Tighter bounds of the First Fit algorithm for the bin-packing problem

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In this paper, we present improved bounds for the First Fit algorithm for the bin-packing problem. We prove $C_{\text{FF}}(L) \leq 17/10 C^*(L) + 7/10$ for all lists $L$, and the absolute performance ratio of $\text{FF}$ is at most $12/7$.

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\section{1. Introduction}

In the classical one-dimensional bin-packing problem, we are given a sequence $L = (a_1, a_2, \ldots, a_n)$ of items, each with a size in $(0, 1]$. We are required to pack them into a minimum number of unit-capacity bins. An excellent survey of the research on this problem is available in [2].

The bin-packing problem was one of the earliest to use an approximation algorithm and worst case analysis. For a given list $L$ and algorithm $A$, let $C^A(L)$ denote the number of bins used when $A$ is applied to list $L$, and $C^*(L)$ denote the optimum number of bins for a packing of $L$. We will omit the mention of $L$ if there is no ambiguity. The \textit{asymptotic performance ratio} for $A$ is defined as

$$\inf \left\{ r \geq 1 \mid \text{for some } N > 0, \frac{C^A(L)}{C^*(L)} \leq r \text{ for all } L \text{ with } C^*(L) \geq N \right\}.$$  

The \textit{absolute performance ratio} for $A$ is defined as

$$\inf \left\{ r \geq 1 \mid \frac{C^A(L)}{C^*(L)} \leq r \text{ for all list } L \right\}.$$  

The bin-packing problem is also one of the few combinatorial optimization problems for which the asymptotic performance ratio and the absolute performance ratio of a given algorithm may not be the same.

For simplicity, we use $a_i$ to denote the size of item $a_i$. The \textit{content} of a bin $B$, which is the total size of items packed in it, is also denoted as $B$, when this causes no confusion. \textit{First Fit (FF for short)} and \textit{First Fit Decreasing (FFD for short)} are two fundamental algorithms for addressing bin-packing problems [6]. The \textit{FF} algorithm can be described as follows: When we are packing $a_i$, we place it in the lowest indexed bin whose current content does not exceed $1 - a_i$. Otherwise, we start a
new bin with \(a_i\), as its first item. Algorithm \(FFD\) first sorts the items in non-increasing order of their sizes and then performs \(FF\).

In Johnson’s pioneer work, he proved that \(C_{\text{FFD}}(L) \leq \frac{11}{9} C^*(L) + 4\) for all lists \(L\) [6]. Note that the asymptotic performance ratio cannot be smaller than \(\frac{11}{9}\) [7]. Later, the additive term was reduced to 3 by Baker [1], and 1 by Yue [11]. Recently, Dósa further reduced it to a tight value \(\frac{9}{10}\) [3]. The absolute performance ratio \(\frac{3}{2}\) of \(FFD\) was obtained by Simchi-Levi [9], and it is also tight since no polynomial time algorithm with absolute performance ratio less than \(\frac{3}{2}\) exists unless \(P = NP\) [5].

For the \(FF\) algorithm, Ullman proved \(C_{\text{FF}}(L) \leq \frac{17}{16} C^*(L) + 3\) for all lists \(L\) [10]. Here the asymptotic performance ratio is asymptotically tight, and there also exists such a list \(L\) that \(C^*(L) = 10\) and \(C_{\text{FF}}(L) = 17\) [7]. The additive term was reduced to 2 in [7], and \(\frac{9}{10}\) in [4]. To the author’s knowledge, no further improvement has been made since then. Simchi-Levi proved that the absolute ratio of \(FF\) is at most \(\frac{2}{3}\) [9], but the bound is not tight. Though \(FF\) has a larger worst case ratio than \(FFD\), it can be used for the online version, where the items arrive in some order and must be packed into a bin as soon as they arrive, without knowledge of the remaining items.

In this paper, we will give both a smaller additive term in the asymptotic performance ratio and a tighter absolute performance ratio of \(FF\). Section 2 gives some definitions and useful lemmas. In Section 3, we prove \(C_{\text{FF}}(L) \leq \frac{17}{10} C^*(L) + \frac{7}{10}\) for all lists \(L\). In Section 4, we prove that the absolute performance ratio of \(FF\) is at most \(\frac{12}{7}\). Thus the gap between upper and lower bounds of the absolute performance ratio decreases by more than 70%.

2. Preliminaries

We define some terminology for convenience. An item greater than \(\frac{1}{2}\) is called large while an item greater than \(\frac{1}{4}\) is called semilarge. The number of large items is denoted as \(L\). Note that a semilarge item also can be bigger than \(\frac{1}{2}\).

Let \(B^*\) be the set of bins used in the optimal packing, and \(B^{\text{FF}}\) be the set of bins used by \(FF\). If a bin \(B_j\) is opened before another bin \(B_k\) in the procedure of \(FF\), then we say that \(B_j\) is before \(B_k\) and \(B_k\) is after \(B_j\). When algorithm \(FF\) terminates, a bin containing exactly one item (two items) is called an i-bin (ii-bin). A bin containing no less than two (three, four) items is called a II-bin (III-bin, IV-bin). An item in an i-bin (ii-bin, II-bin, III-bin, IV-bin) is called an i-item (ii-item, II-item, III-item, IV-item). Let \(B_{\text{i}}(B_{\text{ii}}, B_{\text{ii}}, B_{\text{iii}}, B_{\text{iv}})\) be the set of i-bins (ii-bins, II-bins, III-bins, IV-bins), and \(N_i (N_{\text{ii}}, N_{\text{ii}}, N_{\text{iii}}, N_{\text{iv}})\) be the number of i-bins (ii-bins, II-bins, III-bins, IV-bins). Clearly,

\[
B^{\text{FF}} = B_{\text{i}} \cup B_{\text{ii}} = B_{\text{i}} \cup B_{\text{ii}} \cup B_{\text{iii}}
\]

and

\[
C^{\text{FF}} = N_i + N_{\text{ii}} = N_i + N_{\text{ii}} + N_{\text{iii}}.
\]

Lemma 2.1. \(C^* \geq N_i\).

Proof. Obviously, the total size of any two i-items exceeds 1. Otherwise, \(FF\) will not open a new bin for the item that arrived later. Accordingly, any two of them cannot be packed in one bin in the optimal packing. Hence \(C^* \geq N_i\). \(\Box\)

Lemma 2.2. Given an integer \(k \geq 1\), for any \(M \geq k + 1\), if there are \(M\) bins \(B_1, B_2, \ldots, B_M\) in \(B^{\text{FF}}\) such that each of them contains at least \(k\) items, then \(\sum_{i=1}^{M} B_i > \frac{km}{k+1}\).

Proof. Without loss of generality, assume \(B_i\) is before \(B_t\) for any \(1 \leq s < t \leq M\). For fixed \(B_i\) and \(B_t\), \(s < t\), consider \(k\) arbitrary items \(a_1, a_2, \ldots, a_k\) in \(B_i\). By \(FF\) we have \(B_i + a_j > 1\), \(j = 1, \ldots, k\). Summing the \(k\) inequalities, we get

\[
kB_i + B_t \geq kB_i + \sum_{j=1}^{k} a_j > k.
\]

We will prove the lemma by induction on \(M\). By (2), we have \(kB_i + B_{k+1} > k\), \(i = 1, \ldots, k\). Summing the \(k\) inequalities, we get \(k \sum_{i=1}^{k} B_i + kB_{k+1} > k^2\), i.e., \(\sum_{i=1}^{k+1} B_i > k^2\). The result is true for \(M = k + 1\). Suppose the result is true for \(M = j \geq k + 1\), i.e., \(\sum_{i=1}^{j} B_i > \frac{k^2}{k+1}\). By (2), we have \(kB_i + B_{j+1} > k\), \(i = 1, \ldots, j\). Summing the \(j\) inequalities, we get \(k \sum_{i=1}^{j} B_i + jB_{j+1} > jk\).

Combining this with the induction hypothesis, we have

\[
\sum_{i=1}^{j+1} B_i = \frac{j}{j+1} B_{j+1} + B_{j+1} + (j - k) \sum_{i=1}^{j} B_i > \frac{jk + (j - k)}{j} \frac{k^2}{k+1} = k(j+1)\frac{k+1}{k+1}.
\]

The result is also true for \(M = j + 1\). The lemma is thus proved. \(\Box\)
Corollary 2.1. (i) If $B \subseteq B^{FF}$ and $|B| \geq 2$, then $\sum_{B \in B} B > \frac{1}{2} |B|$. 
(ii) If $B \subseteq B_{ill}$ and $|B| \geq 3$, then $\sum_{B \in B} B > \frac{3}{4} |B|$. 
(iii) If $B \subseteq B_W$ and $|B| \geq 4$, then $\sum_{B \in B} B > \frac{4}{5} |B|$. 
(iv) If $B \subseteq B_{ill}$ and $|B| \geq 5$, then $\sum_{B \in B} B > \frac{5}{6} |B|$. 

3. The asymptotic performance ratio

We use the weighting function defined in [4], that is

$$W(x) = \begin{cases} 
\frac{6}{5} x, & 0 \leq x \leq \frac{1}{6}, \\
9 \frac{x - 1}{10}, & \frac{1}{6} < x \leq \frac{1}{3}, \\
6 \frac{x - 1}{10} + \frac{1}{3}, & \frac{1}{3} < x \leq \frac{1}{2}, \\
6 \frac{x - 1}{10}, & 2 \frac{1}{2} < x \leq 1. 
\end{cases} \tag{3}$$

Clearly, $W(x)$ is an increasing function and

$$W(x) \geq \frac{6}{5} x. \tag{4}$$

Moreover, $W(a) > 1$ if $a$ is a large item. The weight of bin $B$, $W(B)$, is defined as the total weight of the items packed in it.

Lemma 3.1 ([4]). For every bin $B$, $W(B) \leq \frac{17}{10}$. Moreover, if $B$ does not contain large items, then $W(B) \leq \frac{3}{2}$.

Let $\overline{W} = \sum_{a \in L} W(a)$ be the total weight of all items. By Lemma 3.1, we have

$$\overline{W} = \sum_{a \in L} W(a) = \sum_{B \in B^{*}} W(B) \leq \sum_{B \in B^{*}} \frac{17}{10} = \frac{17}{10} C^*.$$

Lemma 3.2 ([4]). $\overline{W} > C^{FF} - 1 + \sum_{B \in B^{FF}} \max(0, W(B) - 1)$. 

Let $C = \{B|B \in B^{FF}$ and $W(B) < 1\}$ and $m = |C|$. Label the bins in $C$ as $C_1, C_2, \ldots, C_m$ such that $C_j$ is before $C_i$ for any $1 \leq i < j \leq m$. For each $C_j \in C$, define $\alpha_j = \max(\alpha|\text{ for some } j. 1 \leq j < i, C_j = 1 - \alpha)$ with $\alpha_1$ taken to be 0.

Lemma 3.3 ([4]). If $m \geq 2$, then $\sum_{i=1}^{m-1} (1 - W(C_i)) \leq \frac{6}{5} \alpha_m$.

The main result of this section is as follows.

Theorem 3.1. For every list $L$, $C^{FF}(L) \leq \frac{17}{10} C^*(L) + \frac{7}{10}$. 

Proof. Suppose $C^{FF} > \frac{17}{10} C^* + \frac{7}{10}$, i.e., $C^{FF} > \frac{17}{10} C^* + \frac{4}{5}$. We have the following claims.

Claim 3.1. If $C^{FF} \geq \frac{17}{10} C^* + \frac{4}{5}$, then there does not exist any large item in II-bins.

Proof. Suppose there exists a large item $b_1$ in a II-bin $B'$. Since $B'$ is a II-bin, it contains another item $b_2$. If $B' \geq \frac{2}{3}$, by $b_1 > \frac{1}{2}$, (3) and (4), we have

$$W(B') \geq \left(\frac{6}{5} b_1 + \frac{4}{10}\right) + \frac{6}{5} (B' - b_1) = \frac{6}{5} B' + \frac{4}{10} \geq \frac{6}{5}.$$ 

Thus $W \geq C^{FF} - 1 + \left(\frac{6}{5} - 1\right) = C^{FF} - \frac{4}{5} \geq \frac{17}{10} C^* $ by Lemma 3.2, which contradicts (5). Therefore, $B' < \frac{2}{3}$ and $b_2 \leq B' - b_1 < \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

For any bin $B$ before $B'$, $B > \frac{2}{3}$ since $b_2 < \frac{1}{3}$, so $W(B) \geq \frac{9}{10} B > 1$ by (4). For any II-bin $B$ after $B'$, any item in $B$ is greater than $\frac{1}{3}$ as $B' < \frac{2}{3}$. Hence $W(B) \geq 2 W \left(\frac{1}{3}\right) = 1$. For any i-bin $B$ after $B'$, at most one bin does not containing large items. Recall that $B' < \frac{2}{3}$, the weight of $B$ is at least 1 if it contains a large item, and $W \left(\frac{1}{3}\right)$ otherwise. In other words, all bins except one in $B^{FF}$ have weight at least 1, and the remaining one is at least $W \left(\frac{1}{3}\right) < 1$. Therefore, by (5),

$$\frac{17}{10} C^* \geq \overline{W} = \sum_{B \in B^{FF}} W(B) \geq \sum_{i=1}^{c^{FF}-1} 1 + W \left(\frac{1}{3}\right) = C^{FF} - 1 + \frac{1}{2} = C^{FF} - \frac{1}{2} \geq \frac{17}{10} C^* + \frac{3}{10},$$

which is impossible. The claim is thus proved.

Claim 3.2. If $C^{FF} \geq \frac{17}{10} C^* + \frac{4}{5}$, then $l \leq N_l - 1$, where we recall that $l$ is the number of large items.
Proof. By Claim 3.1, all large items are packed in i-bins by FF. Since no two large items can be packed into the same bin, we have \( l \leq N_i \). Suppose \( l = N_i \); then any i-bin contains exactly one large item, and its weight is at least 1. That is to say, each bin in \( C \) is a II-bin. We distinguish two cases according to the value of \( m \).

Case 1. \( m \geq 2 \).

Note that \( \alpha_m \geq \frac{1}{6} \). Otherwise, by the definition of \( \alpha_m \) and (4), there exists \( C_j \in \mathcal{C}, 1 \leq j < m \), such that \( C_j = 1 - \alpha_m \) and \( W(C_j) \geq \frac{2}{5} C_j = \frac{2}{5}(1 - \alpha_m) \geq 1 \), which is a contradiction.

Since \( C_m \) is a II-bin, let \( b_1, b_2 \) be two items in \( C_m \). Clearly, \( b_1, b_2 \geq \alpha_m \). By Lemma 3.3, (4), (5) and \( \alpha_m > \frac{1}{6} \), we have

\[
\frac{17}{10} C^* \geq \frac{W = \sum_{B \in B} W(B) + \sum_{i=1}^{m-1} W(C_i) + W(C_m)}{B \notin C_m} \geq \sum_{i=1}^{m-1} (1 - W(C_i)) + W(C_m) \geq (C^F - 1) - \sum_{i=1}^{m-1} (1 - W(C_i)) + (W(b_1) + W(b_2)) \geq C^F - 1 - \frac{6}{5} \alpha_m + \frac{6}{5} \alpha_m + \frac{6}{5} \alpha_m = C^F - 1 + \frac{6}{5} \alpha_m > C^F - \frac{4}{5},
\]

which is impossible.

Case 2. \( m = 1 \).

If \( W(C_1) \geq \frac{1}{5} \), then by (5),

\[
\frac{17}{10} C^* \geq \frac{W = \sum_{B \in B} W(B) + W(C_1)}{B \notin C_1} > C^F - 1 + \frac{1}{5} = C^F - \frac{4}{5},
\]

which is a contradiction. Hence \( W(C_1) \leq \frac{1}{5} \) and thus \( C_1 \leq \frac{6}{5} \). If \( l \geq 1 \), then there exists an i-item \( a \geq \frac{5}{6} \) since \( N_i = l \geq 1 \) and \( C_1 \leq \frac{6}{5} \). Note that all bins except \( C_1 \) have weight at least 1. By (5),

\[
\frac{17}{10} C^* \geq \frac{W = \sum_{B \in B} W(B) + W(a) + W(C_1)}{B \notin C_1} > C^F - 2 + W \left( \frac{5}{6} \right) = C^F - \frac{3}{5},
\]

which is a contradiction. Then we know that \( l = 0 \), but this implies that \( W = \sum_{B \in B^*} W(B) \leq \frac{3}{5} C^* \) by Lemma 3.1, and finally yields \( C^F - 1 < \frac{W}{C^*} \leq \frac{2}{3} C^* \leq \frac{17}{10} C^* - \frac{1}{2} < \frac{17}{10} C^* - \frac{1}{2} \) by Lemma 3.2. That is impossible as well. Then Claim 3.2 follows.

Applying Lemma 2.1 and Claim 3.2, we obtain \( l \leq C^* - 1 \). In other words, there is at least one bin \( B' \in B^* \) which does not contain any large items. By Lemmas 3.1 and 3.2,

\[
C^F - 1 < \frac{W}{C^*} = \sum_{B \in B^* \setminus \{B'\}} W(B) + W(B') \leq \frac{17}{10} (C^* - 1) + \frac{3}{2} = \frac{17}{10} C^* - \frac{1}{2}.
\]

It follows that \( C^F \leq \frac{17}{10} C^* + \frac{1}{2} \). The proof of Theorem 3.1 is thus completed.

4. The absolute performance ratio

In this section, we prove that the absolute performance ratio of FF is no more than \( \frac{12}{7} \).

Lemma 4.1. If \( N_i \leq 1 \) or \( N_i \geq C^F - 2 \), then \( C^F \leq \frac{5}{3} C^* \).

Proof. As \( C^F = 1 \) when \( C^* = 1 \) and \( \frac{2}{3} C^* \geq 3 \) when \( C^* \geq 2 \), our assertion is straightforward if \( C^* = 1 \) or \( C^F \leq 3 \). Now assume \( C^* \geq 2 \) and \( C^F \geq 4 \).

If \( N_i \leq 1 \), then \( N_{\|} = C^F - N_i \geq 3 \). By Corollary 2.1, we have

\[
C^* \geq \sum_{a \in \mathcal{L}} a = \sum_{B \in B_1} B + \sum_{B \in B_2} B \geq \sum_{B \in B_1} B > \frac{2}{3} N_\| = \frac{2}{3} (C^F - N_i) \geq \frac{2}{3} (C^F - 1),
\]

i.e., \( 3C^* \geq 2C^F - 1 \). Recalling that \( C^F \geq 4 \), we have \( C^* \geq 3 \) and thus \( C^F \leq \frac{3}{2} C^* + \frac{1}{2} \leq \frac{5}{3} C^* \).
If $N_i \geq C_{FF}^2 - 2$, then by Lemma 2.1, $C_{FF} \leq N_i + 2 \leq C^* + 2$. If $C^* = 2$, then $C_{FF} = 4$ and $N_i = 2$ since we assume $C_{FF} \geq 4$. Consider the two i-items $b_1, b_2$ which clearly follow $b_1 + b_2 > 1$. Then

$$\sum_{b \in a_i} B = \sum_{a \in L} a - (b_1 + b_2) \leq C^* - (b_1 + b_2) < 1,$$

which contradicts $N_i = C_{FF} - N_i = 2$. Hence $C^* \geq 3$ and we obtain $C_{FF} \leq C^* + 2 \leq \frac{5}{3} C^*$. □

**Lemma 4.2.** If $C_{FF} \geq \frac{12}{10} C^*$, then $4N_i \geq 18C_{FF} - 27C^* - 1$.

**Proof.** Since $C_{FF} \geq \frac{12}{10} C^* > \frac{3}{5} C^*$, we obtain $N_i \geq 2$ and $N_i = C_{FF} - N_i \geq 3$ by Lemma 4.1. In view of Corollary 2.1, we have

$$C^* \geq \sum_{a \in L} a = \sum_{b \in a_i} B + \sum_{b \in a_{ii}} B \geq \frac{1}{2} N_i + \frac{2}{3} N_i$$

$$= \frac{1}{2} N_i + \frac{2}{3} (C_{FF} - N_i) \geq \frac{2}{3} C_{FF} - \frac{1}{6} N_i.$$

By Lemma 2.1, we further have $C^* > \frac{2}{3} C_{FF} - \frac{1}{5} C^*$, i.e., $C_{FF} < \frac{7}{4} C^*$, which is equivalent to

$$C_{FF} \leq \left[ \frac{7}{4} C^* \right] - 1.$$

(7)

Direct calculation shows that $\left[ \frac{7}{4} C^* \right] - 1 < \frac{12}{10} C^*$ when $C^* \leq 6$. Hence we assume $C^* \geq 7$ in the following. Note that (6) is equivalent to

$$N_i = C_{FF} - N_i > 9C_{FF} - 12C^* - 3N_i.$$

(8)

If $9C_{FF} - 12C^* - 3N_i \leq 2$, then $C_{FF} \leq \frac{4}{3} C^* + \frac{1}{3} N_i + \frac{2}{3} \leq \frac{5}{3} C^* + \frac{2}{3} 3C_{FF} < \frac{17}{10} C^*$ by Lemma 2.1 and $C^* \geq 7$, which contradicts $C_{FF} \geq \frac{12}{10} C^*$. Therefore,

$$N_i > 9C_{FF} - 12C^* - 3N_i \geq 3.$$

(9)

**Claim 4.1.** If $C_{FF} \geq \frac{12}{10} C^*$, then the last $9C_{FF} - 12C^* - 3N_i$ II-bins contain only similar large items.

**Proof.** Suppose there exists an item which is not similar in one of the last $9C_{FF} - 12C^* - 3N_i$ II-bins. Then the content of each of the first $N_i - (9C_{FF} - 12C^* - 3N_i) = 12C^* - 8C_{FF} + 2N_i$ II-bins is at least $\frac{3}{4}$. Combining this with $N_i \geq 2$, (9) and Corollary 2.1, we have

$$C^* > \frac{1}{2} N_i + \frac{3}{4} (12C^* - 8C_{FF} + 2N_i) + \frac{2}{3} (9C_{FF} - 12C^* - 3N_i) = C^*,$$

which is a contradiction. □

**Claim 4.2.** If $C_{FF} \geq \frac{12}{10} C^*$, then all i-items are similar large items.

**Proof.** Note that all the i-items are large except at most one. If the remaining one is not similar, then each of the remaining $N_i - 1$ i-items should be greater than $\frac{3}{4}$. Then by Corollary 2.1 and $C^* \geq 7$, we have

$$C^* \geq \frac{3}{4} (N_i - 1) + \frac{2}{3} N_i = \frac{3}{4} (N_i - 1) + \frac{2}{3} (C_{FF} - N_i) \geq \frac{2}{3} C_{FF} - \frac{3}{4} \geq \frac{2}{3} C_{FF} - \frac{3}{28} C^*.$$

which leads to $C_{FF} < \frac{91}{56} C^* < \frac{17}{10} C^*$. □

**Claim 4.3.** If $C_{FF} \geq \frac{12}{10} C^*$, then there are at most $3C^* - 2N_i + 1$ similar large II-items.

**Proof.** Consider the packing of similar large II-items in the optimal packing. Each of the $N_i - 1$ bins containing a large i-item can accommodate at most one similar large II-item. The bin containing the remaining i-item, which is similar by Claim 4.2, can accommodate at most two more similar II-items. Each of the remaining $C^* - N_i$ bins can accommodate at most three similar II-items. Consequently, there are at most $(N_i - 1) + 2 + 3(C^* - N_i) = 3C^* - 2N_i + 1$ similar large II-items. □

By Claims 4.1 and 4.3, we have $2(9C_{FF} - 12C^* - 3N_i) \leq 3C^* - 2N_i + 1$, i.e., $4N_i \geq 18C_{FF} - 27C^* - 1$. The lemma is thus proved. □

**Lemma 3.3.** If $N_i = C^*$, then $C^* \geq 2N_i$. 

Suppose there exists such a list. By Lemma 4.2

$$b_1 + b_2 + a \leq 1.$$ \hfill (10)

Then $b_1$ and $b_2$ are packed in different bins by $FF$. Otherwise, $FF$ will pack $a$, $b_1$ and $b_2$ together, contradicting the definition of an i-item. Let the two bins containing $b_1$, $b_2$ in $B^*$ be $B_1$, $B_2$ respectively, and $B_2$ be after $B_1$ without loss of generality. Since $b_1$ is a ii-item, there exists another item in $B_1$, say $b'_1$. Since $b_2$ is packed in a bin after $B_1$, we have

$$b_1 + b'_1 + b_2 > 1.$$ \hfill (11)

It follows that $b'_1$ is not in $B^*$. Let $a'$ be the i-item which is packed in the same bin with $b'_1$ in $B^*$; then

$$b'_1 + a' \leq 1.$$ \hfill (12)

Since $a$ and $a'$ are both i-items,

$$a + a' > 1.$$ \hfill (13)

Therefore, by (12), (13) and (10),

$$b_1 + b_2 + b'_1 \leq b_1 + b_2 + (1 - a') < b_1 + b_2 + a \leq 1,$$

which contradicts (11). \hfill $\Box$

Lemma 4.4. If $N_i = C^*$ and $C^{FF} > \frac{12}{7}C^*$, then

$$C^{FF} \leq \min \left\{ \frac{1}{9} \left\lceil \frac{C^*}{2} \right\rceil + \frac{5}{3}C^* - \frac{1}{9} \left\lceil \frac{C^*}{2} \right\rceil + C^* + 3 \right\}.$$

Proof. Given $C^* \leq 10$, we get $C^{FF} \leq \left\lceil \frac{2C^*}{7} \right\rceil - 1 \leq \frac{12}{7}C^*$ by (7) and direct calculation. Hence we can assume $C^* \geq 11$.

Moreover, given $C^{FF} \leq C^* + 7$, we get $C^{FF} \leq C^* + 7 \leq \frac{12}{7}C^*$ by $C^* \geq 11$. Then we can assume $C^{FF} \geq C^* + 8$ as well.

If $N_i \leq 2$, there are at least $C^{FF} - N_i - N_i \geq C^{FF} - C^* - 2 \geq 4$ III-bins. By Corollary 2.1, we have

$$C^* \geq \frac{3}{4}(C^{FF} - C^* - 2) + \frac{1}{2}(C^* + 2) = \frac{3}{4}C^{FF} - \frac{1}{4}C^* - \frac{1}{2},$$

i.e. $3C^{FF} < 5C^* + 2$. Hence $3C^{FF} \leq 5C^* + 1$ and thus $C^{FF} \leq \frac{5}{3}C^* + \frac{1}{3} \leq \frac{12}{7}C^*$. Therefore, we only need to consider the case when $N_i \geq 3$.

If $N_i \leq C^{FF} - C^* - 4$, then by Corollary 2.1 and $N_i = C^*$, we have

$$C^* \geq \frac{1}{2}N_i + \frac{1}{2}N_i - N_i = \frac{3}{4}(C^{FF} - N_i) = \frac{3}{4}C^{FF} - \frac{1}{4}C^* \geq \frac{1}{12}N_i,$$

i.e. $9C^{FF} < N_i + 15C^*$. Hence $9C^{FF} \leq N_i + 15C^* - 1$. Applying Lemma 4.3, we obtain $C^{FF} \leq \frac{1}{3}C^{FF} + \frac{2}{3}C^* - \frac{1}{3}$. If $N_i \geq C^{FF} - C^* - 3$, then $C^{FF} \leq \left( \frac{C^*}{2} \right) + C^* + 3$ by Lemma 4.3. Combining this with the two inequalities, the lemma is thus proved. \hfill $\Box$

Lemma 4.5. There is not such a list that

(i) $C^* = 11$ and $C^{FF} = 19$,

(ii) $C^* = 32$ and $C^{FF} = 55$,

(iii) $C^* = 39$ and $C^{FF} = 67$.

Proof. (i) Suppose there exists such a list. By Lemma 4.2, we have $4N_i \geq 18C^{FF} - 27C^* - 1 = 44 = 4C^*$. Therefore, $N_i = C^*$ by Lemma 2.1 and thus $N_i \leq 5$ by Lemma 4.3. According to (8), $N_i > 9C^{FF} - 12C^* - 3N_i = 6 \geq N_i + 1$. It follows that there is at least one III-bin in the last six II-bins. Hence, there are at least $2 \times (6 - 1) + 3 \times 1 = 16$ IV-bins in the last six II-bins, and these items are all semilarge by Claim 4.1. On the other hand, Claim 4.3 implies that there are at most $3C^* - 2N_i + 1 = 12$ semilarge II-items, which is a contradiction.

(ii) Suppose there exists such a list. By Lemma 4.2, we have $4N_i \geq 18C^{FF} - 27C^* - 1 = 125$. Therefore, by Lemma 2.1, $N_i = C^* = 32$ and thus $N_i = C^{FF} - N_i = 23$. Label all the II-bins as $B_1, B_2, \ldots, B_{23}$, so that $B_i$ is before $B_j$ for any $1 \leq i < j \leq 23$. The last $9C^{FF} - 12C^* - 3N_i = 15$ II-bins contain only semilarge II-items by Claim 4.1, but the number of semilarge II-items is no greater than $3C^* - 2N_i + 1 = 33$ by Claim 4.3. Then there are at most $33 - 15 \times 2 = 3$ semilarge items packed in the first $23 - 15 = 8$ II-bins. It follows that at least $8 - \left\lfloor \frac{7}{2} \right\rfloor = 7$ of the first eight II-bins contain not only semilarge items, and at least $8 - 5 = 3$ bins do not contain semilarge items. Let $B_1, B_2, \ldots, B_i$ be all the bins containing
not only semilarge items, where \( i_1 < i_2 < \cdots < i_t \) with \( t = 7 \) or 8. Choose five bins each of which does not contain any semilarge item, say \( B_1, B_2, \ldots, B_5 \), and \( j_1 < j_2 < \cdots < j_5 \). It is obvious that \( J = \{ j_1, j_2, \ldots, j_5 \} \subseteq \{ i_1, i_2, \ldots, i_t \} = I \).

If \( j_5 < i_t \), then \( \frac{B_{j_5}}{B_{i_t}} > \frac{3}{4} \) for \( 1 \leq k \leq 5 \) as \( B_{j_5} \) has items not greater than \( \frac{1}{4} \) in it. For \( 1 \leq k \leq 5 \), since \( B_{j_k} \) does not contain any semilarge item, it must be a IV-bin. Accordingly, by Corollary 2.1,

\[
32 = C^* > \frac{2}{3} \sum_{k=1}^{5} B_{j_k} + \frac{2}{3} (N_{ii} - 5) + \frac{1}{2} N_l > \frac{4}{5} \times \frac{9}{2} \times 5 + \frac{2}{3} \times 18 + \frac{1}{2} \times 32 = 32,
\]

which is a contradiction. Hence \( j_5 = i_t \). Similarly to before, the \( B_{j_k} \), \( 1 \leq k \leq 4 \), which do not contain any semilarge item, must be IV-bins since \( B_{j_k} \) has items not greater than \( \frac{1}{4} \) in it. In the light of \( t \geq 7 \), there exist \( s_1, s_2 \in I \setminus J \). \( B_{j_1}, B_{j_2} \) are before \( B_{j_3} \) since \( j_5 = i_t \). If \( B_{j_3} \) is a III-bin, then by (2)

\[
B_{j_1} + B_{j_2} + \frac{5}{3} B_{j_3} + \frac{2}{3} (N_{ii} - 7) > \frac{1}{2} N_l > \frac{27}{5} \times \frac{3}{2} \times 16 + \frac{1}{2} \times 32 > \frac{481}{15},
\]

but that will lead us to

\[
32 = C^* > B_{j_4} + B_{j_5} + \frac{5}{3} B_{j_6} + \frac{2}{3} (N_{ii} - 7) + \frac{1}{2} N_l > \frac{13}{5} \times \frac{13}{80} \times 4 + \frac{3}{16} \times 4 + \frac{13}{240} \times 3 + \frac{67}{240} \times 3 = \frac{27}{5},
\]

which is a contradiction. Therefore \( B_{j_6} \) must be a ii-bin, so its content is less than \( \frac{1}{4} \) since it does not contain any semilarge items. It follows that items in a II-bin after \( B_{j_6} \) are all large, which causes its content to be greater than 1.

(iii) Suppose there exists such a list. By Lemma 4.2, we have \( 4N_{ii} \geq 18C^* - 27C^* - 1 = 152 = 4C^* - 4 \). We distinguish two cases according to the value of \( N_{ii} \).

Case 1. \( N_{ii} = C^* = 39 \).

By Claims 4.1 and 4.3, the last 9C^* - 12C^* - 3N_{ii} = 18 II-bins contain only semilarge items, while the number of semilarge II-items cannot exceed 3C^* - 2N_{ii} = 40. Moreover, \( N_{ii} = C^* - N_{ii} = 28 \). Then there are at most \( 40 - 18 \times 2 = 4 \) semilarge items in the first \( 28 - 18 = 10 \) II-bins. Therefore, at least \( 10 - 4 = 6 \) of the first ten II-bins do not contain any semilarge items. Among these bins, the content of each of the first five bins is at least \( \frac{3}{4} \), and they must be IV-bins as a consequence. On the other hand, by Lemma 4.3, \( N_{ii} \leq 19 \) and thus \( N_{iii} \geq 9 \). Therefore, by Corollary 2.1.

\[
39 = C^* > \frac{4}{5} \times 5 + \frac{3}{4} (N_{iii} - 5) + \frac{2}{3} N_{ii} + \frac{2}{3} N_{ii} > \frac{4}{5} \times 5 + \frac{3}{4} \times 4 + \frac{2}{3} \times 19 + \frac{1}{2} \times 39 > \frac{235}{6},
\]

which is a contradiction. 

Case 2. \( N_{ii} = C^* - 1 = 38 \).

By Claim 4.1, there are at least \( 9C^* - 12C^* - 3N_{ii} = 21 \) II-bins, and thus at least \( 42 \) semilarge II-items. If all the 38 i-items are large, then each bin in \( 2^* \) containing an i-item can contain at most one semilarge II-item in the optimal packing. However, the remaining \( 42 - N_l = 4 \) semilarge II-items cannot be packed in the remaining \( C^* - N_l = 1 \) bin. Hence there exists an i-item which is not large, but it is still semilarge by Claim 4.2. Moreover, there aren’t any large II-items. Otherwise, there are at least 37 large i-items and one large II-item. Each of them can be packed with at most one semilarge II-item in the optimal packing. However, the remaining \( 42 - (37 + 1) - 1 = 3 \) semilarge II-items and one semilarge i-item cannot be packed in the remaining \( C^* - (37 + 1) = 1 \) bin, which is a contradiction. Therefore, we have \( l \leq N_l = 1 = C^* - 2 \). Thus by Lemmas 3.1 and 3.2, we have

\[
66 = C^* - 1 < 4W \leq \frac{17}{10} (C^* - 2) + \frac{2}{3} \times 4 = \frac{659}{10},
\]

which is a contradiction. \( \square \)

In order to get Theorem 4.1, we also need the following lemma concerning the diophantine equation.

**Lemma 4.6** ([8] Diophantine Equation). If \( a \) and \( b \) are coprime, \( u \) is an integer. The linear diophantine equation \( ax + by = u \) has infinitely many solutions. If the pair \( (x_0, y_0) \) is one integral solution, then all others are of the form

\[
x = x_0 + bu, \quad y = y_0 - au,
\]

where \( u \) is an integer.

**Theorem 4.1.** For every list \( I \), \( C^*(L) \leq \frac{12}{7} C^*(I) \).
Proof. If $C^{\text{FF}} \leq \frac{17}{10} C^*$ or $C^* \leq 10$, the result clearly follows by the previous discussion. We assume $C^{\text{FF}} > \frac{17}{10} C^*$ and $C^* \geq 11$ in the following. Let
\[ 31C^* - 18C^{\text{FF}} = u \] (14)
be a diophantine equation relating to $C^*$ and $C^{\text{FF}}$, where $u$ is an integer and
\[ u = 31C^* - 18C^{\text{FF}} \geq 27C^* + 4N_i - 18C^{\text{FF}} \geq -1 \] (15)
by Lemmas 2.1 and 4.2. Since $(7u, 12u)$ is a solution of (14), any integral solution of (14) can be written as
\[
\begin{align*}
C^* &= 7u + 18v, \\
C^{\text{FF}} &= 12u + 31v,
\end{align*}
\] (16)
by Lemma 4.6, where $v$ is an integer. Taking the expressions for $C^*$ and $C^{\text{FF}}$ in Theorem 3.1, this requires
\[ u + 4v \leq 7. \] (17)
When $u \geq 4$ we get $v \leq 0$ from (17), so by (16)
\[
\frac{C^{\text{FF}}}{C^*} = \frac{12u + 31v}{7u + 18v} = \frac{31}{18} - \frac{1}{18(1 + \frac{18}{u})} \leq \frac{31}{18} - \frac{1}{18 \times 7} = \frac{12}{7}. 
\]
When $u \leq 3$, due to (15) and (17), the possible pairs $(u, v)$ are
\[ (-1, 1), (-1, 2), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), \]
and the corresponding pairs $(C^*, C^{\text{FF}})$ are
\[ (11, 19), (29, 50), (18, 31), (7, 12), (25, 43), (14, 24), (32, 55), (21, 36), (39, 67). \]
Lemma 4.5 excludes the possibility of $(11, 19), (29, 50), (32, 55), (39, 67)$. For the pairs of $(29, 50), (18, 31), (25, 43)$, we have
\[ 4N_i \geq 18C^{\text{FF}} - 27C^* - 1 \geq 4C^* - 3, \]
by Lemma 4.2 and direct calculation. So $N_i = C^*$, and thus Lemma 4.4 implies that such a list will not exist. The remaining pairs all fulfill $C^{\text{FF}} = \frac{12}{7} C^*$. Then we complete the proof of Theorem 4.1. \[ \square \]

Since there exists such a list that $C^* = 10$ and $C^{\text{FF}} = 17$ [7], Theorem 4.1 shows that the gap between the lower and upper bounds of the absolute performance ratio of FF is less than 0.0143. We conjecture that the absolute performance ratio of FF is exactly $\frac{17}{10}$, which implies that the absolute performance ratio and asymptotic performance ratio of FF are identical. This is not common among bin-packing algorithms. To settle the conjecture, the first step is to determine whether there exists such a list that $C^* = 7$ and $C^{\text{FF}} = 12$, or not.

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