A SIMPLE PROOF OF THE INEQUALITY $\mathrm{FFD}(L) \le \frac{11}{9}\mathrm{OPT}(L) + 1, \quad \forall L$ FOR THE FFD BIN-PACKING ALGORITHM*†

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Abstract

The first fit decreasing (FFD) heuristic algorithm is one of the most famous and most studied methods for an approximative solution of the bin-packing problem. For a list L, let OPT(L) denote the minimal number of bins into which L can be packed, and let FFD(L) denote the number of bins used by FFD. Johnson[1] showed that for every list L, FFD(L) $\le$ 11/9OPT(L) + 4. His proof required more than 100 pages. Later, Baker[2] gave a much shorter and simpler proof for FFD(L) $\le$ 11/9 OPT(L)+3. His proof required 22 pages. In this paper, we give a proof for FFD(L) $\le$ 11/9 OPT(L)+1. The proof is much simpler than the previous ones.

In bin-packing, a list L of pieces, i.e. numbers in the range $(0, 1]$, are to be packed into bins, each of which has a capacity 1, and the goal is to minimize the number of bins used. The minimal number of bins into which L can be packed is denoted by OPT(L) for the list L. The first-fit-decreasing (FFD) algorithm first sorts the list into a non-increasing order and then processes the pieces in that order by placing each piece into the first bin into which it fits. More precisely, suppose the sorted pieces are $p_1 \ge p_2 \ge \cdots \ge p_n$, where p_i denotes the piece and its size as well, and that the bins are indexed as $B_1, B_2, \cdots$, FFD processes the pieces in the order $p_1, p_2, \cdots, p_n$. For $1 \le i \le n$, if j is the least k such that B_k holds a total of amount $\le 1 - p_i$ when p_i is to be packed, then FFD places p_i in B_j. For a list L, let FFD(L) denote the number of bins used by FFD. Johnson[1] showed that for every list L, FFD(L)$\le \frac{11}{9}$OPT(L)+4. Unfortunately, his proof required more than 100 pages. Later, Baker[2] gave a much shorter and simpler proof for FFD(L)$\le \frac{11}{9}$OPT(L)+3. However, Baker's proof required still 22 pages and is rather complicated. In this paper, we

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give a proof for

$$\mathrm{FFD}(L) \le \frac{11}{9}\mathrm{OPT}(L) + 1. \tag{1}$$

Since it is easy to show that there exist examples (L) for which $\mathrm{FFD}(L) \ge \frac{11}{9}\mathrm{OPT}(L) + \frac{5}{9}$, our result seems to arrive at the final stage.

For a given list L, let P and P^* denote the FFD packing and an optimal packing of L respectively. Let G be the set of pieces in L with size $> \frac{1}{2}$. A piece in G is denoted also by G. A bin containing a G is called a G-bin. For a bin $B = \{(G, \cdot, \cdot), (G, *, *)\}$, where $(G, \cdot, \cdot)$ and $(G, *, *)$ are bins containing G in the FFD packing and OPT packing respectively, we denote $(G, \cdot, \cdot)$ by $B - P$ and $(G, *, *)$ by $B - P^*$. Sometimes we use $p(B, i, P)$ and $p(B, i, P^*)$ for the ith piece in $B - P$ and $B - P^*$ respectively. Let x be the least piece of L. The size of a piece x_i is also denoted by x_i if no confusion can be made. A G-bin is called a G-ij-bin if $B - P$ contains i pieces and $B - P^*$ contains j pieces in total. Our proof is based on a combination of the weighting function method and the minimal counter-example method. Such a combination has been used by many authors such as Coffman et al[3] and Yue[4]. For a piece p we give it a "weight" $w(p) \le p$. $w(p)$ is called a weighting function. With a given w, we divide all the pieces of L into classes. Denote $R_i = \{y | w(y) = w_i\}$, which is called a region of w, or simply region i. Pieces belonging to R_i are denoted by x_i, if no confusion arises. E.g., we use $\{G, x_3, x_3\}$ for a bin with its 2nd and 3rd pieces belonging to R_3, though these two pieces may have different sizes. Generally, $x_i > x_j$ if $i < j$. We write $w(x_i + x_j)$ for $w(x_i) + w(x_j)$ for simplicity.

In the following we assume that L is a minimal counter-example to (1), i.e., for this L,

$$\mathrm{FFD}(L) > \frac{11}{9}\mathrm{OPT(L)} + 1 \tag{2}$$

holds, and that any list L' satisfying (2) must have $|L'| \ge |L|$. By definition, we can assume that the last FFD bin of L consists of the piece x only.

Lemma 1. Every optimal bin contains at least 3 pieces.

Proof. Let (y, y') be an optimal bin with $y \ge y'$. Let $B = (y, y^0, \cdot)$ be the FFD-bin, into which y falls. If $y^0 \ge y'$, we delete all pieces in B from the list L. Let $L' = L \backslash B$. Evidently, the FFD packing for L' is identical to those for L except that the bin B will be missing. So we have FFD(L')=FFD(L)-1. As for OPT(L), we put y' in the place occupied originally by y^0 after the deletion of B. We have OPT(L')$\le$OPT(L)-1. Thus we have $\mathrm{FFD}(L') = \mathrm{FFD}(L) - 1 > \frac{11}{9}(\mathrm{OPT}(L)\text{-}1) + 1 \ge \frac{11}{9}\ \mathrm{OPT}(L') + 1$. L cannot be a minimal counter-example to (1). If $y^0 < y'$, by the FFD rule, y' must have been put into an FFD-bin $B' = (z, y', \cdot)$ with $z \ge y$ before y^0 was put into B. Deleting all the pieces in B' from L and applying the same argument as above, we have the same conclusion.

Lemma 2. Let B' be a G-23-bin such that the sum of the two least pieces in $B' - P^*$ has a size $\ge \frac{1}{2}(1 - x)$. Then for any G-23-bin B with $p(B, 2, P) \le \frac{1}{2}(1 - x)$, we have $p(B', 2, P) > p(B, 2, P^*)$.

Proof. Let $B = \{(G_0, \bar{x}), (G_0, x', x'')\}$, $B' = \{(G, x_0), (G, x_0', x_0'')\}$. Suppose $x_0 \le x'$. Then we have $\bar{x} > x_0$ and $G_0 < G$, otherwise B' cannot be a G-23-bin. By the FFD rule, we have $\bar{x} + G > 1$. Thus we have $\frac{1}{2}(1 - x) \ge \bar{x} > x_0' + x_0''$. This is impossible.

As we said above, L is a minimal counter-example to (1). Our aim is to prove that this statement cannot be true and therefore no counter-example exists. Our proof is divided into 3 cases according to whether

(a) $\frac{1}{4} < x \le \frac{1}{3}$,
(b) $\frac{1}{5} < x \le \frac{1}{4}$,
(c) $\frac{2}{11} < x \le \frac{1}{5}$.

When $x < \frac{2}{11}$ or $x > \frac{1}{3}$, the truth of (1) follows from Lemma 1 and simple calculations. For a given L, let $w(L)$ be the total weight of L. Our aim is to establish the inequalities

$$(1-x)\mathrm{FFD}(L) \le w(L) + A \le \frac{11}{9}(1-x)\mathrm{OPT}(L) + a, \tag{3}$$

where A and a are two constants, $a \le 1-x$. If every FFD bin has a weight $\ge 1-x$ and every OPT bin has a weight $\le \frac{11}{9}(1-x)$, we set $A = a = 0$ and achieve our goal. Unfortunately, there is an FFD G-23-bin whose weight may be $< 1-x$. Let $B = \{(G,y),(G,y',y'')\}$ be a G-23-bin. If $G+y > 1-x$ and $w(G+y) < 1-x$, we call $d = 1-x-w(G+y)$ the *shortage* of the FFD G-bin B, or simply, the shortage of y, and y is called a piece with shortage. Notice that such a y arises only in G-23-bins. A piece p is called a regular piece if FFD packs it into a B_i at a time when all higher-numbered bins are empty, otherwise p is a fallback piece. A bin is a k-bin if it contains exactly k pieces in it.

Lemma 3. Suppose $i \ge 2, x_i$ and x_{i+l} $(l > 0)$ are pieces with shortage, $B = \{(G,x_i),(G,x_j,x_k)\}$ and $B' = \{(G',x_{i+l}),(G',x_p,x_q)\}$, where $x_p + x_q \ge \frac{1}{2}(1-x)$ in Case (c) (the condition is unnecessary, if $i \ge 4$). Then we have $x_j < x_{i+l}$ and $j \ge i+l$, and both x_j and x_k cannot be pieces with shortage, and

$$w(G+x_j+x_k) + (1-x-w(G+x_i)) = 1-x+w(x_k) - (w(x_i)-w(x_j)).$$

Proof. By the FFD rule, we must have $G \ge G'$, otherwise $G'+x_i > 1$ and $x_i > x_p + x_q$. This is impossible since $i \ge 2$ (and $x_p + x_q \ge \frac{1}{2}(1-x)$ in Case (c)). Since $G'+x_{i+l} > 1-x$, we have $x_j < x_{i+l}$ and $j \ge i+l$. Notice that the truth of the equality in the Lemma needs no assumption.

If x_i is a piece with shortage, and

$$1-x+w(x_k) - w((x_i)-w(x_j)) \le \begin{cases} 1-\dfrac{1}{45}-\dfrac{11}{9}\Delta & \text{in Case (b)} \\ 1-\dfrac{11}{9}\Delta & \text{in Case (c)} \end{cases}$$

we say that the piece x_i can be *balanced* by itself. The empty space(s) in the optimal bin(s) where a quantity equal to the shortage of x_i will be put is called the balance of the shortage of x_i.

Case (a). $\frac{1}{4} < x \le \frac{1}{3}$, $x = \frac{1}{4} + \Delta$, $0 < \Delta \le \frac{1}{12}$.

In this case the weighting function is defined as the following table:

Table 1

line	typical piece	R_i	$w(p)$	total weight of B
0	G	$\left(\frac{1}{2},1\right]$		
1	x_1	$\left(\frac{1-x}{2},\frac{1}{2}\right]$	$\frac{3}{8}-\frac{\Delta}{2}$	$=\frac{3}{4}-\Delta$
2	x_2	$\left[x,\frac{1-x}{2}\right]$	$\frac{1}{4}-\frac{\Delta}{3}$	$=\frac{3}{4}-\Delta$

Since $G+2x_2>1$, we cannot have a G-3-bin. Thus by Lemma 1, for a minimal counter-example, there is no G-bin at all. Evidently, as an optimal bin, there are at most four possibilities: (x_1,x_1), (x_1,x_2), (x_1,x_2,x_2) and (x_2,x_2,x_2). Among them only (x_1,x_2,x_2) needs to be considered. Since $w(x_1+x_2+x_2)=\frac{3}{8}-\frac{\Delta}{2}+\frac{1}{2}-\frac{2}{3}\Delta=\frac{7}{8}-\frac{7}{6}\Delta<\frac{11}{9}(\frac{3}{4}-\Delta)$, every optimal bin has a weight $<\frac{11}{9}(\frac{3}{4}-\Delta)$. There may be an FFD bin $B=\{x_1,x_2\}$, which has a weight $\geq\frac{3}{8}-\frac{\Delta}{2}+\frac{1}{4}-\frac{\Delta}{3}=\frac{3}{4}-\Delta-(\frac{1}{8}-\frac{\Delta}{6})$. The last FFD bin has a weight $\frac{1}{4}-\frac{\Delta}{3}=\frac{3}{4}-\Delta-(\frac{1}{2}-\frac{2}{3}\Delta)$. Since $\frac{1}{8}-\frac{\Delta}{6}+\frac{1}{2}-\frac{2}{3}\Delta=\frac{5}{8}-\frac{5}{6}\Delta<\frac{3}{4}-\Delta$, we have (3) with $A=0$ and $a=\frac{5}{8}-\frac{5}{6}\Delta$.

Case (b). $\frac{1}{5}<x\leq\frac{1}{4}$. Let y be the smallest regular piece in $(\frac{1-x}{3},\frac{1}{3}]$ if such a piece exists, and $\frac{1}{3}$ otherwise. Define a weighting function by Table 2 below.

$$x=\frac{1}{5}+\Delta,\ 0<\Delta\leq\frac{1}{20},\ \theta=\frac{5}{4}\Delta,\ \delta=\frac{1}{45}-\frac{\Delta}{36},\ \theta+\delta=\frac{1}{45}+\frac{11}{9}\Delta.$$

Table 2

line	typical piece	R_i	$w(p)$	type	total weight of a bin
0	G	$\left(\frac{1}{2},1-x\right]$	$G-\delta$		
1	x_1	$\left(\frac{1-x}{2},\frac{1}{2}\right]$	$\frac{2}{5}-\frac{\Delta}{2}$	(r,r)	$=\frac{4}{5}-\Delta$
2	x_2	$\left(\frac{1-y}{2},\frac{1-x}{2}\right]$	$\frac{1-y}{2}-\frac{5}{12}\Delta$	$(r,r,f),f\in I$	$>\frac{4}{5}-\Delta$
3	x_3	$\left(\frac{1}{3},\frac{1-y}{2}\right]$	$\frac{13}{45}-\frac{13}{36}\Delta$	(r,r,f)	$>\frac{4}{5}-\Delta$
4	x_4	$\left(\frac{1-x}{3},\frac{1}{3}\right]$	$\frac{4}{15}-\frac{\Delta}{3}$	(r,r,r)	$=\frac{4}{5}-\Delta$
5	x_5	$\left[x,\frac{1-x}{3}\right]$ I	$\frac{1}{5}-\frac{\Delta}{4}$	(r,r,r,r)	$=\frac{4}{5}-\Delta$

Table 3

x_4	x_4	x_4	x_4	x_4	x_4	x_5
x_3	x_4	x_3	x_4	x_3	x_4	x_2
x_1	x_1	x_2	x_2	x_3	x_4	x_2

x_5	x_5
x_5	x_5
x_4	x_5
x_4	x_4

It is a simple calculation to verify that only x_3, x_4 and x_5 can be pieces with shortage. If both x_4' and x_5' are pieces with shortage, then the bin B containing x_4' must be one of the forms $B' = \{(G, x_4'), (G, x_5, x_5)\}$. For, suppose $B = \{(G, x_4'), (G, x_4, x_5)\}$ and suppose $B' = \{(G', x_5'), (G', x_5, x_5)\}$ be the bin into which x_5' falls. By the FFD rule, we have $G \geq G'$. But then B' cannot be a G-23-bin. Since $w(G + x_5 + x_5) + (1 - x - w(G + x_4')) = 1 - \theta - (\frac{1}{15} - \frac{\Delta}{12}) < 1 - \theta - \delta$, the bin $B - P^*$ has enough space for holding the shortage of x_4. Thus in the optimal bins (x_5, x_5, x_4, x_4) and (x_5, x_5, x_5, x_4), we consider x_4 or x_5 only, not both. Since $G + 2x_4 > \frac{1}{2} + \frac{2}{3}(1 - x) \geq 1$, the possible G-3-bins can only be (G, x_2, x_5), (G, x_3, x_5), (G, x_4, x_5) and (G, x_5, x_5). Since $G + x_2 + x_5$ and $G + x_3 + x_5$ have sizes > 1, (G, x_4, x_5) and (G, x_5, x_5) are the only possibilities. Thus we have

Lemma 4. If bin $B - P$ of $B = \{(G, x_i), (G, x', x'')\}$ is a bin with shortage, then x' must be an x_4 or an x_5 and x'' be an x_5 and

$$w(G + x' + x'') + 1 - x - w(G + x_i) = 1 - \theta - (w(x_i) - w(x')).$$

Corollary.
(i) $w(G + x_4 + x'') + (1 - x - w(G + x_3)) = 1 - \theta - \delta.$
(ii) $w(G + x_4 + x'') + (1 - x - w(G + x_4)) = 1 - \theta - \delta + \delta.$
(iii) $w(G + x_5 + x'') + (1 - x - w(G + x_5)) = 1 - \theta - \delta + \delta.$

If an $x_i (i = 4, 5)$ is a piece with shortage, since $2x > \frac{1}{2}(1 - x)$, by Lemma 2, this piece falls either into a bin of form $B = \{(G, x_1), (G, x_i, x_5)\}$ or into a non-G-bin. In the former case, since $w(G + x_1) \geq \frac{1}{2} - \delta + \frac{2}{5} - \frac{\Delta}{2} = \frac{4}{5} - \Delta + \frac{1}{10} - \delta + \frac{\Delta}{2} > \frac{4}{5} - \Delta + 2\delta$, we subtract 2δ from $w(L)$ to keep the weight of $B - P \geq \frac{4}{5} - \Delta$ and reduce the weight of $B - P^*$ to a quantity $\leq 1 - \theta - \delta - 2\delta$. Since $w(G) = G - \delta$ and every optimal G-bin must contain an x_5, every optimal G-bin has a weight $\leq 1 - \delta - \theta$.

Now we are going to consider the non-G-bin.

$$\begin{aligned} w(x_1 + x_2 + x_5) &= \frac{2}{5} - \frac{\Delta}{2} + \frac{1 - y}{2} - \frac{5}{12}\Delta + \frac{1}{5} - \frac{\Delta}{4} = \frac{11}{10} - \frac{y}{2} - \frac{7}{6}\Delta \\ &= 1 - \theta - \delta - \left(\frac{y}{2} - \frac{11}{90} - \frac{\Delta}{18}\right). \end{aligned} \tag{1'}$$

Since $x_1 + x_2 + x_5 \leq 1$, we have $x_5 \leq x + \frac{1}{2}(y - x)$. Let $B = \{(G, x_5), (G, x_5', x_5'')\}$ be the bin into which x_5 falls. We have $G > 1 - \frac{3x+y}{2}$, since $G + \frac{1}{2}(y + x) \geq G + x_5 > 1 - x$. Since $2x \geq \frac{1}{2}(1 - x) > x_2$, x_2 must fall into a G-bin $B' = \{(G', x_2), (G', \cdot, \cdot)\}$ with $G' \geq G$ by the FFD rule. Thus we have

$$\begin{aligned} w(G' + x_2) &\geq 1 - \delta - \frac{3x + y}{2} + \frac{1 - y}{2} - \frac{5}{12}\Delta \\ &= \frac{6}{5} - y - \delta - \frac{23}{12}\Delta = \frac{4}{5} - \Delta + \left(\frac{2}{5} - y - \delta - \frac{11}{12}\Delta\right). \end{aligned}$$

Since $(\frac{2}{5} - y - \delta - \frac{11}{12}\Delta) + (\frac{y}{2} - \frac{11}{90} - \frac{\Delta}{18}) = \frac{5}{18} - \frac{y}{2} - \frac{35}{36}\Delta - \delta \geq \frac{1}{9} - \frac{35}{36}\Delta - \delta > \delta$, the two bins (G', x_2) and (x_1, x_2, x_5) can provide enough space for the shortage δ of bin (G, x_5) shown in Corollary (iii) of Lemma 4.

$$w(x_1 + x_3 + x_4) = \frac{2}{5} + \frac{5}{9} - \left(\frac{1}{2} + \frac{13}{36} + \frac{1}{3}\right)\Delta = 1 - \theta - \delta - \delta. \tag{2'}$$

By Corollary (i) of Lemma 4, x_3 can be balanced by itself, and only x_4 is to be considered. From Corollary (ii) above, bins (G, x_4, x'') and (x_1, x_3, x_4) together have enough space for the shortage x_4.

$$w(x_1+x_4+x_4)=\frac{2}{5}-\frac{\Delta}{2}+\frac{8}{15}-\frac{2}{3}\Delta=\frac{14}{15}-\frac{7}{6}\Delta=1-\theta-\delta-2\delta, \tag{3'}$$

$$w(x_4+x_4+x_4)=\frac{4}{5}-\Delta=1-\theta-\delta-8\delta, \tag{4'}$$

$$w(x_2+x_2+x_5)=1-y-\frac{5}{6}\Delta+\frac{1}{5}-\frac{\Delta}{4}\le 1-\theta-\delta-\delta, \tag{5'}$$

$$w(x_4+x_4+x_5+x_5)=\frac{8}{15}-\frac{2}{3}\Delta+\frac{2}{5}-\frac{\Delta}{2}=1-\theta-\delta-2\delta, \tag{6'}$$

$$w(x_4+x_5+x_5+x_5)=\frac{13}{15}-\frac{13}{12}\Delta=1-\theta-\delta-3\delta. \tag{7'}$$

Thus we have

Lemma 5. For every bin (G,x_4) (or (G,x_5)) with shortage we can identify a place from an optimal G-bin or/and an optimal non-G-bin which is enough for holding its shortage.

Table 4

	generic piece	R_i	$w(p)$	type	total weight
0	G	$\left(\frac{1}{2},1-2x\right]$	$G-\delta$	(r,f)	
1	x_1	$\left(\frac{1-x}{2},\frac{1}{2}\right]$	$\frac{9}{22}-\frac{\Delta}{2}$	(r,r)	$=\frac{9}{11}-\Delta$
2	x_2	$\left(\frac{1-z}{2},\frac{1-x}{2}\right]$	$\frac{7}{22}-\frac{5}{12}\Delta$	$(r,r,f), f\in I$	$>\frac{9}{11}-\Delta$
3	x_3	$\left(\frac{1}{3},\frac{1-z}{2}\right]$	$\frac{7}{22}-\frac{7}{18}\Delta$	(r,r,f)	$=\frac{9}{11}-\Delta$
4	x_4	$\left(\frac{1-x}{3},\frac{1}{3}\right]$	$\frac{3}{11}-\frac{\Delta}{3}$	(r,r,r)	$=\frac{9}{11}-\Delta$
5	x_5	$\left(\frac{1-z}{3},\frac{1-x}{3}\right]$	$\frac{1}{4}-\frac{11}{36}\Delta$	$(r,r,r,f), f\in I$	$>\frac{9}{11}-\Delta$
6	x_6	$\left(z,\frac{1-z}{3}\right]$	$\frac{9}{44}-\frac{\Delta}{4}$	(r,r,r,r)	$=\frac{9}{11}-\Delta$
7	x_7	$[x,z)$ I	$\frac{2}{11}-\frac{2}{9}\Delta$	(r,r,r,r,r)	$=\frac{10}{11}-\frac{10}{9}\Delta$

Let A be the sum of all shortages in the FFD G-bins (Some modifications should be made if there are some pieces with shortage falling into optimal G-bins. In such cases, certain quantity, a δ or $\frac{y}{2}-\frac{11}{90}-\frac{\Delta}{18}$, as the case may be, should be subtracted from $w(L)$ for each such a piece.) Adding A to the total weight $w(L)$ of the given list L, every FFD G-bin has a weight $\ge \frac{4}{5}-\Delta$ and the weight of every OPT bin is still kept within the bound $1-\theta-\delta=\frac{11}{9}(\frac{4}{5}-\Delta)$. Considering that the last FFD bin has a weight $=\frac{1}{5}-\frac{\Delta}{4}=\frac{4}{5}-\Delta-(\frac{3}{5}-\frac{3}{4}\Delta)$ and that there may be two bins in the FFD packing, namely the bin between regions 1 and 2 and the bin between regions 4 and 5, which may have shortages, the former one (x_1,x_2) has a weight $\frac{2}{5}-\frac{\Delta}{2}+\frac{1-y}{2}-\frac{5}{12}\Delta=\frac{4}{5}-\Delta+\frac{1}{10}-\frac{y}{2}+\frac{\Delta}{12}$, and the latter one $((x_4,x_4,x_5)$ or $(x_4,x_5,x_5))$

has a weight $\geq \frac{10}{15} - \frac{5}{6}\Delta = \frac{4}{5} - \Delta - \frac{2}{15} + \frac{\Delta}{6}$. We have

$$\left(\frac{4}{5} - \Delta\right) \text{FFD}(L) - \frac{3}{5} + \frac{3}{4}\Delta + \frac{1}{10} - \frac{y}{2} + \frac{\Delta}{12} - \frac{2}{15} + \frac{\Delta}{6}$$
$$\leq w(L) + A \leq \frac{11}{9}\left(\frac{4}{5} - \Delta\right) \text{OPT}(L),$$

or

$$\text{FFD}(L) \leq \frac{11}{9}\text{OPT}(L) + 1,$$

which contradicts our assumption (2). Thus no counter-example exists.

Case (c). $\frac{2}{11} < x \leq \frac{1}{5}$. Let z be the smallest regular piece in $(\frac{1-x}{4}, \frac{1}{4}]$ if such a piece exists, $\frac{1}{4}$ otherwise. Let $x = \frac{2}{11} + \Delta$, $0 < \Delta \leq \frac{1}{55}$, $\delta = \frac{11}{9}\Delta$, $\phi = \frac{1}{44} - \frac{\Delta}{36}$. The weighting function and the possible optimal bins hard to deal with are given below.

Table 5

#	worst cases of possible combinations in a bin	total weight of a bin $= 1 - \delta - p$ with p
1	$x_1x_2x_7$	$> 2\phi + \delta$
2	$x_1x_4x_4$	$= 2\phi$
3	$x_1x_3x_5$	$= \phi$
4	$x_2x_3x_4$	$= 2\phi$
5	$x_1x_6x_7x_7$	$= \phi$
6	$x_2x_5x_7x_7$	$> 2\phi$
7	$x_2x_6x_6x_7$	$= 3\phi$
8	$x_3x_4x_6x_7$	$= \phi$
9	$x_3x_6x_6x_6$	$= 3\phi$
10	$x_4x_4x_6x_6$	$= 2\phi$
11	$x_4x_5x_5x_7$	$> \phi$
12	$x_4x_5x_6x_6$	$\geq 2\phi$
13	$x_6x_6x_6x_7x_7$	$= \phi$*)
14	$x_6x_6x_7x_7x_7$	$= 2\phi$

*) For this bin we want to show that among the three x_6's there is at most one requiring an empty space. For, as it is easily seen, if an x_6 with shortage falls into a bin $B = \{(G, x_6), (G, x_6', x_7)\}$, this x_6 can be balanced by itself. Thus we consider only those x_6 which fall into a bin of form $B = \{(G, x_6), (G, x_6', x_6'')\}$. In this case, $G \leq 1-2z$. From $G+x_6 > 1-x$, we have $x_6 > 2z-x$. If there are two such x_6 in $\{x_6, x_6, x_6, x_7, x_7\}$, we would have $x_6+x_6+x_6+x_7+x_7 \geq 4z-2x+z+2x = 5z > 1$.

Lemma 6.

(i) For a given L, if both x_4 and x_6 (or x_7) are pieces with shortage , x_4 can be balanced by itself. The statement is true also for x_5 and x_7.

(ii) If both x_4 and x_5 are pieces with shortage, then x_4 can be balanced by itself and $x_5 > \frac{1-z}{3} - \delta$.

(iii) If both x_5 and x_6 are pieces with shortage, then x_5 can be balanced by itself and $x_6 > \frac{1-z}{3} - \delta$.

Proof.

(i) Assume that both x_4 and x_6 are pieces with shortage. Let

$$B_1 = \{(G_1, x_4), (G_1, y_1, y_2)\} \text{ and } B_2 = \{(G_2, x_6), (G_2, x', x'')\}$$

be the G-bins into which x_4 and x_6 fall respectively.

From Lemma 3, y_1 and y_2 must be an x_6 or an $x_7(y_1 \geq y_2)$ and both y_1 and y_2 cannot be pieces with shortage. From $1-x-w(G_1+x_4)+w(G_1+y_1+y_2) \leq 1-x+\frac{9}{22}-\frac{\Delta}{2}-\frac{3}{11}+\frac{\Delta}{3} = 1-2\phi-\delta$ and $1-x-w(G+x_6)+w(G+2x_6) = 1-\delta+\phi$, we see that x_4 and x_6 can be balanced by themselves. Similarly, for x_7, we have $1-x-w(G_1+x_4)+w(G_1+y_1+y_2) = 1-4\phi-\delta$.

(ii) Let $B = \{(G,x_4),(G,x_j,x_k)\}$ and $B' = \{(G,x_5),(G,x_p,x_q)\}$ be the bins into which x_4 and x_5 fall . From Lemma 3, we have $j \geq 5$ and

$$\begin{aligned}
&w(G+x_j+x_k)+(1-x-w(G+x_4)) \\
=&1-x+w(x_k)-(w(x_4)-w(x_j)) \\
\leq&1-x+w(x_6)-w(x_4)+w(x_5) \\
=&1-x+\frac{9}{44}-\frac{\Delta}{4}-\left(\frac{3}{11}-\frac{\Delta}{3}-\frac{1}{4}+\frac{11}{36}\Delta\right) \\
=&1-\delta.
\end{aligned}$$

The inequality $x_5 > \frac{1-x}{3}-\delta$ can be derived directly from $G-\delta+\frac{1-x}{3} < 1-x$ and $G'+x_5 > 1-x$.

(iii) The proof is quite the same as (ii).

In the following we will show that all the pieces x_5 with shortage and all the pieces x_7 with shortage can be in aggregation balanced by themselves.

Lemma 6 shows that for the pieces with shortage we can assume that all of them either came from R_4 or from R_5 or from R_6 or from R_7, but not from any two of them. Our scheme is as follows. We divide all pieces with shortage into groups. For each group we find its total shortage, α say. We add α to $w(L)$ to make every FFD bin in this group have a weight $\frac{9}{11}-\Delta$. From this process, the corresponding OPT bins obtain an amount α. For some group, these OPT bins have not so large a space to hold α that the weight of each bin does not exceed $1-\delta$. For such a case we find out the quantity of the supernumery, β say. Suppose the group has m bins in total. For each $\frac{\beta}{m}$ we want to identify an optimal bin such that if an x_i with shortage falls into it, it can provide enough space for this x_i and the quantity $\frac{\beta}{m}$.

(a) Now assume first that some $x_4's$ are pieces with shortage. For an FFD G-23-bin (G,x_4), its OPT bin can only be one of (G,x_4,x_6), (G,x_4,x_7), (G,x_5,x_6), (G,x_5,x_7), (G,x_6,x_6) and (G,x_6,x_7). By Lemma 3 (with $k=7$), only bins with no x_7 in it need to be considered. Let

$$A_1 = \{B \in G | B = \{(G,x_4),(G,x_4',x_6)\}, x_4 \text{ in } (G,x_4) \text{ is a piece with shortage}\}.$$

Let $A_1' = \sum w(G+x_4)$ and $A_1'' = \sum w(G+x_4'+x_6)$, where the sums are taken over bins in A_1. Evidently, $A_1'' = A_1' + (\frac{9}{44}-\frac{\Delta}{4})|A_1|$. Let $A_1' = (\frac{9}{11}-\Delta)|A_1|-\alpha$. α is the total shortage of set A_1. (The α will be used later. Needless to say, its value varies with the given set.) Then

$$\begin{aligned}
A_1'' &= \left(\frac{9}{11}-\Delta\right)|A_1|-\alpha+\left(\frac{9}{44}-\frac{\Delta}{4}\right)|A_1| \\
&=(1-\delta)|A_1|-\alpha+\left(\frac{1}{44}-\frac{\Delta}{36}\right)|A_1|.
\end{aligned}$$

When we add α to the total weight $w(A_1)$ of all bins in $|A_1|$, we can make the weight of every FFD bin in A_1 up to $\frac{9}{11}-\Delta$. However, from this process, the corresponding OPT bins in A_1 have a total supernumerary $(\frac{1}{44} - \frac{\Delta}{36})|A_1|$. Later we will show that, for each x_4 with shortage, the optimal bin containing it will provide a space $(\frac{1}{44} - \frac{\Delta}{36})$ for it.

Similarly, for the sets $A_2 = \{B \in G | B = \{(G, x_4), (G, x_5, x_6)\}\}$ and $A_3 = \{B \in G | B = \{(G, x_4), (G, x_6, x_6)\}\}$, where the x_4 in bin (G, x_4) is a piece with shortage, we have

$$A_2'' = (1-\delta)|A_2| - \alpha - \frac{\Delta}{36}|A_2|,$$

$$A_3'' = (1-\delta)|A_3| - \alpha - \left(\frac{1}{22} - \frac{1}{18}\Delta\right)|A_3|.$$

In these cases, bins in each set can be, in aggregation, balanced by themselves.

(b) Assume that some of the x_5's are pieces with shortage. From $G + x_5 > 1 - x$, we have $G > \frac{2}{3}(1-x)$ and $G + \frac{1-z}{3} + z \geq 1 + \frac{2}{3}(z-x) > 1$. Therefore, no combination(G, x_5, x_6) is possible. Only bins of form $\{(G, x_5), (G, x_6, x_6)\}$ need to be considered. As before, let

$$A_4 = \{B \in G | B = \{(G, x_5), (G, x_6, x_6)\}, x_5 \text{ is a piece with shortage}\},$$

we have

$$A_4'' = (1-\delta)|A_4| - \alpha - \left(\frac{1}{44} - \frac{11}{18}\Delta\right)|A_4|.$$

(c) Assume that some of the x_6's are pieces with shortage. Let

$$A_5 = \{B \in G | B = \{(G, x_6), (G, y, y')\}, \text{ where the } x_6\text{'s are pieces with shortage }\}.$$

Since $G + x_6 > 1 - x, y$ and y' can be x_6 or x_7 only. By Lemma 3, we only consider $B = \{(G, x_6), (G, x_6, x_6)\}$. For this case, we have directly

$$A_5'' = (1-\delta)|A_5| - \alpha + \left(\frac{1}{44} - \frac{\Delta}{36}\right)|A_5|.$$

(d) Assume that some of the x_7's are pieces with shortage. Let

$$A_6 = \{B \in G | B = \{(G, x_7), (G, x_7, x_7)\}, \text{the } x_7 \text{ in}(G, x_7) \text{ is a piece with shortage}\}.$$

By a simple calculation, we have

$$A_6'' = (1-\delta)|A_6| - \alpha.$$

From what we proved above what we want to do is to provide every x_4(or x_6) with shortage with a space of size $\geq \frac{1}{44} - \frac{\Delta}{36}$.

(e) From Lemma 2, if a piece x_i with shortage does not fall into a non-G-bin, it must fall into (i) a bin of form $B = \{(G, x_1), (G, \cdot, \cdot)\}$ or (ii) a G-33-bin, or (iii) a bin $B = \{(G, x_j), (G, x_i, \cdot)\}$ with $j \geq 2$ and $y + y' < \frac{1}{2}(1-x)$, where $B' = \{(G', x_i), (G', y, y')\}$ is the bin from which x_i comes.

(i,a) Assume that x_4 falls into a bin $B = \{(G, x_1), (G, x_4, y)\}$. From Lemma 6, we need not consider whether y is a piece with shortage or not. Since $x_4 + x_5 > \frac{1}{2}$, we consider the case $y = x_6$ only. In this case, the total weight of bins (G, x_1) and (G', x_4) is

$$\geq w\left(G + \frac{9}{22} - \frac{\Delta}{2} + G' + \frac{1-x}{3}\right) > 1 - 2\delta + \frac{15}{22} - \frac{5}{6}\Delta > 2\left(\frac{9}{11} - \Delta\right).$$

(i,b) Assume that x_6 falls into a bin $B = \{(G, x_1), (G, x_6, y)\}$. Since $w(G + x_1) > \frac{1}{2} - \delta + \frac{9}{22} - \frac{\Delta}{2} = \frac{9}{11} - \Delta + \frac{1}{11} - \delta + \frac{\Delta}{2} > \frac{9}{11} - \Delta + 3(\frac{1}{44} - \frac{\Delta}{36})$, the shortage of x_6 and the shortage of y, if y is a piece with shortage, can be balanced by B.

(ii,a) Assume that x_4 falls into a G-33-bin $B = \{(G, x', x''), (G, x_4, y)\}$. Since $w(G + 2x_7) > \frac{1}{2} - \delta + \frac{4}{11} - \frac{4}{9}\Delta = \frac{9}{11} - \Delta + \frac{1}{22} - \delta + \frac{5}{9}\Delta > \frac{9}{11} - \Delta + (\frac{1}{44} - \frac{\Delta}{36})$, the shortage of x_4 can be balanced by $B - P$.

(ii,b) Assume that x_6 falls into a G-33-bin $B = \{(G, x', x''), (G, x_6, y)\}$. In this case, y may be a piece x_6 with shortage. Since

$$w(G + 2x_7) = G - \delta + \frac{4}{11} - \frac{4}{9}\Delta = \frac{9}{11} - \Delta + \left(G - \frac{5}{11} - \delta + \frac{5}{9}\Delta\right)$$

and

$$w(G + 2x_6) = G - \delta + \frac{9}{22} - \frac{\Delta}{2} = 1 - \delta - \left(\frac{13}{22} - G + \frac{\Delta}{2}\right),$$

we have, if $G < \frac{13}{22} + \frac{\Delta}{2}$, the sum of the superfluity of $B - P$ and the empty space of $B - P^*$

$$\geq \left(G - \frac{5}{11} - \delta + \frac{5}{9}\Delta\right) + \left(\frac{13}{22} - G + \frac{\Delta}{2}\right) = \frac{3}{22} - \delta + \frac{19}{18}\Delta > 2\left(\frac{1}{44} - \frac{\Delta}{36}\right);$$

If $G \geq \frac{13}{22} + \frac{\Delta}{2}$, the superfluity of $B - P$

$$\geq \left(G - \frac{5}{11} - \delta + \frac{5}{9}\Delta\right) > 2\left(\frac{1}{44} - \frac{\Delta}{36}\right).$$

In either case the shortages of x_6 and y can be balaned by B.

(iii) Assume x_i ($i = 4$ or 6) falls into a G-bin $B = \{(G, x_j), (G, x_i, \cdot)\}$ with $j \geq 2$, and $B' = \{(G, x_i), (G, x', x'')\}$ is the bin from which x_i comes. By Lemma 2, we have $x' + x'' \leq \frac{1}{2}(1 - x)$. Thus we have $x'' = x_7$ and $x' = x_6$ or x_7. By Lemma 3 (with $k = 7$), x_i can be balanced by themselves.

(f) Now we consider x_3. By the definition of the weighting function, it may happen that $w(G + x_3) < 1 - x$. This happens only when $G < \frac{1}{2} + \delta - \frac{11}{18}\Delta$. In such a case, the maximal shortage is $\delta - \frac{11}{18}\Delta$. It is easy to check that for the optimal bin of such a G, the only possible combinations are (G, x_5, x_6), (G, x_6, x_6) and (G, x_6, x_7). In either case, its weight is $\leq 1 - \delta - 2\delta$.

Now we want to consider those pieces with shortage which fall into some non-G-bins. The possible worst combinations for an optimal non-G-bin and the corresponding total weights are listed in Table 5. From Cases (a)–(d) considered above and Lemma 6, we consider x_6's and x_4's only. Notice that, for a given list L, among x_4 and x_6 only one type can be pieces with shortage. From Table 5, we see that all optimal non-G-bins can provide enough room for the pieces with shortage which fall into it.

Let A be the sum of all shortages. (Modifications should be made for the special cases mentioned above. E.g., in Case (ii,b), what we add to A is not the shortage of x_6, but this shortage minus the superfluity of $B - P$). In the definition of the weighting function, there may be three bins: the bin B_1 between R_1 and R_2, B_2 between R_4 and R_5, and B_3 between R_6 and R_7, in which pieces come from different regions. E.g., B_1 may contain an x_1 and an x_2, etc. For B_1, B_2 and B_3, we define the weight of each piece in them equal to its size and call them irrgular pieces. If B_i has $i + 1$ pieces in it, we define the weight of each piece as those given in Table 4. There are at most 9 irregular pieces in total. When an irregular piece falls into an optimal bin, this bin may have a weight $1 = 1 - \delta + \delta$. Noticing that the last FFD-bin contains a piece, x, only, its weight $= \frac{2}{11} - \frac{2}{9}\Delta = \frac{9}{11} - \Delta - \frac{7}{11} + \frac{7}{9}\Delta$.

There may be a bin B_0 between R_3 and R_4, $B_0 = \{x_3, x_4, x_i\}$, $i \in \{4, 5, 6\}$, which may have a weight $\frac{9}{11} - \Delta - \phi$, if $i = 6$. For B_0, if $x_3 + x_4 \geq 1 - x - z + 3\delta$, we define the weight of each piece in B_0 as its size. If $x_3 + x_4 < 1 - x - z + 3\delta$, we have $x_3 + x_4 \leq \frac{2}{3}$, so that i can be 4 or 5. If no x_4 or x_5 exists, there is no B_2. We define the weight of each item in $B_0 = \{x_3, x_4, x_6\}$ as those given in Table 4, so that $w(x_3 + x_4 + x_6) = \frac{9}{11} - \Delta - \phi$. Thus we have

$$\left(\frac{9}{11} - \Delta\right) \mathrm{FFD}(L) - \frac{7}{11} + \frac{7}{9}\Delta$$
$$\leq w(L) + A \leq (1 - \delta)\mathrm{OPT}(L) + 6\delta + \begin{cases} 3\delta, & \text{if } B_2 \text{ exists}, \\ \phi, & \text{otherwise}. \end{cases} \tag{4}$$

If B_2 does not exist, (4) becomes

$$\left(\frac{9}{11} - \Delta\right) \mathrm{FFD}(L) \leq (1 - \delta)\mathrm{OPT}(L) + 6\delta + \phi + \frac{7}{11} - \frac{7}{9}\Delta.$$

It is easy to verify that $6\delta + \phi + \frac{7}{11} - \frac{7}{9}\Delta \leq \frac{9}{11} - \Delta$. Thus we have (1). Now we assume B_2, and therefore x_4 exists.

In the following we want to show that, if B_1 exists (otherwise we can omit 2δ from the righthand side of (4)), either we have a surplus δ on the lefthand side of (4), or we can omit a δ from the righthand side of (4). If $B_1 = \{x_1, x_2\}$, it means x_2 exists. From Table 5, we see that all optimal bins containing an x_2 has a room $\geq \delta$. So we can take one δ from the 9δ and put it into the optimal bin containing x_2, and then the righthand side of (4) becomes $(1 - \delta)\mathrm{OPT}(L) + 8\delta$. Let $B_1 = \{x_1, x_3\}$ or $\{x_1, x_4, \cdot\}$. For this x_1, we assume that the bin (x_1, x_4, x_4) is a possible combination in the OPT packing, otherwise every optimal bin containing x_1, has a room $\geq \delta$. Thus we have $x_1 \leq 1 - \frac{2}{3}(1 - x) = \frac{5}{11} + \frac{2}{3}\Delta$. When it is the turn of x_1 to be processed in the FFD packing, there are two possibilities: (i) no G left, i.e. all FFD G-bins are of form (G, x_1') which has a weight $> \frac{9}{11} - \Delta + \delta$, or (ii) all pieces G left are too large so that $G + x_1 > 1$, and therefore $G \geq \frac{6}{11} - \frac{2}{3}\Delta$. Thus we have $w(G + x_3) \geq \frac{9}{11} - \Delta + \delta$, $\forall x_3$. If no x_3 exists, we have $B_1 = \{x_1, x_4', x_i\}$, $i \in \{4, 5, 6, 7\}$, since $x_4' + x \leq \frac{2}{3}(1 - x) \leq x_4 + x_4$. In this case we define the weight of each piece in B_1 as those given in Table 4, It makes B_1 have a total weight $> \frac{9}{11} - \Delta + \delta$. Thus our assertion has been proved. And therefore (4) becomes

$$\left(\frac{9}{11} - \Delta\right) \mathrm{FFD}(L) - \frac{7}{11} + \frac{7}{9}\Delta \leq (1 - \delta)\mathrm{OPT}(L) + 8\delta.$$

From this (1) follows immediately since $\frac{7}{11} - \frac{7}{9}\Delta + 8\delta \leq \frac{9}{11} - \Delta$. Thus no counter-example to (1) exists.

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