the loss of $(2 + 2 + 1 + 3)/2 = 4$ points.

According to this remarks let m_1 resp. m_2 denote the multiplicity of the equality $q_k = 1 \pmod 3$ resp. $q_k = 2 \pmod 3$.

Definition 25.43 *A victory satisfying sequence $q = (q_1, q_2, \ldots, q_n)$ is called* ***strong****, iff*

$$\sum_{i=1}^{k} q_k + (n-k)q_k \le 3B_n - L''_k \quad (k = 1, 2, \ldots, n). \tag{25.63}$$

Lemma 25.44 (Lucz, Iványi, Sótér [74]) *Every football sequence is strong.*

Proof The assertion follows from the fact that any point matrix of a football tournament order the draws into pairs. ■

The following algorithm Strong-Test exploits Lemma 25.8.1.

Input. n: the number of teams ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a loss satisfying sequence of length n

Output. W: a logical variable. Its value is True, if the input vector is Landau type, and False otherwise.

Working variables. i: cycle variable;
$L' = (L'_0, L'_1, \ldots, L_n)$: modified loss coefficients, global variables;
S_n: sum of the elements of the sequence q, global variable;
$L'' = (L''_0, L''_1, \ldots, L''_n)$: strongly modified loss coefficients.

```
Strong-Test(n, q, W)
m_1 = m_2 = 0
for i = 1 to n
    if q_i = 1 (mod 3)
    m_1 = m_1 + 1
    if q_i == 2 (mod 3)
    m_2 = m_2 + 1
L'' = L'
if m_1 == 1 and m_2 = 0
   L'' = max(L', 8)
if m_1 == 0 and m_2 = 1
   L'' = max(L', 5)
if m_1 == 1 and m_2 = 1
   L'' = max(L', 6)
if m_1 == 0 and m_2 = 2
   L'' = max(L', 5)
if m_1 == 1 and m_2 = 2
   L'' = max(L', 4)
if S_n < 3B_n - L''
   W = False
   return
W = True
return W
```

Strong-Test runs in all cases in $\Theta(n)$ time.

We remark that L'' is in the following a global variable.

The number of strong sequences will be denoted by $\tau(n)$.

Testing of the sport property

One of the typical form to represent a football tournament is its point matrix as it was shown in Figure 25.3.

Definition 25.45 *A victory satisfying sequence* $q = (q_1, \ldots, q_n)$ *is called* ***sport sequence*** *iff it can be transformed into a sport matrix.*

Lemma 25.46 (Lucz, Iványi, Sótér [74]) *Every football sequence is a sport sequence.*

Proof This assertion is a consequence of the definition of the football sequences. ■

If a loss sequence q can be realized as a sport matrix, then the following algorithm SPORT-TEST constructs one of the sport matrices belonging to q.

If the team T_i has q_i points, then it has at least $d_i = q_i \pmod 3$ draws, $v_i = \max(0, q_i - n + 1)$ wins and $l_i = \max(0, n - 1 - q_i)$ losses. These results are called ***obligatory wins, draws***, resp. ***losses***. SPORT-TEST starts its work with the computation of v_i, d_i and l_i. Then it tries to distribute the remaining draws.

Input. n: the number of players ($n \geq 2$);
$q = (q_1, q_2, \ldots, q_n)$: a victory satisfying sequence of length n.

Output. W: a logical variable. Its value is TRUE, if the input sequence is sport sequence, and FALSE otherwise;

Working variables. i: cycle variable;
$v,\ d,\ l$: columns of the sport matrix;
$V,\ D,\ L$: sum of the numbers of obligatory wins, draws, resp. losses;
$B_n,\ S_n$: global variables;
$S_n = \sum_{i=1}^{n} q_i$: the sum of the elements of the input sequence;
$VF,\ DF,\ LF$: the exact number of wins, draws, resp. losses.

SPORT-TEST(n, q)

$V = D = L = 0$
for $i = 1$ **to** n
    $v_i = \max(0, q_i - n + 1)$
    $V = V + v_i$
    $d_i = q_i \pmod 3$
    $D = D + d_i$
    $l_i = \max(0, n - 1 - q_i)$
    $L = L + l_i$
$DF = 3B_n - S_n$
if $D > DF$ or $2DF - D \neq 0 \pmod 3$
    $W =$ FALSE
    return W
$VF = S_n - 2B_n$
$LF = VF$
for $i = 1$ **to** n
    while $DF > 0$ or $VF > 0$ or $LF > 0$
        $x = \min(\frac{q_i - d_i - 3v_i}{3}, \lfloor \frac{3(n-1) - q_i - d_i}{6} \rfloor)$
        $d_i = d_i + 3x$
        $DF = DF - 3x$
        $v_i = \frac{q_i - d_i}{3}$

```
            VF = VF − v_i
            l_i = n − 1 − d_i − v_i
            LF = LF − l_i
            if l_i ≠ v_i
               W = FALSE
               return W
            if DF ≠ 0 or VF ≠ 0 or LF ≠ 0
               W = FALSE
               return
W = TRUE
return W
```

SPORT-TEST runs in $\Theta(n)$ time in all cases. The number of the sport sequences is denoted by $\sigma(n)$

Concrete examples

Let us consider short input sequences illustrating the power of the linear testing algorithms.

If $n = 2$, then according to Lemma 25.34 we have $\beta(2) = 4^4 = 16$ and $\mu(2) = \binom{5}{2} = 10$. The monotone sequences are $(0, 0)$, $()0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$, $)1, 2$, $(1, 3)$, $(2, 2)$, $(2, 3)$, $(3, 3)$. Among the monotone sequences there are 4 interval sequences: $(0, 2)$, $(0, 3)$, $(1, 1)$, and $(1, 2)$, so $\gamma(2) = 4$. LOSS-TEST does not help, therefore $\lambda(2) = 4$. VICTORY-TEST excludes $(1, 2)$, so $v(2) = 3$. Finally SPORT-TEST can not construct a sport matrix for $(0, 2)$ and so it concludes $\sigma(2) = 2$. After further unsuccessful tests FOOTBALL reconstructs $(0 : 3)$ and $(1, 1)$, proving $\varphi(2) = 2$.

If $n = 3$, then according to Lemma 25.34 we have $\beta(3) = 7^3 = 343$ and $\mu(3) = \binom{9}{3} = 84$. Among the 84 monotone sequence there are 27 interval sequences, and these sequences at the same time have also the loss property, so $\gamma(3) = \lambda(3) = 27$. These sequences are the following: $(0, 2, 4)$, $(0, 2, 5)$, $(0, 2, 6)$, $(0, 3, 3)$, $(0, 3, 4)$, $(0, 3, 5)$, $(0, 3, 6)$, $(0, 4, 4)$, $(0, 4, 5)$, $(1, 1, 4)$, $(1, 1, 5)$, $(1, 1, 6)$, $(1, 2, 3)$, $(1, 2, 4)$, $(1, 2, 5)$, $(1, 2, 6)$, $(1, 3, 3)$, $(1, 3, 4)$, $(1, 3, 5)$, $(1, 4, 4)$, $(2, 2, 2)$, $(2, 2, 3)$, $(2, 2, 4)$, $(2, 2, 5)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(3, 3, 3)$. From these sequences only $(0, 3, 6)$, $(0, 4, 4)$, $(1, 1, 6)$, $(1, 2, 4)$, $(1, 3, 4)$, $(2, 2, 2)$, $(3, 3, 3)$ are paired sport sequences, so $\pi(3) = 7$. The following tests are unsuccessful, but FOOTBALL reconstructs the remained seven sequences, therefore $\varphi(3) = 7$.

If $n = 4$, then according to Lemma 25.34 we have $\beta(4) = 10^4 = 10\,000$ and $\mu(4) = \binom{13}{4} = 715$. The number of paired sport sequences is $\pi(4) = 40$. We now that $\varphi(4) = 40$, so our linear algorithms evaluate the input sequences correctly up to $n = 4$.

If $ngeq5$, then

25.8.2. Polynomial testing algorithms of the draw sequences

Earlier we used a greedy approach to check whether the necessary number of draws is allocatable.

Definition 25.47 *A sequence $1 \le d_1 \le d_2 \le \cdots \le d_n \le n - 1$ is called* ***potential***

n-***draw sequence***. *The number of potential n-draw sequences is denoted by $\pi(n)$.*

Lemma 25.48 (Iványi, Lucz, Sótér [54]) *If $n \geq 1$, then $\pi(n) = \binom{2n-2}{n}$.*

Proof The proof is similar to the proof of Lemma 25.34. ■

Let us suppose we get a potential draw sequence. In this subsection we describe the testing algorithms Quick-Havel-Hakimi and Linear-Erdős-Gallai.

Quick Havel-Hakimi algorithm

Algorithm Quick-Havel-Hakimi-Test is based on the following classical theorem [39, 44, 73].

Theorem 25.49 (Havel [44], Hakimi [39]). *If $n \geq 3$, then a nonincreasing sequence $q = (q_1, \ldots, q_n)$ of positive integers is the outdegree sequence of a simple graph G if and only if $q' = (q_2 - 1, q_3 - 1, \ldots, q_{q_1} - 1, q_{q_1+1} - 1, q_{q_1+2}, \ldots, q_{d_n})$ is the outdegree sequence of some simple graph G'.*

Proof See [39, 44]. ■

If G is for example a complete simple graph, then it contains $\Theta(n^2)$ edges and the direct application of Havel-Hakimi theorem requires $\Theta(n^2)$ time. We make an attempt to decide in linear time the pairability of a sequence of positive integers.

The first simple observation is the necessity of the condition $d_i \leq n - 1$ for all $i = 1, 2, \ldots, n$. We have not to test this property since all our draw allocation algorithms guarantee its fulfilment. Another interesting condition is

Lemma 25.50 (Iványi, Lucz, Sótér [54]) *If a nonincreasing sequence $d = (d_1, \ldots, d_n)$ of positive integers is the outdegree sequence of a simple graph G, then*

$$\sum_{i=1}^{n} d_i \quad \text{is even.} \tag{25.64}$$

and

$$\sum_{i=1}^{k} d_i - \min\left(2\binom{k}{2}, \sum_{i=1}^{k} d_i\right) \leq \sum_{i=k+1}^{n} d_i \quad (k = 1, 2, \ldots, n). \tag{25.65}$$

Proof The draw request of the teams $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_k$ must be covered by inner and outer draws. The first sum on the right side gives the exact number of usable outer draws, while the sum of the right side gives the exact number of the reachable inner draws. The minimum on the left side represent an upper bound of the possible inner draws. ■

If we substitute this upper bound with the precise value, then our formula becomes a sufficient condition, but the computation of this value by Havel-Hakimi theorem is dangerous for the linearity of the method.

Let's take a few example. If $n = 2$, then we have only one potential draw-sequence, which is accepted by Havel-Hakimi algorithm and satisfies (25.64) and (25.65).

If $n = 3$, then there are $\binom{4}{3} = 4$ potential draw sequence: (2,2,2), (2,2,1), (2,1,1) and (1,1,1). From these sequences Havel-Hakimi algorithm and the conditions of Lemma 25.48 both accept only (2,2,2) and (1,1,1).

If $n = 4$, then there are $\binom{6}{4} = 15$ potential draw sequences. Havel-Hakimi algorithm and the conditions of Lemma 25.48 both accept the following 7: (3,3,3,3), (3,3,2,2), (3,2,2,1), (3,1,1,1), (2,2,2), (2,2,1,1), and (1,1,1,1).

If $n = 5$, then there are $\binom{8}{5} = 56$ potential draw sequences. The methods are here also equivalent.

From one side we try to find an example for different decisions or try to find an exact proof of the equivalence of these algorithms.

Linear Erdős-Gallai algorithm

For given nondecreasing sequence $q = (q_1, \ldots, q_n)$ of nonnegative integers the first i elements of the sequence is called the ***head*** of the sequence and last $n - i$ elements are called ***the tail*** belonging to the ith element of the sequence. The sum of the elements of the head is denoted by H_i, while the sum of the element of the tail is denoted by T_i. The sum $\sum_{k=i+1}^{n} \min(i, b_k)$ is denoted by C_i and is called the capacity of the tail belonging to q_i. If H_n is even, then $\mathbf{q}$ is called ***even,*** otherwise the sequence is called ***odd sequence.***

Another classical theorem on the testing of the potential draw sequences whether they are graphical is the theorem proved by Erdős and Gallai in 1960 [24].

Theorem 25.51 (Erdős, Gallai, [24]) *If $n \geq 1$, the n-regular sequence $(q_1 \ldots, q_n)$ is graphical if and only if*

$$H_n \quad \textit{is even} \tag{25.66}$$

and

$$H_i - i(i-1) \leq C_i \quad (i = 1, \ldots, n-1). \tag{25.67}$$

Proof See [20, 24, 118, 129] ∎

Recently we could improve this theorem [54]. The algorithm Erdős-Gallai-Linear exploits, that q is monoton. It determines the C_i capacities in constant time. The base of the quick computation is thesequence $m(q)$ containing pointers.

For given sequence q let $m(q) = (m_1, \ldots, m_{n-1})$, where m_i points to the element of q_k having the maximal index among such elements of q which are greater or equal with i.

Theorem 25.52 (Iványi, Lucz, Sótér [54]) *If $n \geq 1$, the n-regular sequence $(q_1 \ldots, q_n)$ is graphical if and only if*

$$H_n \quad \textit{is even} \tag{25.68}$$

and if $i > m_i$, *then*

$$H_i \leq i(i-1) + H_n - H_i, \tag{25.69}$$

further if $i \leq m_i$, *then*

$$H_i \leq i(i-1) + i(m_i - i) + H_n - H_{m_i}, \tag{25.70}$$

Proof (25.68) is the same as (25.66).

During the testing of the elements of q by Erdős-Gallai-Linear there are two cases:

- if $i > m_i$, then the contribution of the tail of q equals to $H_n - H_i$, since the contribution C_k of the element q_k is only q_k.
- if $i \leq m_1$, then the contribution of the tail of q consists of two parts: $C_{i+1}, \ldots, C_{m_i}$ equal to i, while $C_j = b_j$ for $j = m_i + 1, \ldots, n$.

Therefore in the case $n - 1 \geq i > m_i$ we have

$$C_i = i(i-1) + H_n - H_i, \tag{25.71}$$

and in the case $1 \leq i \leq m_i$

$$C_i = i(i-1) + i(m_i - i) + H_n - H_{m_i}. \tag{25.72}$$

■

The following program is based on Theorem **??** a **??**. It decides on arbitrary n-regular sequence whether it is graphicakl or not.

Input. n: number of vertices ($n \geq 1$);
$q = (q_1, \ldots, q_n)$: n-regular sequence.

Output. L: logical variable, whose value is True, if the input is graphical, and it is False, if the input is not graphical.

Work variables. i and j: cycle variables;
$H = (H_1, \ldots, H_n)$: H_i is the sum of the first i elements of the tested q;
$m = (m_1, \ldots, m_{n-1})$: m_i is the maximum of the indices of such elements of q, which are not smaller than i; $H_0 = 0$: help variable to compute of the other elenments of the sequence H;
$q_0 = n - 1$: help variable to compute the elements of the sequence m.

Erdős-Gallai-Linear(n, b)

01 H_0 ▷ Line 01: initialization
02 **for** $i = 1$ **to** n ▷ Lines 02–03: computation of the elements of H
03 $\quad H_i = H_{i-1} + q_i$
04 **if** H_n odd ▷ Lines 06–08: test of the parity
05 $\quad L =$ False
06 $\quad$ **return**
07 $q_0 = n - 1$ ▷ Line 07: initialization of b_0
08 **for** $j = n$ **downto** $q_1 + 1$ ▷ Lines 08–09: setting of some pointers

Team	Wins	Draws	Losses	Points
T_1	3	0	0	9
T_2	1	0	2	3
T_3	1	0	2	3
T_4	0	2	1	2

25.1. Table Sport table belonging to the sequence $q = (2, 3, 3, 9)$

```
09        m_j = 0
10 for i = 1 to n                      ▷ Lines 10–16: calculation of the pointers
11        if q_i < q_{i-1}
12            for j = q_{i-1} downto q_i + 1
15                m_j = i − 1
16        m_{q_i} = i
17 for j = q_n − 1 downto 1              ▷ Lines 17–18: setting of some pointers
18        m_j = n − 1
19 for i = 1 to n − 1                                 ▷ Lines 19–25: test of q
20        if i > m_i and H_i > i(i − 1) + H_n − H_i
21            L = FALSE
22            return L
23        if i ≤ m_i and H_i > i(i − 1) + H_n − H_i
24            L = FALSE
25            return L
26 L = TRUE                    ▷ Lines 26–27: the program ends with TRUE value
27        return L
```

Theorem 25.53 (Iványi, Lucz [53], Iványi, Lucz, Sótér [54] *Algorithm* Erdős-Gallai-Linear *decides in $O(n)$ time, whether an n-regular sequence $q = (q_1, \ldots, , q_n)$ is graphical or not.*

Proof Line 1 requires $O(1)$ time, lines 2–3 $O(n)$ time, steps 4–6 $O(1)$ time, line 07 $O(1)$ time, line 08–09 $O(1)$ time, lines 10–18 $O(n)$ time, lines 19–25 $O(n)$ time and lines 26–27 $O(1)$ time, therefore the total time requirement of the algorithm is $O(n)$. ■

Testing of the pairing sport property at cautious allocation of the draws

Sport-Test investigated, whether the scores allow to include $S_n - 2\binom{n}{2}$ draws into the sport matrix.

Let us consider the sport sequence $(2, 3, 3, 9)$. In a unique way we get the sport matrix

Here T_4 has no partners to make two draws, therefore q is not a football sequence. Using the Havel-Hakimi algorithm [39, 44, 73] we can try to pair the draws of any

n	$\zeta(n)$	$\beta(n)$	$\varphi(n)$	$\gamma(n)$	$\gamma(n+1)/\gamma(n)$
1	**0**	**1**	**0**	**1**	2.000000
2	**1**	**2**	**2**	**2**	2.000000
3	**4**	**4**	**4**	**4**	2.750000
4	**11**	**4**	**11**	**11**	2.818182
5	???	**31**	**31**	**31**	3.290323
6	???	103	**102**	**102**	3.352941
7	???	349	343	**342**	3.546784
8	???	1256	???	1213	3.595218
9	???	4577	???	4361	3.672552
10	???	17040	???	16016	3.705544
11	???	63944	???	59348	3.742620
12	???	242218	???	222117	3.765200
13	???	922369	???	836315	3.786674
14	???	???	???	3166852	3.802710
15	???	???	???	12042620	3.817067
16	???	???	???	45967479	3.828918
17	???	???	???	176005709	3.839418
18	???	???	???	675759564	3.848517
19	???	???	???	2600672458	3.856630
20	???	???	???	10029832754	3.863844
21	???	???	???	38753710486	3.870343
22	???	???	???	149990133774	3.876212
23	???	???	???	581393603996	3.881553
24	???	???	???	2256710139346	3.886431
25	???	???	??	8770547818956	3.890907
26	???	???	???	34125389919850	???
27	???	???	???	$ZETA$: 97684354869695	???
28	???	???	???	???	???
29	???	???	???	???	???

Figure 25.6 Nullamentes, binomiális, fejfelező és jó sorozatok száma, valamint a jó sorozatok szomszédos helyeken vett értékeinek hányadosa.

sport matrix. If we received the sport matrix in a unique way, and Havel-Hakimi algorithms can not pair the draws, then the investigated sequence is not a football sequence.

We can increase the chance to get such negative result thinking on the method of allocation of the draws. Sport-Test allocated the draws in a greedy way. The following lemma shows that the uniform as possible allocation strategy is increases the percent of sequences refused by a testing algorithm.

Lemma 25.54 *If*

Proof ■

Now consider the football sequence $f = (6^6, 9, 21, 24, 27, \ldots, 54, 57, 57, 69^7)$, which is the result of a tournament of 7 weak, 14 medium and 7 strong teams. the weak player play draws among themselves and loss against the medium and strong teams. The medium teams form a transitive subtournament and loss against the strong teams. The strong teams play draws among themselves. We perturbate this simple structure: one of the weak teams wins against the best medium team instead of to lost the match. There are 42 draws in the tournament, therefore the sum of the v_i multiplicities of the sport matrix has to be 84. A uniform distribution results $v_i = 3$ for all i determining the sport matrix in a unique way.

Let us consider the matches in the subtournament of T_1, T_2, ..., T_7. This subtournament consists of 21 matches, from which at most $\lfloor \frac{7 \cdot 3}{2} \lfloor = 10$ can end with draw, therefore at least 11 matches have a winner, resulting at least $2 \cdot 10 + 3 \cdot 11 = 53$ inner points. But the seven teams have only $6 \times 6 + 9 = 45$ points signalizing that that the given sport matrix is not a football matrix.

In this case the concept of inner draws offers a solution. Since $f_1 + f_2 + \ldots + f_6 = 36$ and $3\binom{6}{2} = 45$, the teams T_1, T_2, ..., T_6 made at least 9 draws among themselves. "Cautious" distribution results a draw sequence (3^6), which can be paired easily. Then we can observe that $f_1 + f_2 + \ldots + f_6 + f_7 = 45$, while $3 \cdot \binom{7}{2} = 63$, so the teams T_1, T_2, ..., T_7 have to made at least 18 draws. Cautious distribution results a draw sequence $(6^5, 3, 3)$. Havel-Hakimi algorithm finishes the pairing with the draw sequence (2,2), so 2 draws remain unpaired. If we assign a further draw pack to this subtournament, then the uniform distribution results the draw sequence $(6^6, 3)$ consisting of 13 draw packs instead of 12. Since $3 \cdot 13 = 39$ is an odd number, this draw sequence is unpairable—the subtournament needs at least one outer draw. ???

25.9. Reconstruction of the tested sequences

The reconstruction begins with the study of the inner draws. Let us consider the following sequence of length 28: $q = (6^6, 9, 21, 24, 27, 30, \ldots, 54, 57, 57, 69^7)$. This is the score sequence of a tournament, consisting of seven weak, 14 medium and 7 strong teams. The weak teams play only draws among themselves, the medium teams win against the weak teams and form a transitive subtournament among themselves, the strong teams win against the weak and medium teams and play only draws among themselves. Here a good management of obligatory draws is necessary for the successful reconstruction.

In general the testing of the realizabilty of the draw sequence of a sport matrix is equivalent with the problem to decide on a given sequence d of nonnegative integers whether there exists a simple nondirected graph whose degree sequence is d.

Let us consider the following example: $q = (6^4, 12, 15, 18, 21, 24, 27, 30^5, 33)$. This is the score sequence of a tournament of 4 ”week“, 8 ”medium“ and 4 ”strong“ teams. The week teams and also the strong teams play only draws among themselves. The medium teams win against the weak ones and the strong teams win against the medium ones. T_25 wins again T_1, T_{26} wins against T_2, T_{27} wins against T3, and T_{28} wins against T_4, and the remaining matches among weak and strong teams end with draw.

In this case the 16 teams play 120 matches, therefore the sum of the scores has to be between 240 and 360. In the given case the sum is 336, therefore the point matrix has to contain 96 wins and 24 draws. So at uniform distribution of draws every team gets exactly one draw pack.

How to reconstruct this sequence? At a uniform distribution of the draw packs we have to guarantee the draws among the weak teams. The original results imply nonuniform distribution of the draws but it seems not an easy task to find a quick and successful method for a nonuniform distribution.

Exercises

25.9-1 How many

Problems

25-1 Football score sequences

Let

Chapter Notes

A nondecreasing sequence of nonnegative integers $D = (d_1, d_2, \ldots, d_n)$ is a score sequence of a $(1, 1, 1)$-tournament, iff the sum of the elements of D equals to B_n and the sum of the first i $(i = 1, 2, \ldots, n-1)$ elements of D is at least B_i [72].

D is a score sequence of a (k, k, n)-tournament, iff the sum of the elements of D equals to kB_n, and the sum of the first i elements of D is at least kB_i [62, 81].

D is a score sequence of an (a, b, n)-tournament, iff (25.17) holds [49].

In all 3 cases the decision whether D is digraphical requires only linear time.

In this paper the results of [49] are extended proving that for any D there exists an optimal minimax realization T, that is a tournament having D as its outdegree sequence and maximal G and minimal F in the set of all realization of D.

In a continuation [51] of this chapter we construct balanced as possible tournaments in a similar way if not only the outdegree sequence but the indegree sequence is also given.

[3] [4] [7] [8] [13] [16] [19] [18] [37]
[39] [44]
[49] [51] [50] [52] [56]
[68] [72] [81] [82] [84]
[96] [95]

There are further papers on imbalances in different graphs [61, 84, 96, 115].

Many efforts was made to enumerate the different types of degree and score sequences and connected with them sequences, e.g. by Ascher [2], Barnes and Savage

[6, 5], Hirschhorn and Sellers [46], Iványi, Lucz and Sótér [55, 54], Metropolis [78], Rødseth, Sellers and Tverberg [112], Simion [119], Sloane and Plouffe [120, 121, 124, 123, 122].

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