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ON A NUMBER-THEORETIC BIN PACKING CONJECTURE

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ABSTRACT

In the standard bin packing problem, we are given a list of items, with sizes between 0 and 1, and are asked to partition them into as few sets as possible so that no set contains items whose sizes sum to more than 1. It has been shown that the "first-fit" algorithm constructs partitions which never contain more than $\frac{17}{10}$ times the minimum possible number of sets. There is a number-theoretic explanation of this fractional bound $\frac{17}{10}$, which we conjecture also yields the correct bound over an entire spectrum of bin packing problems, of which the standard one is merely a special case. We discuss this conjecture and present partial results in its support.

1. INTRODUCTION

The "bin-packing" problem studied in [2], [3], [4], [5], [6], [7] is among the most simply structured, yet mathematically complex, packing problems encountered in computer science and operations research. It models a wide variety of real problems, involving such tasks as stock-cutting, computer memory allocation, and multiprocessor scheduling [2], [3], [4]. Stated informally, we assume that we have a collection of identical, fixed-capacity bins and a set of items of various sizes. Our goal is to pack the items into as few bins as possible so that no bin contains items whose sizes sum to more than the bin capacity. Because this problem is known to be "NP-complete" [1], most efforts have concentrated on finding and analyzing simple algorithms that construct "near-optimal" packings [4], [5], [6], [7]. In this paper we shall be concerned with further investigating the properties of one such algorithm, called the "first-fit algorithm", which is known never to use more than $\frac{17}{10}$ times the minimum possible number of bins. In particular, we shall be concerned with the effect of applying the first-fit algorithm to pack items already packed in bins of one fixed capacity into bins of a different capacity. As we shall see, this problem gives rise to a variety of interesting number-theoretic results and conjectures, which also provide some additional insight into the derivation of the number $\frac{17}{10}$ in the previously mentioned bound.

We begin with some basic definitions. For any number $\alpha > 0$, we define an *α-list* to be an ordered set $L = \{a_1, a_2, \ldots, a_n\}$ of *items* along with a size function s which assigns to each $a_i \in L$ a *size* $s(a_i)$ satisfying $0 < s(a_i) \leqslant \alpha$. A *packing* of an α-list L into "bins" of capacity α ("α-bins") is a partition $P = \langle A_1, A_2, \ldots, A_m \rangle$ of L such that $\sum_{a \in A_j} s(a) \leqslant \alpha$ for each j, $1 \leqslant j \leqslant m$. The *number of bins* used by the packing P is $m = |P|$. For an α-list L, the *optimal* number of bins of capacity α is defined by

$$\mathrm{OPT}(L, \alpha) = \min\{|P| : P \text{ is a packing of } L \text{ into } \alpha\text{-bins}\},$$

and a particular packing P is called an optimal packing if $|P| = \mathrm{OPT}(L, \alpha)$.

The *first-fit* (FF) *packing* P_{FF} of an arbitrary α-list L is constructed as follows. Define the *level* of any set $A \subseteq L$ to be level $(A) = \sum_{a \in A} s(a)$. The construction begins with n empty sets A_i, $1 \leqslant i \leqslant n$, i.e., each set having level 0. It then proceeds to assign each item in L to one of the sets, the assignments taking place one at a time, in the sequence given by the ordering of L. Informally, each item a_i is assigned in its turn to the "first" set into which it will "fit". More formally, to place a_i, we set j^* to be the least $j \geqslant 1$ for which the current level of A_j added to $s(a_i)$ does not exceed α, and we then update the set A_{j^*} to $A_{j^*} \cup \{a_i\}$. The construction of P_{FF} is completed once a_n has been assigned, and the final packing consists only of those A_j having positive levels. Notationally we define $\mathrm{FF}(L, \alpha) = |P_{\mathrm{FF}}|$.

We shall be interested in comparing the values of $\mathrm{OPT}(L, \alpha)$ and $\mathrm{FF}(L, \alpha')$ for various lists L and bin capacities α and α'. By normalizing appropriately, there is no loss of generality in restricting our attention to the case of $\alpha' = 1$. Thus we define the following worst-case measures:

$$(1.1) \qquad R_N[\mathrm{FF}, \alpha] = \max \left\{ \frac{\mathrm{FF}(L, 1)}{\mathrm{OPT}(L, \alpha)} : \begin{array}{l} L \text{ is a } \min\{\alpha, 1\}\text{-list} \\ \text{and } \mathrm{OPT}(L, \alpha) = N \end{array} \right\}$$

$$(1.2) \qquad R_\infty[\mathrm{FF}, \alpha] = \limsup_{N \to \infty} R_N[\mathrm{FF}, \alpha].$$

The standard bin-packing problem is concerned only with the case of $\alpha = 1$, that is, the case in which the same bin capacity is used for both the optimal and FF packings. For this case, the following intriguing result has been obtained [4], [5], [7]:

Theorem 1. $R_\infty[\mathrm{FF}, 1] = \frac{17}{10}$.

This result says that asymptotically the FF packing uses no more than 70% more 1-bins than an optimal packing and, furthermore, that there exist lists which cause FF to use this many additional bins in the limit. The reason we called this result "intriguing" is that the number $\frac{17}{10}$ seems to appear out of mid-air in the proof, with no apparent intuitive justification. One motivation for turning to the generalized problem is to seek further insight into why the number $\frac{17}{10}$ occurs in this bound.

Of course the generalized problem has practical implications in its own right. For example, suppose $R_\infty\left[\mathrm{FF}, \frac{1}{2}\right] = \frac{7}{10}$ (as indeed it does). This tells us that by doubling the bin capacity and packing using first-fit we will be guaranteed to save at least 30% over the number of bins used by *any* packing into bins of the original capacity. Thus the function $R_\infty[\mathrm{FF}, \alpha]$ provides bounds on the reduction (or increase) in number of bins used when FF is applied to pack the items into bins of a different capacity.

We now proceed to describe a number-theoretic conjecture concerning the precise value of $R_\infty[\mathrm{FF}, \alpha]$ as a function of α. The conjecture involves certain decompositions of the number α into a sum of unit fractions.

For any real number $\alpha > 0$, a *feasible decomposition* of α is any sequence $D(\alpha) = \left\langle \frac{1}{p_1}, \frac{1}{p_2}, \ldots \right\rangle$ where the p_i are integers satisfying:

(i) $\sum_{i \geqslant 1} \frac{1}{p_i} = \alpha$

(ii) $2 \leqslant p_1 \leqslant p_2 \leqslant \ldots$

(iii) at least two p_i's exceed 2.

For any sequence $D = \left\langle \frac{1}{q_1}, \frac{1}{q_2}, \ldots \right\rangle$ of unit fractions with all $q_i \geqslant 2$, the *augmented sum* AS (D) is defined to be

$$\mathrm{AS}\,(D) = \sum_{i \geqslant 1} \frac{1}{q_i - 1}.$$

The function $W\colon (0, \infty) \to (0, \infty)$ is then defined as follows:

$$W(\alpha) = \sup\,\{\mathrm{AS}\,(D(\alpha))\colon D(a) \text{ is a feasible decomposition of } \alpha\}.$$

We can now state our conjecture about the value of $R_\infty[\mathrm{FF}, \alpha]$.

Conjecture. For all $\alpha > 0$, $R_\infty[\mathrm{FF}, \alpha] = W(\alpha)$.

This may not appear at first glance to be a particularly informative conjecture, because no method is given for evaluating $W(\alpha)$. However, in the next section we shall remedy this defect by showing that for all α there is a specific, easily described, feasible decomposition of α whose augmented sum is just $W(\alpha)$.

In the remainder of the paper we present a number of results supporting the conjecture. In Section 3 we provide examples to show that for all α, $R_\infty[\mathrm{FF}, \alpha] \geqslant W(\alpha)$. In Section 4 we prove that the conjecture holds for infinitely many values of α.

II. EVALUATING $W(\alpha)$

The following observation will be useful later. If $D = \left\langle \frac{1}{p_1}, \frac{1}{p_2}, \ldots \right\rangle$ is a sequence of unit fractions with $2 \leqslant p_1 \leqslant p_2 \leqslant \ldots$ then for any k for which p_k is defined, we have

$$\sum_{i \geqslant k} \frac{1}{p_i} < \mathrm{AS}\,(D) - \sum_{1 \leqslant i < k} \frac{1}{p_i - 1} \leqslant \frac{p_k}{p_k - 1} \sum_{i \geqslant k} \frac{1}{p_i}. \tag{2.1}$$

We now introduce, for each $\alpha > 0$, a special feasible decomposition $D^*(\alpha) = \left\langle \frac{1}{p_1^*}, \frac{1}{p_2^*}, \ldots \right\rangle$, called the *modified greedy decomposition,* which will be used in computing $W(\alpha)$. (Interested readers should be able to deduce what the greedy decomposition is, and why it had to be modified so that it always is a *feasible* decomposition.)

We define $D^*(\alpha)$ inductively. Suppose $p_1^*, p_2^*, \ldots, p_{k-1}^*$ are defined and let $\delta = \alpha - \sum_{i=1}^{k-1} \frac{1}{p_i^*}$. If $\delta = 0$, then $D^*(\alpha)$ is complete and $|D^*(\alpha)| = k - 1$. Otherwise, let

$$p^* = \min\left\{p \geqslant 2\colon p \text{ is an integer and } \delta \geqslant \frac{1}{p}\right\}.$$

If $\delta = \frac{1}{p^*}$ and either $k = 1$ or $p_{k-1}^* = 2$, then we set $p_k^* = p^* + 1$ and $p_{k+1}^* = p^*(p^* + 1)$, completing the specification of $D^*(\alpha)$ with $|D^*(\alpha)| = k + 1$. Otherwise, we set $p_k^* = p^*$.

It is not hard to see that $D^*(\alpha)$ is finite iff α is rational (e.g., see [8]). Also, it is clear that $D^*(\alpha)$ is always a *feasible* decomposition of α. As examples, we have

$$D^*(1) = \left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \right\rangle$$

and

$$D^*(\pi) = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{61}, \frac{1}{5020}, \ldots \right\rangle.$$

It follows from the definition of $D^*(\alpha)$ that for $\alpha > 1$ and $k \geqslant 1$,

$$(2.2) \qquad \sum_{i > k} \frac{1}{p_i^*} \leqslant \frac{1}{p_k^* - 1}.$$

The main result of this section is the following.

Theorem 2. *For any* $\alpha > 0$,

$$W(\alpha) = \text{AS}\,(D^*(\alpha)).$$

Proof. For $\alpha > 0$, $D^*(\alpha)$ is a feasible decomposition of α and so, by the definition of W,

$$W(\alpha) \geqslant \text{AS}\,(D^*(\alpha)).$$

It remains to show the reverse inequality.

Suppose $D'(\alpha) = \left\langle \frac{1}{q_1}, \frac{1}{q_2}, \ldots \right\rangle$ is some other feasible decomposition for α. Let $k \geqslant 0$ be the greatest integer such that $q_i = p_i^*$ for $1 \leqslant i \leqslant k$, and define

$$\beta = \sum_{1 \leqslant i \leqslant k} \frac{1}{p_i^*} = \sum_{1 \leqslant i \leqslant k} \frac{1}{q_i}.$$

Since by hypothesis $D'(\alpha)$ and $D^*(\alpha)$ are different, then $\alpha > \beta$, and hence p_{k+1}^* and q_{k+1} must be defined.

Suppose $q_{k+1} \leqslant p_{k+1}^* - 1$. By the definition of $D^*(\alpha)$, this implies that

$$\alpha - \beta = \frac{1}{q_{k+1}} = \frac{1}{p_{k+1}^*} + \frac{1}{p_{k+2}^*}$$

and

$$p_{k+1}^* = q_{k+1} + 1.$$

However, since $D'(\alpha)$ is feasible, then we must have $k \geqslant 1$ and $q_k > 2$. But this implies that $p_k^* = q_k > 2$ so that the modified greedy decomposition would in fact have $p_{k+1}^* = q_{k+1}$, which is a contradiction.

Thus $q_{k+1} > p^*_{k+1} - 1$. In this case, by the definition of k we actually have

$$q_{k+1} \geqslant p^*_{k+1} + 1.$$

Thus

$$\sum_{i \geqslant k+1} \frac{1}{p^*_i - 1} \geqslant \frac{1}{p^*_{k+1} - 1} + \sum_{i > k+1} \frac{1}{p^*_i} =$$

$$= \frac{1}{p^*_{k+1}(p^*_{k+1} - 1)} + \sum_{i \geqslant k+1} \frac{1}{p^*_i} = \frac{1}{p^*_{k+1}(p^*_{k+1} - 1)} + \alpha - \beta,$$

and, by (2.1),

$$\sum_{i \geqslant k+1} \frac{1}{q_i - 1} \leqslant \left(\frac{q_{k+1}}{q_{k+1} - 1}\right) \sum_{i \geqslant k+1} \frac{1}{q_i} \leqslant \left(\frac{p^*_{k+1} + 1}{p^*_{k+1}}\right) (\alpha - \beta).$$

Subtracting, we obtain

$$\mathrm{AS}\,(D^*(\alpha)) - \mathrm{AS}\,(D'(\alpha)) = \sum_{i \geqslant 1} \frac{1}{p^*_i - 1} - \sum_{i \geqslant 1} \frac{1}{q_i - 1} \geqslant$$

$$\geqslant \frac{1}{p^*_{k+1}(p^*_{k+1} - 1)} - \frac{(\alpha - \beta)}{p^*_{k+1}} \geqslant 0$$

since

$$\alpha - \beta \leqslant \frac{1}{p^*_{k+1} - 1}$$

by (2.2). Therefore, $\mathrm{AS}\,(D'(\alpha)) \leqslant \mathrm{AS}\,(D^*(\alpha))$ and, since $D'(\alpha)$ was an arbitrary feasible decomposition of α (differing from $D^*(\alpha)$),

$$W(\alpha) = \sup \{\mathrm{AS}\,(D(\alpha)) \colon D \ \text{feasible}\} \leqslant \mathrm{AS}\,(D^*(\alpha)).$$

This proves the theorem.∎

III. LOWER BOUND EXAMPLES

In this section we shall prove that for all $\alpha > 0$,

(3.1) $\qquad R_\infty[\mathrm{FF}, \alpha] \geqslant W(\alpha).$

This will follow as a straightforward corollary of the following result.

Theorem 3. *For all* $\alpha > 0$, *if* D *is a feasible decomposition of* α, *then for any* $\epsilon > 0$ *and any* N_0 *there is an* $N \geqslant N_0$ *such that*

$$R_N[\mathrm{FF}, \alpha] > \mathrm{AS}\,(D) - \epsilon. \tag{3.2}$$

Proof. Suppose $D = \left\langle \frac{1}{p_1}, \frac{1}{p_2}, \ldots \right\rangle$. By the definition of $\mathrm{AS}\,(D)$ there is an integer $M \geqslant 2$ such that

$$S_M = \sum_{i=1}^{M} \frac{1}{p_i - 1} > \mathrm{AS}\,(D) - \frac{\epsilon}{2}. \tag{3.3}$$

Let K be a large multiple of l.c.m. $\{p_1 - 1, \ldots, p_M - 1\}$. We shall construct a list L such that

$$N_0 \leqslant N = \mathrm{OPT}\,(L, \alpha) \leqslant K + 1 \tag{3.4}$$

and

$$\mathrm{FF}\,(L, 1) \geqslant S_M K. \tag{3.5}$$

This will imply by the definition of $R_N[\mathrm{FF}, \alpha]$ that

$$R_N[\mathrm{FF}, \alpha] \geqslant \frac{S_M K}{K+1} = S_M - \frac{S_M}{K+1} > \mathrm{AS}\,(D) - \epsilon$$

for K sufficiently large (using (3.3)). Thus, the theorem will be proved once we construct L and show that it obeys (3.4) and (3.5).

Our list L will consist of M types of elements, with $K + 1$ elements of each type. The j-th element of type i will be denoted by $a_i[j]$ and will have size specified as follows (with $\gamma = \frac{1}{Mp_M}$):

For $1 \leqslant j \leqslant K + 1$,

$$s(a_M[j]) = \begin{cases} \dfrac{1}{p_M} + \gamma^{2j}, & j \text{ odd}, \\[2ex] \dfrac{1}{p_M} - \gamma^{2j-1}, & j \text{ even}, \end{cases}$$

$$s(a_{M-1}[j]) = \begin{cases} \dfrac{1}{p_{M-1}} - \gamma^{2j-1}, & j \quad \text{odd}, \\ \dfrac{1}{p_{M-1}} + \gamma^{2j}, & j \quad \text{even}, \end{cases}$$

and, for $1 \leqslant i < M - 1$,

$$s(a_i[j]) = \frac{1}{p_i} + \gamma^{2j}.$$

Even though we have yet to specify the order of the elements $a_i[j]$ in L, we can already show that (3.4) holds. To see that $\text{OPT}(L, \alpha) \leqslant \leqslant K + 1$, simply note that for each j, $1 \leqslant j \leqslant K + 1$,

$$\sum_{i=1}^{M} s(a_i[j]) = \sum_{i=1}^{M} \frac{1}{p_i} - \gamma^{2j-1}(1 - (M-1)\gamma) < \alpha$$

since $\gamma < \frac{1}{M}$. To guarantee that $\text{OPT}(L, \alpha) \geqslant N_0$, all we need do is choose a sufficiently large K, since

$$\text{OPT}(L, \alpha) \geqslant \frac{1}{\alpha} \sum_{j=1}^{K+1} \sum_{i=1}^{M} s(a_i[j]).$$

Thus (3.4) holds for sufficiently large K.

To complete the definition of the list L, we shall now specify an order for the elements $a_i[j]$. The elements of the ordered list L will be denoted by

$$L' = \langle b_M[1], b_M[2], \ldots, b_M[K], b_{M-1}[1], \ldots$$
$$\ldots, b_{M-1}[K], b_{M-2}[1], \ldots$$
$$\ldots, b_1[K], b_0[1], b_0[2], \ldots, b_0[M] \rangle.$$

The $b_0[j]$ are identified as follows:

$$b_0[j] = \begin{cases} a_j[K+1], & 1 \leqslant j \leqslant M-2 \\ a_{M-1}[1], & j = M-1 \\ a_M[2], & j = M. \end{cases}$$

For $1 \leqslant i \leqslant M-2$,

$$b_i[j] = a_i[j], \qquad 1 \leqslant j \leqslant K.$$

The only complicated part of the ordering concerns the $b_{M-1}[j]$'s and $b_M[j]$'s, which are identified as follows

$$b_{M-1}[j] = \begin{cases} a_{M-1}[j+2], & 1 \leqslant j \leqslant K-1 \text{ and } j \text{ odd} \\ a_{M-1}[j], & 2 \leqslant j \leqslant K \text{ and } j \text{ even} \\ a_{M-1}[K+1], & j = K \text{ and } K \text{ odd} \end{cases}$$

$$b_M[j] = \begin{cases} a_M[j], & 1 \leqslant j \leqslant K \text{ and } j \text{ odd} \\ a_M[j+2], & 2 \leqslant j \leqslant K-1 \text{ and } j \text{ even} \\ a_M[K+1], & j = K \text{ and } K \text{ even.} \end{cases}$$

The reader may verify that the FF-packing is constructed as follows. Let $B_i = \langle b_i[1], \ldots, b_i[K] \rangle$, $1 \leqslant i \leqslant M$. Suppose $M \geqslant j \geqslant 1$ and all elements in all B_i with $i > j$ have been assigned. Then the elements of B_j all go in new bins, the first $p_j - 1$ elements in the first new bin, the second $p_j - 1$ elements in the second new bin, and so on, until $\dfrac{K}{p_j - 1}$ new bins have been formed, each containing $p_j - 1$ elements from B_j. The packing proceeds in this way until only the $b_0[j]$ elements remain to be assigned, at which point there will already be $\sum_{i=1}^{M} \dfrac{K}{p_i - 1} = S_M K$ non-empty bins. Thus, no matter what happens to these last M elements, we will have $\text{FF}(L, 1) \geqslant S_M K$, which shows that (3.5) holds.

Thus, as was argued earlier, if K is sufficiently large, (3.2) must hold. The theorem is proved.∎

IV. SPECIAL CASE UPPER BOUND RESULTS

In light of (3.1), all that we need do to show that the Conjecture holds for a particular value of α is to prove that $R_\infty[\text{FF}, \alpha] \leqslant W(\alpha)$. Although we have as yet been unable to prove this for arbitrary $\alpha > 0$, we have been able to establish it in two interesting special cases.

The first result is presented without proof, as it follows directly by means of the methods for proving Theorem 1 in [4], [5], [7].

Theorem 4. *Suppose* $\alpha = \frac{K}{2}$ *for some positive integer* K. *Then*

$$R_\infty[\mathrm{FF}, \alpha] = W(\alpha) = K - \frac{3}{10}.$$

The second result represents our only successful attempt to generalize the proof method for Theorem 1.

Theorem 5. *Suppose the modified greedy decomposition* $D^*(\alpha) = D$ *satisfies* $|D| = 2$. *Then*

$$R_\infty[\mathrm{FF}, \alpha] = W(\alpha).$$

Proof. Observe first that in order for $|D|$ to be 2, we must have $\alpha \leqslant \frac{1}{2}$. Suppose $D = \left\langle \frac{1}{q}, \frac{1}{p} \right\rangle$. By Theorem 2 we then have $W(\alpha) = \frac{1}{q-1} + \frac{1}{p-1}$. Moreover, by the definition of the modified greedy decomposition we have $p > q > 2$ and by (2.2) we have

$$p \geqslant q(q-1). \tag{4.1}$$

We shall proceed by proving a sequence of lemmas. We first define a weighting function $f\colon (0, \alpha] \to (0, W(\alpha)]$ as follows (see Figure 1)

$$f(x) = \begin{cases} \dfrac{p}{p-1}\, x, & 0 \leqslant x \leqslant \dfrac{1}{p} \\[2mm] \dfrac{pqx - 1}{(p-1)(q-1)}, & \dfrac{1}{p} \leqslant x \leqslant \dfrac{1}{q} \\[2mm] \dfrac{1}{q-1} + \left(x - \dfrac{1}{q}\right)\dfrac{p}{p-1} = & \\[2mm] \quad = \dfrac{p}{p-1}\, x + \dfrac{q(p-1) - p(q-1)}{q(p-1)(q-1)}, & \dfrac{1}{q} \leqslant x \leqslant \alpha. \end{cases}$$

If L is an α-list and $A \subseteq L$, we define $f^*(A) = \sum_{a \in A} f(s(a))$. Our

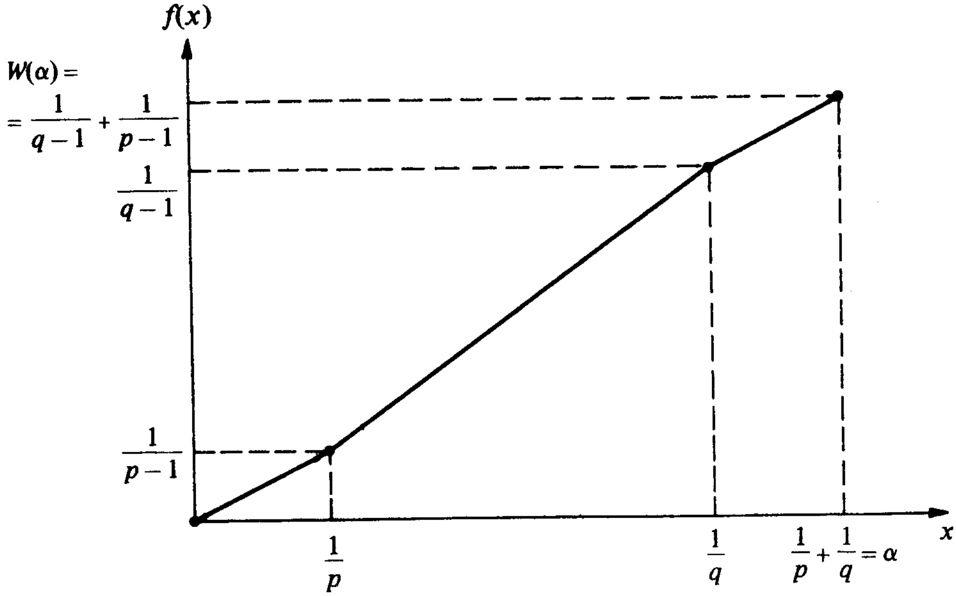

Fig. 1. The function f: $(0, \alpha] \to (0, W(\alpha)]$

lemmas deal with the relations between $|A|$, level(A), and $f^*(A)$ for such subsets A.

Lemma 4.1. *Suppose* L *is an* α*-list and* $A \subseteq L$ *is such that level* $(A) \leqslant \alpha$. *Then* $f^*(A) \leqslant W(\alpha)$.

Proof of lemma. Suppose L and $A = \{a_1, a_2, \ldots, a_m\} \subseteq L$ give a counterexample. We can make certain normalizing assumptions about A, given the nature of f. Observe that the following two properties follow from the definition of f.

(i) If $0 < x \leqslant \frac{1}{p}$, then $f\left(\left(\frac{1}{q}\right) + x\right) = f\left(\frac{1}{q}\right) + f(x)$.

(ii) If $0 < \epsilon \leqslant x \leqslant y \leqslant \frac{1}{q} - \epsilon$, then $f(x) + f(y) \leqslant f(x - \epsilon) + f(y + \epsilon)$.

Using these properties and the fact that level$(A) \leqslant \alpha$, it is a straightforward matter to show that we may assume the following about our counterexample A.

(1) $\left|\left\{a \in A \colon \frac{1}{q} < s(a) \leqslant \alpha\right\}\right| = 0$

(2) $\left|\left\{a \in A \colon \frac{1}{p} < s(a) \leqslant \frac{1}{q}\right\}\right| \leqslant 1$

(3) $\left|\left\{a \in A \colon 0 < s(a) \leqslant \frac{1}{p}\right\}\right| \leqslant 1$

Hence, the highest possible value for $f^*(A)$ is $f\left(\frac{1}{q}\right) + f\left(\frac{1}{p}\right) = \frac{1}{q-1} + \frac{1}{p-1} = W(\alpha)$, a contradiction.∎

Lemma 4.2. *Suppose L is an α-list and $A \subseteq L$ is such that level $(A) > 1 - \alpha$. Then $|A| \geqslant q - 1$.*

Proof of lemma. *If* $|A| \leqslant q - 2$, then

$$\text{level}\,(A) \leqslant (q-2)\alpha = (q-1)\left(\frac{1}{p} + \frac{1}{q}\right) - \alpha \leqslant$$
$$\leqslant (q-1)\left(\frac{1}{q(q-1)} + \frac{1}{q}\right) - \alpha = 1 - \alpha,$$

by (4.1), a contradiction.∎

Lemma 4.3. *Suppose L is an α-list, $A = \{a_1, a_2, \ldots, a_m\} \subseteq L$, and $c \in [0\ \alpha)$ are such that level $(A) \geqslant 1 - c$ and $s(a_i) > c$, for $1 \leqslant i \leqslant q - 1$. Then $f^*(A) \geqslant 1$.*

Proof of lemma. We first observe that the hypotheses of the lemma are consistent, since $m \geqslant q - 1$ by Lemma 4.2. Our proof divides into three cases.

Case 1. $0 \leqslant c \leqslant \frac{1}{p}$.

In this case $f^*(A) \geqslant \frac{p}{p-1}\,\text{level}\,(A) \geqslant \frac{p}{p-1}\left(1 - \frac{1}{p}\right) = 1$.

Case 2. $\frac{1}{p} < c < \frac{1}{q}$.

We first observe that $c < \frac{1}{q}$ implies $(q-1)c < 1 - c$. Thus from Lemma 4.2 and the definition of f we have

$$f^*(A) \geqslant (q-1)\left[\frac{cpq-1}{(p-1)(q-1)}\right] + \frac{p}{p-1}[1-c-(q-1)c] =$$
$$= \frac{1}{p-1}[cpq - 1 + p - pqc] = 1.$$

Case 3. $\frac{1}{q} \leqslant c < \alpha$.

By Lemma 4.2 and the definition of f we have

$$f^*(A) \geqslant (q-1)\left(\frac{1}{q-1}\right) = 1.$$

This exhausts the possibilities.∎

Lemma 4.4. *Suppose* L *is an* α*-list,* $A \subseteq L$, *and* $c \in (0, \alpha]$, *and* $\beta > 0$ *are such that* $s(a) > c$ *for all* $a \in A$ *and* $level\,(A) = 1 - c - \beta > 1 - \alpha$. *Then* $f^*(A) \geqslant 1 - \left(\frac{p}{p-1}\right)\beta$.

Proof of lemma. Since $\text{level}\,(A) > 1 - \alpha$, we have $|A| \geqslant q - 1$ by Lemma 4.2. Consider an α-list L' such that $L' \supseteq A \cup Y$, where $Y = \{y_1, y_2, \ldots, y_p\}$ and $s(y_i) = \frac{\beta}{p}$, $1 \leqslant i \leqslant p$. Since $\text{level}\,(A \cup Y) = 1 - c$ and the elements of A constitute at least $q - 1$ elements of $A \cup Y$ with size exceeding c, Lemma 4.3 applies and we conclude

$$f^*(A \cup Y) = f^*(A) + f^*(Y) \geqslant 1.$$

However, since all elements of Y have size $\frac{\beta}{p} < \frac{1}{p}$, we have $f^*(Y) = \left(\frac{p}{p-1}\right)\beta$. The conclusion of the lemma follows.∎

We now resume the direct line of our proof of Theorem 5. Clearly we will be done if we can prove that the following inequality holds for all α-lists L:

(4.2) $\quad \text{FF}\,(L, 1) \leqslant W(\alpha) \cdot \text{OPT}\,(L, \alpha) + 1.$

So suppose L is an α-list. We first observe that by applying Lemma 4.1 to the bins of an optimal packing of L into α-bins, we obtain

(4.3) $\quad f^*(L) \leqslant W(\alpha) \cdot \text{OPT}\,(L, \alpha).$

Now consider the FF packing $P = \langle A_1, A_2, \ldots, A_{\mathrm{FF}(L,1)} \rangle$ of L into 1-bins. For each i, $2 \leqslant i \leqslant \mathrm{FF}\,(L, 1)$, define the *coarseness of bin* A_i, denoted by $c[i]$, by

$$c[i] = \max \{1 - \mathrm{level}\,(A_j) \colon\ 1 \leqslant j < i\},$$

and set $c[1] = 0$ by convention. Note that coarseness is a non-decreasing function of bin index, and all elements of A_i must have size exceeding $c[i]$ by the definition of FF, and that no bin can have coarseness as large as α. Thus $\mathrm{level}\,(A_i) > 1 - \alpha$ for all i, $1 \leqslant i \leqslant \mathrm{FF}\,(L, 1) - 1$, and each of these bins must satisfy the hypotheses of either Lemma 4.3 (with $c = c[i]$) or Lemma 4.4 (with $c = c[i]$ and $\beta = c[i+1] - c[i]$). Since $A_{\mathrm{FF}(L,1)}$ must contain at least one item of size exceeding $c[\mathrm{FF}\,(L, 1)]$, we thus can conclude

$$f^*(L) \geqslant \sum_{i=1}^{\mathrm{FF}(L,1)-1} f^*(A_i) + \frac{p}{p-1}\, c[\mathrm{FF}\,(L, 1)] \geqslant$$

$$\begin{aligned} &\geqslant \mathrm{FF}\,(L,1) - 1 - \frac{p}{p-1} \sum_{i=2}^{\mathrm{FF}(L,1)} (c[i] - c[i-1]) + \\ &+ \frac{p}{p-1}\, c[\mathrm{FF}\,(L,1)] = \end{aligned} \tag{4.4}$$

$$= \mathrm{FF}\,(L,1) - 1 - \frac{p}{p-1} (c[\mathrm{FF}\,(L,1)] - c[1]) +$$

$$+ \frac{p}{p-1}\, c[\mathrm{FF}\,(L,1)] = \mathrm{FF}\,(L,1) - 1$$

Combining this with (4.3) we get (4.2) and hence the theorem is proved. ∎

The two special case theorems presented in this section only establish our basic conjecture for a very small (though infinite) set of values for α. However, the results of our attempts to extend the type of weighting function argument used here to more general situations have been unsuccessful. The twin goals of constructing a weighting function f^* which satisfies inequalities (4.3) and (4.4), even for $|D| = 3$, appear to be irreconcilable. Alternative approaches have also been fruitless. Future researchers might wish to approach the general problem by attempting to prove the following result (currently still open despite its apparently obvious validity).

Conjecture. For all $\alpha > 0$,

$$R_\infty\left[\mathrm{FF}, \alpha + \frac{1}{2}\right] = 1 + R_\infty[\mathrm{FF}, \alpha].$$

A proof of this conjecture might suggest analogous results for unit fractions other than $\frac{1}{2}$, and lead to a proof of the general result.

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