PERFORMANCE BOUNDS FOR SIMPLE BIN PACKING ALGORITHMS

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Introduction

We consider the classical bin packing problem [2, 5, 9]: a collection of unit-capacity bins and pieces (a list of sizes) is given, and the pieces are to be packed into the bins (the sizes of the pieces do not exceed 1). The problem of minimizing the number of required bins is NP-complete [5, 7], therefore approximation algorithms, as FF, BF, NF, FFD, BFD, NFD [1, 2, 5] have been analysed with the objective of characterizing worst-case performance relative to optimal packing.

Let $A(L)$ denote the number of required bins for an algorithm A and a list of real sizes L, let ϱ denote the set of all real lists, and let L_0 denote the minimal number of bins. Some worst-case results are summarized in the following assertions.

Assertion 1 [6]. *For every* $L \in \varrho$

$$NF(L) \leq 2L_0 + 1$$

and for every positive integer n *there exists a list* P *for which*

$$P_0 = n \quad \text{and} \quad NF(P) \geq 2P_0 - 2. \quad \square$$

Assertion 2 [2, 5, 8]. *For every* $L \in \varrho$,

$$FF(L) < \frac{17}{10} L_0 + 1$$

and for every integer number n *there exists a list* Q *for which*

$$Q_0 = n \quad \text{and} \quad FF(Q) \geq \frac{17}{10} Q_0 - 8. \quad \square$$

Assertion 3 [5, 7, 9]. *For every* $L \in \varrho$,

$$BF(L) \leq \frac{17}{10} L_0 + 2$$

and for every positive integer n there exists a list R for which

$$R_0 = n \quad \text{and} \quad BF(R) \ge \frac{17}{10} R_0 - 8. \quad \square$$

Assertion 4 [2, 5, 8]. *For every $L \in \varrho$,*

$$FFD(L) \le \frac{11}{9} L_0 + 4$$

and for every positive integer n there exists a list S for which

$$S_0 = n \quad \text{and} \quad FFD(S) \ge \frac{11}{9} S_0 - 2. \quad \square$$

Assertion 5 [2, 5, 8]. *For every $L \in \varrho$,*

$$BFD(L) \le \frac{11}{9} L_0 + 4$$

and for every positive integer n there exists a list T for which

$$T_0 = n \quad \text{and} \quad BFD(T) \ge \frac{11}{9} S_0 - 2. \quad \square$$

Assertion 6 [1]. *For every $L \in \varrho$*

$$NFD(L) \le \gamma L_0 + 3$$

and for every positive integer n and positive ε there exists a list U for which

$$U_0 = n \quad \text{and} \quad NFD(U) > (\gamma - \varepsilon) U_0,$$

where

$$\gamma = \sum_{i=1}^{\infty} \frac{1}{a_i}, \; a_1 = 1, \; a_{i+1} = a_i(a_i + 1), \; i = 1, 2, \dots \; (\gamma \approx 1{,}691). \quad \square$$

1. Performance of algorithms for binary data

In the memory of computers, data are stored as finite binary fractions. Therefore we investigate the performance of algorithms in the case of binary data.

To illustrate this let us consider a simple example. Let B_i denote the i-th bin and let the sum of the sizes of pieces in B_i be denoted by h_i and called the level of B_i. Let ν denote the set of algorithms which yield a packing where the sum of the levels of arbitrary two consecutive bins is larger than 1. Of course, all the mentioned algorithms belong to ν.

Let $\beta_k = \{u, 2u, 3u, \dots, 2^k u\}$, where $u = 2^{-k}$, $k \ge 0$. Then the set β_k consists of the binary fractions of the interval (0, 1] representable by using k digits.

Theorem 1. *For every $A\in\nu$ and $L\in\varrho$ we have $A(L)\leq 2L_0-1$ and for every positive integer n there exist $C\in\nu$ and $M\in\varrho$, for which $M_0 = n$, $C(M) = 2M_0-1$ and every size in M belongs to β_m, where $m = \lceil\log_2 n\rceil$.* □

Proof. *a)* $A(L)\leq 2L_0-1$ is a consequence of Theorem 5 (this inequality also is mentioned in [3]).

b) Let $M = (t_1, \ldots, t_q)$, where $q = n+2^m$, $t_i = 1$ for $i = 2, 4, \ldots, 2n$ and $t_i = 1/2^m$ for the remaining cases. Then $M_0 = n+1$ and $NF(M) = 2n+1$, i.e. $NF(M) = 2M_0-1$. Of course, the sizes belong to β_m. □

Now we consider another example, where the worst lists require only a fixed number of digits.

Let μ denote the set of lists having an optimal packing into $s\geq 1$ bins in such a way that $a_i, b_i, c_i\in B_i$ with $a_i+b_i+c_i = 1$, $1/2<a_i<2/3$, $1/3<b_i<1/2$, $1/7<c_i<1/6$. The following theorem shows that the lists in Garey and Johnson's book [5] are the worst possible, and the same worst case is obtainable by using a few of digits.

Theorem 2 [6]. *For every $N\in\mu$ we have $FF(N)\leq[5/3\ N_0]$, and for every positive integer number n there exists an $M\in\mu$, for which $M_0 = n$, $FF(M) = BF(M) = [5/3\ M_0]$ and the sizes of the pieces in M belong to β_7.* □

Proof. Let $M = (s_1, \ldots, s_n, m_1, \ldots, m_n, l_1, \ldots, l_n)$, where $s_j = 20/128$, $m_j = 43/128$, $l_j = 65/128$, $j = 1, \ldots, n$. Because of $s_j+m_j+l_j = 1$ we have $M_0 = n$. It is easy to compute that $FF(M) = BF(M) = [5/3\ M_0]$. For example, if $n = 6r$, then the small pieces $s_1, \ldots, s_n$ occupy r bins, the medium elements $3r$ bins and the large ones $6r$ bins. Due to $128 = 2^7$ the sizes belong to β_7.

b) Let us consider the packing of the list N generated by FF. *Let* $q = \max_{1\leq j\leq FF(N)} \{j|B_j$ contains at least one small element$\}$. We call $B_1, \ldots, B_q$ low bins, the remaining ones, $B_{q+1}, \ldots, B_{FF(N)}$, – high bins. Let X denote the number of large pieces in the low bins, and Y the number of medium pieces in the same bins. If $N_0\geq 5$, then we have the following cases: *b*1) $X = 0$, $Y = 0$; *b*2) $X = 0$, $Y = N_0$; *b*3) $X = 0$, $0<Y<N_0$; *b*4) $0<X<N_0$; *b*5) $X = N_0$.

*b*1) Every large and medium piece is in the high bins. $B_1, \ldots, B_{q-1}$ contain at least *6* small elements each, and B_q contains at least *5* ones. The large and medium pieces occupy the maximal number of bins if every large one occupy one bin, and every pair of medium pieces occupy a whole bin, therefore

$$FF(N)\leq N_0+[N_0/2]+[(N_0-5)/6]+1\leq 5/3\ N_0.$$

To prove the second inequality we distingueshed six subcases according to $N = 6r+j$ ($r\geq 0$; $j = 0, 1, \ldots, 5$).

*b*2) All the large pieces are in the high bins, and the medium pieces are in the low bins. The bins $B_1, \ldots, B_{q-1}$ contain at least 6 small pieces each (or one medium piece instead of three small ones) and B_q contains at least 3 small pieces, so that

$$FF(N)\leq N_0+1+[(N_0+3N_0-3)/6]\leq 5/3\ \mathrm{N}_0.$$

$b3$) Every large element is in the high bins, the medium pieces are distributed between the low and high bins, B_q contains at least four small pieces (each medium element can replece three small elements) so that

$$FF(N)\leq N_0+1+[(N_0-Y)/2]+[(N_0+3Y-4)/6]\leq 5/3\ N_0.$$

$b4$) Now, the large pieces being distributed, B_q contains at least three small pieces. Each large piece can replace 4 small ones, so that

$$FF(N)\leq (N_0-X)+1+[(N_0-Y)/2]+[(N_0+4X+3Y-3)/6]\leq 5/3\ N_0.$$

$b5$) The high bins do not contain large pieces. One medium piece can occupy one high bin, and B_q contains at least one small piece. Therefore

$$FF(N)\leq [(N_0-Y)/2]+1+[(N_0+4N_0+3Y-1)/6]\leq 5/3\ N_0.$$

c) If $N_0 = 4$, then we consider the cases $q = 1, \dots, F$.

If $q\geq 3$, then $h_1>5/6$, $h_2>5/6$, and $\sum_{i=3}^{F} h_i>2(F-2)$. Then from $\sum_{i=1}^{F} h_i = 4$ it follows $F\leq 6$.

If $q = 1$, then B_1 has to contain the small pieces and a medium (or large) one too. The remaining pieces can fit at most 5 bins, therefore $F\leq 6$.

If $q = 2$, then B_1 and B_2 contain all the small pieces and at least 3 medium (or large) ones too – the remaining pieces can fit at most 4 bins, therefore $F\leq 6$.

d) If $N_0\leq 3$, then we use Theorem 1.

All cases are considered. Because $FF(N)$ is an integer, $FF(N)\leq 5/3\ N_0$ implies $FF(N)\leq [5/3\ N_0]$. □

We remark that some simplification of the original proof is due to S. O. Botchkov, a student of Moscow State University.

The following theorem shows that there are even binary fraction examples for FF and BF which are worse than the ones known sofar.

Theorem 3 [10]. *For every positive integer n there exists an $L\in\varrho$, for which $L_0 = n$, $FF(L) = BF(L) = [17/10\ L_0]$ and the sizes in L belong to β_{n+15}.* □

In the proof, L is constructed as follows. Let $Q = (s_1, \dots, s_n, m_1, \dots, m_n, l_1, \dots, l_n)$, where $s_i = 1/6+\varepsilon+(-1)^i\ 2^{-5-i}-2^{-10-n}$, $m_i = 1/3-\varepsilon+(-1)^{i+1}2^{-5-i}$, $l_i = 1/2+2^{-10-n}$, $(1/6+\varepsilon)\in(1/6,\ 1/6+2^{-15-n})$, $(1/6+\varepsilon)\in\beta_{15+n}$, $i = 1, \dots, n$. We get L from Q, using some rearrengement of the small and in some subcases of the medium pieces.

We remark that using another construction in the proof of the theorem, we can replace β_{n+15} by β_h, where $h = n/2+5$.

From this theorem, for example, we get a negative answer to a conjecture [8] that for $L_0>20$, $FF(L)<17/10\ L_0$ and $BF(L)<17/10\ L_0$.

We can realize the worst known performance of FFD and BFD using only 6 digits.

Theorem 4. *For every positive integer n there exists an $L\in\varrho$, for which $L_0 = n$, $FFD(L) = BFD(L) = [11/9\ L_0]$ and the sizes belong to β_6.* □

Proof. Let $n = 3m+j$, $0 \le j \le 2$, $m \ge 2$. Suppose $2m+j$ optimal bins contain pieces with sizes *33/64, 17/64, 14/64*, and m optimal bins contain pieces with sizes *18/64, 18/64, 14/64, 14/64*. For example in the case $n = 9s$ in *FFD*- and *BFD*-packings we have $6s$ bins with *33/64, 18/64*, $2s$ bins with *17/64, 17/64, 17/64*, and $3s$ bins with *14/64, 14/64, 14/64, 14/64*. The remaining cases are similar. □

For *NFD* the known worst performance values [1] in Assertion 6 are also obtainable by using binary fractions (the number of required digits depends on L_0 and ε).

2. Dependence of performance on data accuracy

In this section we give the tight worst-case bound for a class of bin packing algorithms as a functions of data accuracy.

Theorem 5. *For every* $A \in \nu$ *and* $M \in \beta_k$

$$A(M) \le 2[M_0/(1+2^{-k})] + \text{sign}\,(M_0/(1+2^{-k}) - [M_0/(1+2^{-k})]) = f(M_0, k),$$

and there exists an element $C \in \nu$ *(for example, NF) such that for every positive integer n there exists* $N \in \beta_6$, *for which* $N_0 = n$ *and* $C(N) = f(N_0, k)$. □

Proof. *a)* In this case the sum of the levels of neighbouring bins is greater than or equal to $1+2^{-k}$, therefore $A(M) \le f(M_0, k) = 2s+d$, where $s = [M_0/(1+2^{-k})]$, $d = \text{sign}\,(M_0/(1+2^{-k}) - [M_0/(1+2^{-k})])$.

b) If $N_0/(1+2^{-k})$ is integer, then $s = N_0/(1+2^{-k})$, In this case $d = 0$ and $s = N_0/(1+2^{-k}) = 2^k N_0/(2^k+1) = 2^k(N_0/(2^k+1))$, that is $N_0/(2^k+1)$ an integer and s is divisible by 2^k.

Let now $N = (t_1, \ldots, t_{2s})$, where $t_i = 2^{-k}$, if $i = 1, 3, \ldots, 2s-1$, and $t_i = 1$, if $i = 2, 4, \ldots, 2s$. In this case we have $NF(N) = 2s$, $N_0 = s+s/2^k$, hence $2s = NF(N) = 2[N_0/(1+2^{-k})]$.

If $N_0/(1+2^{-k})$ is not an integer, then $s = [N_0/(1+2^{-k})]$ and $d = 1$. Let now $N = (t_1, \ldots, t_{2s+1})$, where $t_i = 2^{-k}$, if $i = 1, 3, \ldots, 2s-1$, and $t_i = 1$, if $i = 2, 4, \ldots, 2s$ and $t_{2s+1} = N_0 - s(1+2^{-k})$. In this case $NF(N) = 2s+1$, $N_0 = s+[s/2^k]$, hence $2s+1 = NF(N) = 2[N_0/(1+2^{-k})]+1$. □

Of course, for a fixed N_0 we have

$$\lim_{k\to\infty} f(N_0, k) = 2N_0 - 1$$

(here even equality holds for $k \ge k_0(N_0)$). Therefore it is easy to get Theorem 1 as a corollary of Theorem 5.

Suppose, for example, that $k = 4$ and $N_0 = 17p$. Then from Theorem 5 it follows that

$$A(N) \le 2(17p/(1+2^{-16})) = 32p = 2N_0 - 2/17\, N_0.$$

If, as in the computer BESM−6 [11], we have $k = 47$, then let $N_0 = (2^{47}+1)q$. For these data from Theorem 5 we obtain the inequality

$$A(N) \le 2N_0 - 2/(2^{47}+1)\, N_0.$$

Comparing this result with the estimation in Theorem 1, we can conclude that the inaccuracy of big computers has only a weak influence on the tight worst-case bounds of the algorithm NF. Using similar constructions as in the proof of Theorem 3 and 4, we can draw the same conclusion for the remaining bin packing algorithms, too.

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