A new heuristic algorithm for the machine scheduling problem with job delivery coordination

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A B S T R A C T

In a rapidly changing environment, competition among enterprises has a tendency towards competing between supply chain systems instead of competing between individual companies. Traditional scheduling models which only address the sequence of jobs to be processed at the production stage under some criteria are no longer suitable and should be extended to cope with the distribution stage after production. Emphasizing on the coordination and integration among various members of a supply chain has become one of the vital strategies for the modern manufacturers to gain competitive advantages. This paper studies the NP-hard problem of the two-stage scheduling in which jobs are processed by two parallel machines and delivered to a customer with the objective of minimizing the makespan ($P_2 \rightarrow D, k = 1 | v = 1, c = z | C_{\text{max}}$). The proposed heuristic algorithm is shown to have a worst-case ratio of $63/40$, except for two particular cases.

1. Introduction

Traditional scheduling models only addressed the sequence of jobs to be processed at the production stage under some criteria. Nevertheless, it is no longer suitable and the models should be extended with transportation considerations to cope with the distribution stage after production. Lee and Chen [6] investigated machine scheduling models that impose constraints on both transportation capacity and transportation times, and discussed the computational complexity of various scheduling problems by either showing the NP-hardness or proposing polynomial algorithms for these problems. Chang and Lee [1] further studied the problems in which each job requires different physical space for delivery, whereas Li et al. [7] investigated a problem involving job deliveries to multiple customers at different locations. Lee and Chen [6], and Soukhali et al. [12] analyzed the complexity issues of a class of flow shop problems. He et al. [3] studied a class of single machine with two-stage scheduling problems proposed by Chang and Lee [1] and reduced the worst-case ratio from $5/3$ to $53/35$. Woeginger [13] studied parallel machine environment with equal job arrival times and proposed a heuristic method with worst-case analysis. Other related researches can be found in [10,14,9,2].

This paper focuses mainly on a class of two parallel machines’ problem, in which jobs need to be delivered to a single customer by a vehicle after their production stages. The problem was first proposed by Chang and Lee [1] and can be described as follows: There is a set of $n$ independent jobs, $N = \{j_1, j_2, \ldots, j_n\}$, to be processed without preemption at a manufacturing system consisting of two identical machines, $M_1$ and $M_2$. Each job $j_i, i = 1, 2, \ldots, n$, must be first processed in the manufacturing system by one of the two identical machines and has a processing time $p_i$, and the finished jobs are delivered in batches to the customer by a vehicle. Moreover, a job size $s_j$, which represents the physical space $j$ occupies
Table 1
A list of notation.

<table>
<thead>
<tr>
<th>Algorithm (procedure)</th>
<th>H2</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>LPT</th>
<th>Optimal schedule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schedule</td>
<td>$\sigma^{H2}$</td>
<td>$\bar{\sigma}$</td>
<td>$\bar{\alpha}^{H2}$</td>
<td>$\bar{\alpha}'$</td>
<td>$\bar{T}$</td>
<td>$\sigma^*$</td>
<td></td>
</tr>
<tr>
<td>The number of batches</td>
<td>$b^{H2}$</td>
<td>$b$</td>
<td>$b^{H2}$</td>
<td>$b$</td>
<td>$b^{H2}$</td>
<td>$b^*$</td>
<td></td>
</tr>
<tr>
<td>The makespan</td>
<td>$C^{H2}$</td>
<td>$\bar{C}$</td>
<td>$\bar{C}$</td>
<td>$\bar{C}'$</td>
<td>$\bar{T}$</td>
<td>$C^*$</td>
<td></td>
</tr>
<tr>
<td>The time when machines finish processing the last job</td>
<td>$C(M)$</td>
<td>$\bar{C}(M)$</td>
<td>$\bar{C}(M)$</td>
<td>$\bar{C}(M)$</td>
<td>$\bar{C}(M)$</td>
<td>$C(M)^*$</td>
<td></td>
</tr>
<tr>
<td>The total processing time of jobs in the first batch</td>
<td>$x$</td>
<td>$\bar{x}$</td>
<td>$\bar{x}$</td>
<td>$\bar{y}$</td>
<td>$\bar{y}$</td>
<td>$u$</td>
<td></td>
</tr>
<tr>
<td>The total processing time of jobs in the second batch</td>
<td>$y$</td>
<td>$\bar{y}$</td>
<td>$\bar{y}$</td>
<td>$y$</td>
<td>$v$</td>
<td>$w$</td>
<td></td>
</tr>
</tbody>
</table>

when being loaded in the vehicle, is associated with each job. There is only one vehicle initially located at the manufacturing system and available with a limited capacity $c$ representing the total physical space that the vehicle provides for one delivery at the manufacturing facility. Chang and Lee [1] also presented a polynomial time algorithm with a worst-case ratio of 2.

Zhong et al. [15] presented an improvement algorithm for the problem and reduced the worst-case ratio to 5/3. The purpose of this paper is to propose a new algorithm which leads to a best possible solution with a worst-case ratio of $63/40$, except for the two particular cases below where $C^{MH2}/C^* \leq 8/5$. Table 1 lists the notation defined by Zhong et al. [15] and used in this paper.

1. $b^* = 3$ and $\bar{b} = 4$, with $\bar{C}' = \bar{x}' + \bar{b}'T$ or $\bar{C}' = \bar{y}' + (\bar{b}' - 1)T$ (if it exists).
2. $b^* = 2$ and $\bar{b} = 3$, with $\bar{C}' = \bar{x}' + \bar{b}'T$ or $\bar{C}' = \bar{y}' + (\bar{b}' - 1)T$ (if it exists).

2. Problem: $P2 \rightarrow D, k = 1|v = 1, c = z|C_{\text{max}}$

This section considers the two-stage scheduling problem with a single machine and one customer area, or $P2 \rightarrow D, k = 1|v = 1, c = z|C_{\text{max}}$. Let $P$ be the total processing time of all the jobs and $t$ be the one-way transportation time between the machine and the customer. Hence, each delivery has the same transportation time of $T = 2t$.

This problem was first solved by Chang and Lee [1]. They proposed an algorithm $H2$ with a worst-case ratio of 2. Zhong et al. [15] stated that there are two points preventing the worst-case ratio of $H2$ from getting better. The first point is that $H2$ ignores the processing times of jobs when assigning jobs to batches and the second is that $H2$ does not take the idleness of the other machine into account when assigning jobs in one batch to a machine as a whole. Therefore, they proposed an algorithm $MH2$ to improve on these two points, and the algorithm includes procedures $A$ and $B$ that are executed depending on the values of $b^{H2}$ and $C^{H2}$. The worst-case ratio of $MH2$ is $5/3$.

Let the makespan of $\sigma$ be $\bar{C}$ and the time the machines finish processing the last job be $C(\bar{M})$. On the other hand, let the resulting schedule be $\sigma^{H2}$ with makespan $C^{MH2}$ and the time the machines finish processing the last job be $C(M)$. Next, consider one point which prevents the worst-case ratio of $MH2$ from being better. Algorithm $MH2$ assigns batches to machines according to the shortest processing time ($SPT$) rule; hence, the loads over the two machines may not be well balanced and lead to a larger makespan. The longest processing time ($LPT$) rule is a method developed for $Pm || C_{\text{max}}$ problem [8] and has the effect of balancing the load among various machines. Furthermore, since the $P2||C_{\text{max}}$ problem is a special case of the $Pm || C_{\text{max}}$ problem, we can apply the $LPT$ rule to the problem and then reverse the sequence batches assigned on each machine such that the batches are sorted in the increasing order of their processing times, and the loads over the two machines may be better balanced.

3. A new algorithm $MH3$

This section describes a heuristic algorithm $MH3$ for solving the problem under study.

Algorithm $MH3$

Step 1: If $b^{H2} = 3$ or $b^{H2} = 4$, run procedure $C$; otherwise, run procedure $D$.

Step 2: Output $C^{MH3}$, stop.

Procedure $C$:

Step 1. Construct an instance of the knapsack problem as follows: for each job $j$, $j = 1, 2, \ldots, n$, construct an item with profit $p_j$ and size $s_j$, and let the knapsack capacity be $z$. Run any $FPTAS$ for the knapsack problem with $\varepsilon = 1/5$, and denote by $N_1$ the set of items put into the knapsack.

Step 2. Assign all jobs in $N_1$ to the same batch and assign other jobs to batches by algorithm $FF$ (First Fit). Let the total number of the resulting batches be $\bar{b}'$.

Step 3. Define $P_k$ as the total processing time of the jobs in the $k$th batch, $k = 1, 2, \ldots, \bar{b}'$, and denote the $k$th batch as $B_k$.

Step 4. Re-index these batches so that $P_1 \geq P_2 \geq \cdots \geq P_{\bar{b}'}$. Jobs in each batch can be sequenced in any arbitrary order and let $S = \{B_1, B_2, \ldots, B_{\bar{b}'}\}$.
Step 5. Let $S_1$ and $S_2$ be two sets of batches, which are both initially empty. Let $\overrightarrow{P}_{S_1}$ and $\overrightarrow{P}_{S_2}$ be the periods of time for processing all the batches in $S_1$ and $S_2$, respectively.

Step 6. Set $i = 1$ and $k = 0$.

Step 7. Set $i = i + k$. If $i > B'$, go to step 10.

Step 8. Put batch $B_i$ into the set with smaller total processing time between $S_1$ and $S_2$. Move $B_i$ to be the first batch of the set and eliminate it from $S$.

Step 9. Set $k = k + 1$ and go to step 7.

Step 10. Denote the sequences in $S_1$ and $S_2$ as $\sigma_1'$ and $\sigma_2'$, respectively.

Step 11. Assign batches one by one to machine 1 and machine 2 according to $\sigma_1'$ and $\sigma_2'$, respectively, except for batch $B_1$.

Step 12. Assign the jobs in batch $B_1$ one by one to machines 1 and 2 according to the LPT rule.

Step 13. Dispatch each completed but undelivered batch (all jobs in the same batch are assigned to the same machine, except batch $B_1$) whenever the vehicle becomes available. If multiple batches have been completed when the vehicle becomes available, dispatch the batch with the earliest completed.

Step 14. Let the obtained makespan be $C^{\text{MH3}}$ and the time machines finish processing the last job be $\overrightarrow{C(M)}$.

**Remark 3.1 ([15]).** The jobs corresponding to the items in $N_1$ are assigned to the same batch by algorithm FF in Step 2 of procedure $A$.

**Remark 3.2 ([15]).** For the knapsack problem, among others, Lawler [5] proposed an FPTAS with a time complexity of $O(n \log(1/\varepsilon) + 1/\varepsilon^4)$, where (1 – $\varepsilon$) is the worst-case ratio. Kellerer and Pferschy [4] also proposed an FPTAS with a time complexity of $O(n \min\{\log n, \log(1/\varepsilon)\} + (1/\varepsilon^2) \min\{n, (1/\varepsilon) \log(1/\varepsilon)\}$.

Procedure $D$: 

Step 1: Assign jobs into batches using FFD (First Fit Decreasing) rule. Set the total number of the resulting batches to be $b$.

Step 2: Define $P_k$ as the total processing time of the jobs in the $k$th batch, $k = 1, 2, \ldots, b$, and denote the $k$th batch as $B_k$.

Step 3: Re-index these batches such that $P_0 < P_{b-1} \leq \cdots \leq P_1$. Jobs in each batch can be sequenced in any arbitrary order, and let $S = \{B_1, B_2, \ldots, B_b\}$.

Step 4: Let $S_1$ and $S_2$, be the sets of batches, which are both initially empty. Let $P_{s1}$ and $P_{s2}$ be the periods of time for processing all the batches in $S_1$ and $S_2$, respectively.

Step 5: Set $i = 1$ and $k = 0$.

Step 6: Let $i = i + k$. If $i > b$, go to step 9.

Step 7: Put batch $B_i$ into the set with smaller total processing time between $S_1$ and $S_2$. Move $B_i$ to be the first batch of the set and eliminate it from $S$.

Step 8: Set $k = k + 1$ and go to step 6.

Step 9: Denote the sequences in set $S_1$ and $S_2$ as $\overrightarrow{\sigma_1}$ and $\overrightarrow{\sigma_2}$, respectively.

Step 10: Assign batches one by one to machines 1 and 2 according to $\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}$, respectively, except for batch $B_1$.

Step 11: Assign the jobs in batch $B_1$ one by one to machines 1 and 2 according to the LPT rule.

Step 12: Dispatch each completed but undelivered batch (all jobs in the same batch are assigned to the same machine, except batch $B_1$) whenever the vehicle becomes available. If multiple batches have been completed when the vehicle becomes available, dispatch the batch with the earliest completed.

Step 13: Let the resulting makespan be $C^{\text{MH3}}$ and the time machines finish processing the last job be $\overrightarrow{C(M)}$.

Both procedures $C$ and $D$ schedule batches by LPT rule. Therefore, it follows that $C(M)^{\leq} \geq \overrightarrow{C(M)}$ and $C(M)^{\leq} \geq \overrightarrow{C(M)}$ since the sequences generated are merely different in the composition of batches and $S$.

Next, three examples are presented to illustrate the procedure of the proposed heuristic algorithm.

**Example 3.1.** Consider the instance given in [15]. Let $T = \delta, z = 2, s_1 = s_2 = s_3 = s_4 = 1, p_1 = p_2 = \delta$ and $p_3 = p_4 = 1$.

First run $H2$. We get $b^{H2} = 2, B_1 = \{J_1, J_2\}, B_2 = \{J_3, J_4\}$, $P_1 = 2\delta$ and $P_2 = 2$. The jobs are processed and delivered as shown in Fig. 1.

Next, run $MH2$. From algorithm $H2$, we obtain $C(M) = 2$ and $C^{H2} = C(M) + T = 2 + \delta$. Since $C^{H2} = C(M) + T$, run procedure $B$ and we have $B_i = \{J_4\}$ and $\overrightarrow{C} = 1 + 3\delta$. The jobs are processed and delivered as shown in Fig. 2.

Then, we run algorithm $MH3$ and get $b^{MH} = 2, B_1 = \{J_1, J_2\}, B_2 = \{J_3, J_4\}$, $P_1 = 2\delta, P_2 = 2$ and $C^{MH3} = 1 + 3\delta$. The jobs are processed and delivered as illustrated in Fig. 2. According to Zhong et al. [15], it follows that $C^* = 1 + 2\delta$.

Consequently, we have $C^{MH3}/C^* = (1 + 3\delta)/(1 + 2\delta) < 63/40$, while $C^{MH2}/C^* < 63/40$ and $C^{MH2}/C^* < 2$.

**Example 3.2.** Consider the instance given in [15]. Let $T = 2, z = 7, s_1 = s_2 = 3, s_3 = s_4 = s_5 = s_6 = 2, p_1 = p_3 = 1, p_2 = p_4 = p_5 = \delta$ and $p_6 = 2$.

First run $H2$. We get $b^{H2} = 3, B_1 = \{J_1, J_2\}, B_2 = \{J_3, J_4, J_5\}, B_3 = \{J_6\}$, $P_1 = 1 + \delta, P_2 = 1 + 2\delta, P_3 = 2$ and $C^{H2} = C(M) + T = 7 + \delta$. The jobs are processed and delivered as shown in Fig. 3.

Next, run $MH2$. Since $b^{MH} = 3$ and $C^{H2} \neq C(M) + T, MH2$ goes to step 4 and executes procedure $A$. According to Zhong et al. [15], $B = 2$ and $B_1 = \{J_2, J_4, J_5\}, B_2 = \{J_1, J_3, J_6\}, P_1 = 3\delta, P_2 = 4$ and $C^{MH3} = \min\{C^{H2}, C\} = 7 \delta, 6\delta = 6$. The jobs are processed and delivered as depicted in Fig. 4.
Then, run algorithm $MH_3$. Since $b_{H_2} = 3$, $MH_3$ goes to step 1 and executes procedure $C$. It follows that $B' = 2, B_1' = \{J_1, J_2\}, B_2' = \{J_3, J_4\}, P_1' = 2\delta, P_2' = 2$ and $C_{MH_3} = 4 + 3\delta$. The jobs are processed and delivered as illustrated in Fig. 5. According to Zhong et al. [15], $C^* = 4 + 2\delta$.

Consequently, we have $C_{MH_2}/C^* = (4 + 3\delta)/(4 + 2\delta) \approx 1$, while $C_{MH_2}/C^* \leq 3/2$ and $C_{H_2}/C^* \leq 7/4$.

**Example 3.3.** Consider the instance given in [15]. Let $T = 6, z = 7, s_1 = s_2 = s_3 = s_4 = 3, s_5 = s_6 = s_7 = s_8 = 2, s_9 = 1, p_1 = p_4 = p_8 = 6, p_2 = p_3 = \delta, p_5 = p_6 = p_7 = 2$ and $p_9 = 2\delta$.

First run $H_2$. We have $b_{H_2} = 4, B_1 = \{J_8\}, B_2 = \{J_5, J_6, J_7\}, B_3 = \{J_3, J_4\}, B_4 = \{J_1, J_2, J_9\}, P_1 = 6, P_2 = 6, P_3 = 6 + \delta, P_4 = 6 + 3\delta$ and $C_{H_2} = x + kT = 6 + 4 \times 24 = 30$. The sequences of jobs processed and delivered are shown in Fig. 6.

Next, run $MH_2$. Since $b_{H_2} = 3$ and $C_{H_2} \neq C(M) + T$, $MH_2$ goes to step 2 and output $C_{H_2}$. Namely, $C_{MH_2} = C_{H_2}$. 

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**Fig. 1.** The results of Example 3.1 scheduled by algorithm $H_2$.

**Fig. 2.** The results of Example 3.1 scheduled by algorithm $MH_2$ or $MH_3$.

**Fig. 3.** The results of Example 3.2 scheduled by algorithm $H_2$.

**Fig. 4.** The results of Example 3.2 scheduled by algorithm $MH_2$. 

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Then, we run algorithm \( MH3 \). Since \( b^{H2} = 4 \), \( MH3 \) goes to step 1 and runs procedure \( C \). It follows that \( \overline{b} = 4 \), \( \overline{B_1} = \{J_8, J_1, J_3\} \), \( \overline{B_2} = \{J_4, J_6, J_7\} \), \( \overline{B_3} = \{J_9, J_2, J_3\} \), \( \overline{P_1} = 14 \), \( \overline{P_2} = 10 \), \( \overline{P_3} = 4\delta \) and \( C^{MH3} = 22 + 4\delta \). The jobs are processed and delivered as shown in Fig. 7.

According to Zhong et al. [15], we have \( C^* = 18 + 2\delta \). Consequently, it follows that \( C^{MH3}/C^* = (22 + 4\delta)/(18 + 2\delta) \leq 63/40 \), while \( C^{H2}/C^* \leq 5/3 \) and \( C^{H2}/C^* \leq 5/3 \).

Next, we will prove that except for certain cases, the worst-case performance of algorithm \( MH3 \) is 63/40.

**Theorem 1.** For \( MH3 \), we have \( \overline{C(M)}/C(M)^* \leq 107/80 \), and \( \overline{C(M)} / C(M)^* \leq 107/80 \).

**Proof.** Consider the batches scheduled by \( LPT \) rule, and regard a batch as a job for simplicity. Hence, the problem can be treated as a \( P2\parallel C_{\text{max}} \) [8] and proved by discussing the following conditions:

1. \( P_1 \leq \sum_{i=2}^{b^{H2}} P_i \) and the shortest batch is the last one to finish its processing.

   By contradiction, assume that there exists one or more counterexamples where the ratios are strictly larger than 107/80. If more than one such counterexample exists, there must be a problem which has the smallest number of jobs, say \( n \) jobs. Since \( C(M)^* \geq (\sum_{i=1}^{n} P_i)/2 \), we have

   \[
   C(M)^* \leq \frac{\sum_{i=1}^{n} P_i}{2} + \frac{\sum_{i=1}^{n} P_i}{2} \leq \frac{1}{2} P_n + C(M)^*.
   \]

   The following series of inequalities holds for the counterexample:

   \[
   \frac{107}{80} < \frac{C(M)^*}{C(M)} \leq \frac{1}{2} P_n + C(M)^* \leq 1 + \frac{P_n}{2C(M)^*}
   \]

   \[
   27 \leq \frac{P_n}{2C(M)^*} \implies \frac{27}{40} C(M)^* < P_n \implies \frac{27}{80} (P_1 + P_2 + \cdots + P_n) < P_n.
   \]

   According to Algorithm \( MH3 \), we have \( P_1 \geq P_2 \geq \cdots \geq P_n \). Therefore, for the smallest counterexample, it implies that the scheduling can result in at most two jobs, namely, \( P_1 \) and \( P_n \), i.e., \( P_1 \geq P_n > 27/80P \) and \( P_1 < 53/80P \).

   Thus, we have \( \overline{C(M)} / C(M)^* \leq \overline{C(M)} / C(M)^* \leq \frac{53/80P}{((1/2)P)} < 107/80 \).

2. \( P_1 \leq \sum_{i=2}^{b^{H2}} P_i \) and the shortest batch is not the last one to finish its processing.

   Delete the shortest batches until a shortest batch is the last one to finish its processing. Consequently, \( C(M)^* \) remains the same while \( C(M)^* \) may remain the same or decrease. From part (1), we have \( \overline{C(M)} / C(M)^* \leq \overline{C(M)} / C(M)^* \leq \frac{1}{2} P_1/((1/2)P) = \frac{53/80P}{((1/2)P)} < 107/80 \).
Let $\text{Fig. 7}$ be the results of Example 3.3 scheduled by algorithm MH3.

(3) $P_1 > \sum_{i=2}^{n/2} P_i$.

There are two cases to be considered and they are as follows:

**Case 1.** If batch 1 has only one job, then it is obvious that the schedule is optimal.

**Case 2.** If batch 1 has more than one job, then from parts (1) and step 11 of procedure $D$, we have $C(M) \leq \left( \sum_{i=2}^{n/2} P_i \right) / 2 + \left( \sum_{i=2}^{n/2} p_{1i} \right)/2 + p_{11}$ where $p_{11} = \min\{p_{12}, p_{13}, \ldots, p_{1n}\}$. It follows that

$$C(M)/C(M)^* \leq \left( \left( \sum_{i=2}^{n/2} P_i + \sum_{i=2}^{n/2} p_{1i} \right)/2 + p_{11}/2 \right) \leq 4/3 < 107/80.$$

Therefore, we obtain $C(M)/C(M)^* \leq 107/80$. In a similar way, we can obtain $C(M)/C(M)^* \leq 107/80$. □

**Lemma 3.1** ([15]). For an instance $I$ of the bin-packing problem, let $OPT(I)$, $FF(I)$, $FFD(I)$ be the number of used bins in an optimal solution, the solutions yielded by FF and FFD, respectively. We have

1. ([11]). $FF(I) \leq (7/4)OPT(I)$.
2. ([14]). $FFD(I) \leq (11/9)OPT(I) + 1$.

**Lemma 3.2.** Denote $d$ as the time period while there is only one machine processing jobs. Thus $d = C(M) - (P - C(M))$, and let the batch completed last be $B_k$. For any batch $k$, if $P_{B_k} \leq P_k$, then $d \leq P_k$, $k = 1, 2, \ldots$.

**Proof.** Consider the following two conditions:

1. The shortest batch is the last one to finish its processing. Recall that $P_1 \geq P_2 \geq \cdots \geq P_n$. Since the shortest batch is the last one to finish, it follows that $d \leq P_n \leq P_k$, $k = 1, 2, \ldots$.
2. The shortest batch is not the last one to finish its processing. From part (1), we have $d \leq P_{B_k}$. Moreover, since $P_{B_k} \leq P_k$, hence, $d \leq P_k$, $k = 1, 2, \ldots$. □

**Lemma 3.3.** Let $\sigma_1^j$ and $\sigma_2^j$ denote the inverse sequences of $\sigma_1$ and $\sigma_2$ on machines 1 and 2, respectively. For any two batches $h$ and $k$ processed on different machines, let $\mu_h^j$ and $\mu_k^j$ denote the times machines start processing batch $h$ and $k$ under $\sigma_1^j$ and $\sigma_2^j$, respectively. If $\rho_h + d \leq \rho_h$, then $P_k \leq P_h$.

**Proof.** By contradiction, assume that there exist one or more counterexamples where $P_k$ is larger than $P_h$ and $\rho_k + d \leq \rho_h$, as shown in Fig. 8. There are two cases to be considered and they are as follows:

1. Batch $B_k$ is processed on the machine which completes all its processing first. In this case, we have $\mu_k^j = C(M) - \rho_k - d$ and $\mu_h^j = C(M) - \rho_h$. Since $\rho_h + d \leq \rho_h$, we have $\mu_k^j \geq \mu_h^j$.
2. Batch $B_k$ is not processed on the machine which completes all its processing first. In this case, we have $\mu_k^j = C(M) - \rho_k$ and $\mu_h^j = C(M) - \rho_h - d$. Since $\rho_k + d \leq \rho_h$, we have $\mu_k^j \geq \mu_h^j$.

According to algorithm $MH3$, both $\sigma_1^j$ and $\sigma_2^j$ are scheduled by LPT; therefore, $P_k$ should be less than or equal to $P_h$, and batch $h$ starts processing earlier than batch $k$. This contradiction completes the proof of the lemma. □

**Lemma 3.4.** If there exists a batch $B_k$ such that $\delta_k = \rho_k$ and $k \geq 3$ under algorithm $MH3$, then at least one of the equations $\delta_{k-1} = \rho_{k-1}$, $\delta_{k-1} = \rho_{k-1}$, or $\delta_{k-2} = \rho_{k-2}$ holds, for $i = 1, 2, \ldots, k - 2$.

**Proof.** Note that $\rho_{k+1} \leq \rho_k \leq \rho_{k-1}$. The lemma can be proved by considering the various conditions of batches $B_{k-1}, B_k$ and $B_{k+1}$ processed on the machines:

1. Batches $B_{k-1}, B_k$ and $B_{k+1}$ are all processed on the same machine, while another batch, say $B_n$, is processed on the other machine.
   Since $\delta_k = \rho_k$, it follows that $\delta_{k+1} > \rho_{k+1}, \delta_{k+2} > \rho_{k+2}, \ldots, \delta_k = \rho_k$. Consequently, we have $\rho_k - \rho_{k-1} > T$, $P_{k-1} > T$, and hence $\delta_{k-1} = \rho_{k-1}$. 

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**Fig. 7:** The results of Example 3.3 scheduled by algorithm MH3.
(2) Both the batches $B_k$ and $B_{k-1}$ are processed on the same machine, while $B_{k+1}$ is processed on the other one. Since $\delta_k = \rho_k$ and $B_{k+1}$ is processed on the other machine, from Chang and Lee [1], it follows that $P_k > 2T$. Therefore, we have $\delta_{k-1} = \rho_{k-1}$.

When batches $B_k$ and $B_{k-1}$ are not processed on the same machine, it is necessary to consider the various conditions of batches $B_{k-2}$ and $B_{k-3}$ being processed as follows:

(3) Batches $B_{k-1}$ and $B_{k-2}$ are processed on the same machine, while $B_k$ and $B_{k+1}$ are processed on the other one.

If $\rho_{k-1} - \rho_k \geq T$, then $\delta_{k-1} = \rho_{k-1}$.

If $\rho_{k-1} - \rho_k < T$, then from part (2), we know that $P_{k-1} > 2T$ and thus $P_{k-2} > 2T$. Therefore, it follows that $\delta_{k-2} > \rho_{k-2}$.

(4) Batches $B_{k+1}$, $B_k$ and $B_{k-2}$ are processed on the same machine, while $B_{k-1}$ is processed on the other one.

If $\rho_{k-1} - \rho_k \geq T$, then $\delta_{k-1} = \rho_{k-1}$.

If $\rho_{k-2} - \rho_k \geq 2T$, then $\delta_{k-2} = \rho_{k-2}$.

If $\rho_{k-1} - \rho_k < T$, or $\rho_{k-2} - \rho_k < 2T$, then there are two cases to be considered:

Case 1: $B_{k-3}$ and $B_k$ are not processed on the same machine as shown in Fig. 9.

Since $\rho_{k+1} < \rho_k < \rho_{k-1}$, then we have $P_{k-3} < P_{k-1} > 3T$ from part (2). Therefore, it follows that $\delta_{k-3} = \rho_{k-3}$.

Case 2: $B_{k-3}$ and $B_k$ are processed on the same machine as shown in Fig. 10.

Since $\rho_k = \delta_k$, we have $P_k \geq P_{k-1} > T$, $P_{k-1} \geq 3T$, and $P_k \geq d$. Meanwhile, from Lemmas 3.2 and 3.3, since batches $B_{k-1}$ and $B_{k-3}$ are processed on different machines and $P_{k-3} \geq P_{k-2} > 2T$, we can obtain $\rho_{k-1} + d \leq \rho_{k-1} + P_{k-3} \leq \rho_{k-3}$ and it follows that $3T < P_{k-1} < P_{k-3}$. Thus we have $\delta_{k-3} = \rho_{k-3}$.

(5) $B_k$ and $B_{k-2}$ are processed on the same machine and $B_{k-1}$, $B_{k+1}$ are processed on the other one.

If $\rho_{k-1} - \rho_k \geq T$, then $\delta_{k-1} = \rho_{k-1}$.

If $\rho_{k-1} - \rho_k < T$, then according to part (2), we know that $P_k > 2T$. Thus $P_{k-2} > 2T$ and it follows that $\delta_{k-2} = \rho_{k-2}$.
Lemma 3.5. If $\rho_{k-1} \leq T$, then $\delta_{k-1} = \rho_{k-1}$.

Proof. If $\rho_{k-1} < T$, then from Lemmas 3.2 and 3.3, since batches $B_k$ and $B_{k-2}$ are processed on different machines, we have $\rho_k + d \leq \rho_k + P_{k-2} \leq \rho_{k-2}$. It follows that $P_k \leq P_{k-2}$. Meanwhile, from part (2), we also know that $2T < P_k$, thus we have $\delta_{k-2} = \rho_{k-2}$.

Therefore, we see that if $\delta_k = \rho_k$, then $\delta_{k-1} = \rho_{k-1}$, or $\delta_{k-i-1} = \rho_{k-i-1}$, or $\delta_{k-i-2} = \rho_{k-i-2}$ holds, for $i = 1, 2, \ldots, k-2$. □

Lemma 3.6 ([1]). $C^* \leq \max\{C(M)^* + T, u + b^*T\}$.

Lemma 3.7. For $b^{12} \neq 3$ and 4, if $\bar{C} = x + b^{12}T$ or $\bar{C} = y + (b^{12} - 1)T$, then $C^{MH3}/C^* \leq 3/2$.

Proof. Recall that $b^{12} \leq (11/9)b^*_L + 1 < (11/9)b^* + 1$, and Zhong et al. [15] showed that:

\begin{equation}
\frac{x + b^{12}T}{C^*} < \frac{2}{b^{12}} + \frac{11}{9} b^{12} - \frac{2}{b^{12} - 1} \quad \text{or} \quad < \frac{2}{b^{12}} + \frac{b^{12} - 2}{b^*} \tag{1}
\end{equation}

\begin{equation}
\frac{y + b^{12}T}{C^*} < \frac{2}{b^{12} - 1} + \frac{11}{9} b^{12} - 1 - \frac{2}{b^{12} - 1} \quad \text{or} \quad < \frac{2}{b^{12} - 1} + \frac{b^{12} - 1 - b^{12} - 1}{b^*} \tag{2}
\end{equation}

(3) If $b^{12} = 5, 6$, or 7, then $C^{12}/C^* < 63/40$.

Zhong et al. [15] rewrote Eqs. (1) and (2) as

\begin{equation}
f(x) = \frac{2}{x} + \frac{11}{9} x - \frac{b^{12}}{1 - x} \tag{3}
\end{equation}

\begin{equation}
g(x) = \frac{2}{x} + \frac{11 x - 2}{9} \tag{4}
\end{equation}

From Eqs. (3) and (4), we can verify that $f(9) < 63/40$, $f'(x) < 0$ for $x \geq 9$ and $g(8) < 63/40$, $g'(x) < 0$ for $x \geq 8$.

(3) If $b^{12} = 5, 6$, or 7, then it follows from Eq. (1) that $(x + b^{12}T)/C^* < 2/7 + (7 - 2/7)/6 < 3/2$, and from Eq. (2) that $(y + (b^{12} - 1)T)/C^* < 2/6 + (6 - 2/6)/6 < 3/2$.

(4) On the other hand, if $b^{12} = 8$ and $b^* \geq 7$, it follows from Eq. (1) that $(x + b^{12}T)/C^* < 2/8 + (8 - 2/8)/7 < 3/2$, and from Eq. (2) that $(y + (b^{12} - 1)T)/C^* < 2/7 + (7 - 2/7)/7 < 3/2$.

Hence, we have $(x + b^{12}T)/C^* < 8/5$ and $(y + (b^{12} - 1)T)/C^* < 8/5$ for $b^{12} \neq 3, 4$. □

Lemma 3.8. If $b^* = 2$ and $\bar{b} = 3$ (if it exists), then $\bar{x}' \leq (1/5)C(M)^* + (4/5)u$.

Proof. We have $\bar{P}_3 \geq (4/5)P^* \geq (4/5)v$ from [15]. We know that $P \leq 2u + v$, thus $\bar{P}_3 \geq (4/5)(P - 2u)$. Note that $\bar{x}' \leq \bar{P}$, then it follows that

\begin{equation}
\bar{x}' \leq \frac{P - \bar{P}_3}{2} \leq \frac{P - \frac{4}{5}(P - 2u)}{2} = \frac{1}{10} P + \frac{4}{5} u \leq \frac{1}{5} C(M)^* + \frac{4}{5} u. \tag{5}
\end{equation}
Lemma 3.9. If $b^* = 3$ and $\overline{b'} = 4$ (if it exists), then $\overline{x} \leq (2/15)C(M)^* + (8/15)u + (4/15)v$.

**Proof.** It follows from [15] that $\tilde{p}_4 \geq (4/5)P^* \geq (4/5)w$. In addition, we have $P \leq 2u + v + w$, thus $\tilde{p}_4 \geq (4/5)(P - 2u - v)$. Note that $\overline{x} \leq \overline{p}$, hence

$$\overline{x} \leq \frac{P - \tilde{p}_4}{3} \leq \frac{P - \frac{4}{5}(P - 2u - v)}{3} = \frac{1}{15}P + \frac{8}{15}u + \frac{4}{15}v \leq \frac{2}{15}C(M)^* + \frac{8}{15}u + \frac{4}{15}v. \quad \square$$

Lemma 3.10. If $b^* = 2$, $\overline{b} \leq 3$, $\overline{C(M)} \neq \overline{x} + \tilde{b'}T$ and $\overline{C(M)} \neq \overline{y} + (\tilde{b'} - 1)T$ (if it exists), then $C_{TH}^{\text{SH}}/C^* \leq 63/40$.

**Proof.** From Theorem 1, we have $\overline{C(M)}/C(M)^* \leq 107/80$.

If $b^* = 2$ and $\overline{b} = 2$, or $b^* = 2$ together with $\overline{b} = 3$ and $\overline{C} = \overline{C\text{M})} + T$, then we have

$$\overline{C} \leq \frac{C(M)^* + T}{C(M)^*} \leq \frac{107}{80}C(M)^* + T \leq \frac{107 - \frac{4}{5}T}{80}C(M)^* + T < 63 \leq 40.$$ 

If $b^* = 2$, $\overline{b} = 3$ and $\overline{C} = P - C(M) + 2T$, then we have $T < P - C(M) \leq P/2$, and the two cases that need to be considered are as follows:

1. $C^* = C(M)^* + T \geq u + 2T$

$$\overline{C} \leq \frac{P - C(M)^*}{C(M)^* + T} < \frac{P + 2T}{P + T} \leq \frac{P + 2T}{T} \leq 63 < 40.$$ 

2. $C^* = u + 2T \geq C(M)^*$

$$\overline{C} \leq \frac{P - C(M)^* + 2T}{u + 2T} \leq \frac{P + 2T}{u + 2T} \leq \frac{C(M)^* + 2T}{u + 2T} \leq \frac{u + 3T}{u + 2T} \leq 63 < 40. \quad \square$$

Lemma 3.11. If $b^* = 2$ and $\overline{b} \leq 3$ with $\overline{C} = \overline{x} + \overline{b'T}$ or $\overline{C} = \overline{y} + (\overline{b'} - 1)T$ (if it exists), then $C_{TH}^{\text{SH}}/C^* \leq 8/5$.

**Proof.** If $\overline{b} = 2$ and $b^* = 2$, then from Lemma 3.7, we obtain $\overline{C} / C^* < 63/40$.

If $\overline{b} = 3$, since $(\overline{C} = \overline{x} + \overline{b'T}$ or $\overline{C} = \overline{y} + (\overline{b'} - 1)T$, then from Lemma 3.8, we have $\overline{x} \leq (1/5)C(M)^* + (4/5)u$.

If $u + 2T \geq C(M)^*$, then $u + T \geq C(M)^*$ and $\overline{x} \leq (1/5)C(M)^* + (4/5)u \leq (1/5)(u + T) + (4/5)u = u + (1/5)T$. Hence,

$$\overline{C} \leq \frac{\overline{x} + 3T}{C(M)^* + T} \leq \frac{\overline{x} + 3T}{u + 2T} \leq 1 + \frac{6T}{u + 2T} \leq \frac{8}{5}.$$ 

If $u + 2T < C(M)^*$, then $u + T < C(M)^*$ and $\overline{x} \leq (1/5)C(M)^* + (4/5)u < (1/5)(C(M)^* + (4/5)(C(M)^* - T) = C(M)^* - (4/5)T$. Hence,

$$\overline{C} \leq \frac{\overline{x} + 3T}{C(M)^* + T} \leq \frac{C(M)^* - \frac{2}{5}T + 3T}{C(M)^* + T} \leq 1 + \frac{6T}{C(M)^* + T} \leq \frac{8}{5}. \quad \square$$

Lemma 3.12. If $b^* \geq 4$, $\overline{C} \neq x + b^*HT$ and $\overline{C} \neq y + (b^* - 1)T$, then $C_{TH}^{\text{SH}}/C^* \leq 63/40$.

**Proof.** From Theorem 1, we have $\overline{C(M)}/C(M)^* \leq 107/80$. Recall that $\text{FFD}(I) \leq (11/9)\text{OPT}(I) + 1$ [14], so it follows that $b^* \geq 3$.

If $u + b^*T \leq C(M)^* + T$ and $b^* \geq 3$, then $C(M)^* \geq (b^* - 1)T$, so

$$\overline{C} \leq \frac{C(M)^* + 2T}{C(M)^* + T} \leq \frac{107}{80}C(M)^* + 2T \leq \frac{107 + \frac{53}{80}T}{80}C(M)^* + T \leq \frac{107}{80} + \frac{53}{80}T < 40.$$ 

If $u + b^*T \geq C(M)^* + T$ and $b^* \geq 3$, then $C(M)^* \leq u + (b^* - 1)T$, so

$$\overline{C} \leq \frac{C(M)^* + 2T}{u + b^*T} \leq \frac{(u + b^*T)^* + 2T}{u + b^*T} \leq \frac{107(u + b^*T) + 53T}{80(u + b^*T)} \leq \frac{107}{80} + \frac{53T}{80(u + b^*T)} < 40. \quad \square$$

Lemma 3.13. According to steps 1 and 2 of procedure C, if $b^* = 3$, then $\overline{b'} \leq 4$.

**Proof.** Recall that $\text{FFD}(I) \leq (7/4)\text{OPT}(I)$, thus if $b^* = 3$, then $\overline{b'} \leq 5$. Therefore, we only need to show that if $b^* = 3$, then $\overline{b'} \neq 5$.

For simplicity, assume that the bin (batch) capacity, and the size of each job are scaled such that the capacity is 1, and the size of each job is between 0 and 1.

By contradiction, assume that there exists one or more counterexamples where $\overline{b'} = 5$, then we consider the various cases on the size of the first job assigned to batch 5, $s_i$, as follows:
Lemma 3.14. If $b^* = 3$ and $\bar{b} \leq 4$, with $\bar{C} = \bar{x} + \bar{b}T$ or $\bar{C} = \bar{y} + (\bar{b} - 1)T$ (if it exists), then $C_{MH}^*/C^* \leq 8/5$.

Proof. If $b^* = 3$ and $\bar{b} = 4$, with $(\bar{C} = \bar{x} + \bar{b}T$ or $\bar{C} = \bar{y} + (\bar{b} - 1)T$, then from Lemma 3.9, we have $\bar{x} \leq (2/15)C(M)^* + (8/15)u + (4/15)v$. Furthermore, we have $C^* = \max\{C(M)^* + T, u + 3T\}$ from Lemma 3.6. Therefore, if $u + 3T \geq C(M)^* + T$, then $\bar{x} \leq (2/15)C(M)^* + (8/15)u + (4/15)v \leq (2/15)(u + 2T) + (8/15)u + (4/15)v = (2/3)u + (4/15)v + (4/15)v$. Hence,

$$\frac{\bar{C}}{C^*} \leq \frac{\bar{x} + 4T}{u + 3T} = \frac{\frac{2}{3}u + \frac{64}{15}v + \frac{4}{15}u}{u + 3T} \leq \frac{2}{3} \frac{\frac{14}{15}T + \frac{4}{15}u}{u + 3T}.$$

Denote $a$ as the time period of only one machine which is processing the first batch $u$, then, $a = |d_i^1 - d_i^2|$, where $d_i^1$ denotes completed time of the first batch $u$ on machine $i$. Therefore, we have $a \leq u$. Since $u + 3T \geq C(M)^* + T$, $\rho_2^* < \delta_2^*$, thus $u + (v - a)/2 \leq \rho_2^* < \delta_2^* = u + T$, and consequently, $u < 2T + a$. Hence, it follows that

$$\frac{\bar{C}}{C^*} \leq \frac{\bar{x} + 4T}{C(M)^* + T} \leq \frac{\bar{x} + 4T}{(u + 3T)C(M)^* + T} \leq \frac{\frac{10}{27}C(M)^* + 4T}{729C(M)^* + 540T} \leq \frac{63}{40}.$$

In a similar way, if $b^* = 3$ and $\bar{b} = 3$, then we have $\bar{C}/C^* \leq 63/40$.

Lemma 3.15. If $b^* = 3$, $\bar{b} \leq 4$, $\bar{C} \neq x + b^{H2}T$ and $\bar{C} \neq y + (b^{H2} - 1)T$ (if it exists), then $C_{MH}^*/C^* \leq 63/40$.

Proof. It is similar to Lemma 3.12 that if $(b^* = 3$ and $\bar{b} = 3$) or $(b^* = 3$ and $\bar{b} = 4$), then $C_{MH}^*/C^* \leq 63/40$.

Theorem 2. $C_{MH}^*/C^* \leq 63/40$, except for the two particular cases below where $C_{MH}^*/C^* < 8/5$.

1) $b^* = 3$ and $\bar{b} = 4$, with $\bar{C} = \bar{x} + \bar{b}T$ or $\bar{C} = \bar{y} + (\bar{b} - 1)T$ (if it exists).

2) $b^* = 2$ and $\bar{b} = 3$, with $\bar{C} = \bar{x} + \bar{b}T$ or $\bar{C} = \bar{y} + (\bar{b} - 1)T$ (if it exists).

Proof. This is a direct conclusion from Lemmas 3.7, 3.10–3.12, 3.14 and 3.15.

4. Conclusions and suggestions for future research

This paper studies a two-stage scheduling problem with two parallel machines and one customer to minimize the makespan of jobs $(P_2 \rightarrow D, k = 1|v = 1, c = z|C_{max})$. Since the problem is NP-hard, several heuristic methods have been developed and the best worst-case ratio was $5/3$. This paper presents a new algorithm based on the LPT rule and shows that the proposed method can achieve a worst-case ratio close to $63/40$, except for the two particular cases below where the associated ratio is $8/5$:

1) $b^* = 3$ and $\bar{b} = 4$, with $\bar{C} = \bar{x} + \bar{b}T$ or $\bar{C} = \bar{y} + (\bar{b} - 1)T$ (if it exists).

2) $b^* = 2$ and $\bar{b} = 3$, with $\bar{C} = \bar{x} + \bar{b}T$ or $\bar{C} = \bar{y} + (\bar{b} - 1)T$ (if it exists).

Many topics remain for future exploration. First of all, it is worthwhile investigating the complexity of the more general problem $P_2 \rightarrow D, k = 1|v = 1, c = z|C_j$. Secondly, for the intractable $P_2 \rightarrow D, k = 1|v = k, c = z|\Sigma C_j$ problem, it is justified through developing possibly efficient heuristic algorithms for obtaining approximate solutions. Thirdly, more realistic scheduling models that impose constraints on limited buffer capacities between manufacturing machines, or models involving multiple customer areas with vehicle routing decisions need to be studied. Furthermore, it is interesting to investigate the problems under other performance measures, such as due date related criteria, or under more complicated manufacturing configurations, such as job shop or open shop.
References