7. Game Theory

In many situations in engineering and economy there are cases when the conflicting interests of several decision makers have to be taken into account simultaneously, and the outcome of the situation depends on the actions of these decision makers. One of the most popular methodology and modeling is based on game theory.

Let N denote the number of decision makers (who will be called ***players***), and for each $k = 1, 2, \ldots, N$ let S_k be the set of all feasible actions of player $\mathcal{P}_k$. The elements $s_k \in S_k$ are called ***strategies*** of player $\mathcal{P}_k$, S_k is the ***strategy set*** of this player. In any realization of the ***game*** each player selects a strategy, then the vector $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ ($s_k \in S_k$, $k = 1, 2, \ldots, N$) is called a ***simultaneous strategy vector*** of the players. For each $\mathbf{s} \in S = S_1 \times S_2 \times \cdots \times S_N$ each player has an outcome which is assumed to be a real value. This value can be imagined as the utility function value of the particular outcome, in which this function represents how player $\mathcal{P}_k$ evaluates the outcomes of the game. If $f_k(s_1, \ldots, s_N)$ denotes this value, then $f_k : S \to \mathbb{R}$ is called the ***payoff function*** of player $\mathcal{P}_k$. The value $f_k(\mathbf{s})$ is called the ***payoff*** of player $\mathcal{P}_k$ and $(f_1(\mathbf{s}), \ldots, f_N(\mathbf{s}))$ is called the ***payoff vector***. The number N of players, the sets S_k of strategies and the payoff functions f_k ($k = 1, 2, \ldots, N$) completely determine and define the N-person game. We will also use the notation $G = \{N; S_1, S_2, \ldots, S_N; f_1, f_2, \ldots, f_N\}$ for this game.

The solution of game G is the ***Nash-equilibrium***, which is a simultaneous strategy vector $\mathbf{s}^\star = (s_1^\star, \ldots, s_N^\star)$ such that for all k,

1. $s_k^\star \in S_k$;

2. for all $s_k \in S_k$,

$$f_k(s_1^\star, s_2^\star, \ldots, s_{k-1}^\star, s_k, s_{k+1}^\star, \ldots, s_N^\star) \le f_k(s_1^\star, s_2^\star, \ldots, s_{k-1}^\star, s_k^\star, s_{k+1}^\star, \ldots, s_N^\star) . \qquad (7.1)$$

Condition 1 means that the k-th component of the equilibrium is a feasible strategy of player $\mathcal{P}_k$, and condition 2 shows that none of the players can increase its payoff by unilaterally changing its strategy. In other words, it is the interest of all players to keep the equilibrium since if any player departs from the equilibrium, its payoff does not increase.

		Player $\mathcal{P}_2$	
		N	C
Player $\mathcal{P}_1$	N	$(-2, -2)$	$(-10, -1)$
	C	$(-1, -10)$	$(-5, -5)$

Figure 7.1. Prisoner's dilemma.

7.1. Finite Games

Game G is called finite if the number of players is finite and all strategy sets S_k contain finitely many strategies. The most famous two-person finite game is the ***prisoner's dilemma,*** which is the following.

Example 7.1 The players are two prisoners who committed a serious crime, but the prosecutor has only insufficient evidence to prosecute them. The prisoners are held in separate cells and cannot communicate, and the prosecutor wants them to cooperate with the authorities in order to get the needed additional information. So $N = 2$, and the strategy sets for both players have two elements: cooperating (C), or not cooperating (N). It is told to both prisoners privately that if he is the only one to confess, then he will get only a light sentence of 1 year, while the other will go to prison for a period of 10 years. If both confess, then their reward will be a 5 year prison sentence each, and if none of them confesses, then they will be convicted to a less severe crime with sentence of 2 years each. The objective of both players are to minimize the time spent in prison, or equivalently to maximize its negative. Figure 7.1 shows the payoff values, where the rows correspond to the strategies of player $\mathcal{P}_1$, the columns show the strategies of player $\mathcal{P}_2$, and for each strategy pair the first number is the payoff of player $\mathcal{P}_1$, and the second number is the payoff of player $\mathcal{P}_2$. Comparing the payoff values, it is clear that only (C, C) can be equilibrium, since

$$\begin{aligned} f_2(\mathrm{N,N}) = -2 &\quad < \quad f_2(\mathrm{N,C}) = -1, \\ f_1(\mathrm{N,C}) = -10 &\quad < \quad f_1(\mathrm{C,C}) = -5, \\ f_2(\mathrm{C,N}) = -10 &\quad < \quad f_2(C, C) = -5. \end{aligned}$$

The strategy pair (C, C) is really an equilibrium, since

$$\begin{aligned} f_1(\mathrm{C,C}) = -5 > f_1(\mathrm{N,C}) = -10, \\ f_2(\mathrm{C,C}) = -5 > f_2(\mathrm{C,N}) = -10. \end{aligned}$$

In this case we have a unique equilibrium.

The existence of an equilibrium is not guaranteed in general, and if equilibrium exists, it might not be unique.

Example 7.2 Modify the payoff values of Figure 7.1 as shown in Figure 7.2. It is easy to see that no equilibrium exists:

$$\begin{aligned} f_1(\mathrm{N,N}) = 1 &\quad < \quad f_1(\mathrm{C,N}) = 2, \\ f_2(\mathrm{C,N}) = 4 &\quad < \quad f_2(\mathrm{C,C}) = 5, \\ f_1(\mathrm{C,C}) = 0 &\quad < \quad f_1(\mathrm{N,C}) = 2, \\ f_2(\mathrm{N,C}) = 1 &\quad < \quad f_2(\mathrm{N,N}) = 2. \end{aligned}$$

		Player $\mathcal{P}_2$	
		N	C
Player $\mathcal{P}_1$	N	$(1, 2)$	$(2, 1)$
	C	$(2, 4)$	$(0, 5)$

Figure 7.2. Game with no equilibrium.

If all payoff values are identical, then we have multiple equilibria: any strategy pair is an equilibrium.

7.1.1. Enumeration

Let N denote the number of players, and for the sake of notational convenience let $s_k^{(1)}, \dots, s_k^{(n_k)}$ denote the feasible strategies of player $\mathcal{P}_k$. That is, $S_k = \{s_k^{(1)}, \dots, s_k^{(n_k)}\}$. A strategy vector $\mathbf{s}^\star = (s_1^{(i_1)}, \dots, s_N^{(i_N)})$ is an equilibrium if and only if for all $k = 1, 2, \dots, N$ and $j \in \{1, 2, \dots, n_k\} \setminus i_k$,

$$f_k(s_1^{(i_1)}, \dots, s_{k-1}^{(i_{k-1})}, s_k^{(j)}, s_{k+1}^{(i_{k+1})}, \dots, s_N^{(i_N)}) \le f_k(s_1^{(i_1)}, \dots, s_{k-1}^{(i_{k-1})}, s_k^{(i_k)}, s_{k+1}^{(i_{k+1})}, \dots, s_N^{(i_N)}). \tag{7.2}$$

Notice that in the case of finite games inequality (7.1) reduces to (7.2).

In applying the enumeration method, inequality (7.2) is checked for all possible strategy N-tuples $\mathbf{s}^\star = (s_1^{(i_1)}, \dots, s_N^{(i_N)})$ to see if (7.2) holds for all k and j. If it does, then $\mathbf{s}^\star$ is an equilibrium, otherwise not. If during the process of checking for a particular $\mathbf{s}^\star$ we find a k and j such that (7.2) is violated, then $\mathbf{s}^\star$ is not an equilibrium and we can omit checking further values of k and j. This algorithm is very simple, it consists of $N + 2$ imbedded loops with variables $i_1, i_2, \dots, i_N, k$ and j.

The maximum number of comparisons needed equals

$$\left(\prod_{k=1}^{N} n_k\right)\left(\sum_{k=1}^{N}(n_k - 1)\right),$$

however in practical cases it might be much lower, since if (7.2) is violated with some j, then the comparison must stop for the same strategy vector.

The algorithm can formally be given as follows:

for $i_1 \leftarrow 1$ **to** n_1
  do for $i_2 \leftarrow 1$ **to** n_2
    $\ddots$
      do for $i_N \leftarrow 1$ **to** n_N
        do $key \leftarrow 0$
        for $k \leftarrow 1$ **to** N
          do for $j \leftarrow 1$ **to** n_k
            do if (7.2) fails
              then $key \leftarrow 1$ and go to 10
        if $key = 0$
          then $(s_1^{(i_1)}, \ldots, s_N^{(i_N)})$ is equilibrium
          and give message accordingly

Consider next the two-person case, N=2, and introduce the $n_1 \times n_2$ real matrixes $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ with (i, j) elements $f_1(i, j)$ and $f_2(i, j)$ respectively. Matrixes $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are called the ***payoff matrixes*** of the two players. A strategy vector $\left(s_1^{(i_1)}, s_2^{(i_2)}\right)$ is an equilibrium if and only if the (i_1, i_2) element in matrix $\mathbf{A}^{(1)}$ is the largest in its column, and in matrix $\mathbf{A}^{(2)}$ it is the largest in its row. In the case when $f_2 = -f_1$, the game is called ***zero-sum***, and $\mathbf{A}^{(2)} = -\mathbf{A}^{(1)}$, so the game can be completely described by the payoff matrix $\mathbf{A}^{(1)}$ of the first player. In this special case a strategy vector $(s_1^{(i_1)}, s_2^{(i_2)})$ is an equilibrium if and only if the element (i_1, i_2) is the largest in its column and smallest in its row. In the zero-sum cases the equilibria are also called the ***saddle points*** of the games. Clearly, the enumeration method to find equilibria becomes more simple since we have to deal with a single matrix only.

The simplified algorithm is as follows:

for $i_1 \leftarrow 1$ **to** n_1
  do for $i_2 \leftarrow 1$ **to** n_2
    do $key \leftarrow 0$
    for $j \leftarrow 1$ **to** n_1
      do if $a_{ji_2}^{(1)} > a_{i_1 i_2}^{(1)}$
        then $key \leftarrow 1$
        and go to 12
    for $j \leftarrow 1$ **to** n_2
      do if $a_{i_1 j}^{(2)} > a_{i_1 i_2}^{(2)}$
        then $key \leftarrow 1$
        and go to 12
    if $key = 0$
      then give message that $(s_{i_1}^{(1)}, s_{i_2}^{(2)})$ is equilibrium

7.1.2. Games Represented by Finite Trees

Many finite games have the common feature that they can be represented by a finite directed tree with the following properties:

1. there is a unique root of the tree (which is not the endpoint of any arc), and the game starts at this node;
2. to each node of the tree a player is assigned and if the game reaches this node at any time, then this player will decide on the continuation of the game by selecting an arc originating from this node. Then the game moves to the endpoint of the chosen arc;
3. to each terminal node (in which no arc originates) an N-dimensional real vector is assigned which gives the payoff values for the players if the game terminates at this node;
4. each player knows the tree, the nodes he is assigned to, and all payoff values at the terminal nodes.

For example, the chess-game satisfies the above properties in which $N = 2$, the nodes of the tree are all possible configurations on the chessboard twice: once with the white player and once with the black player assigned to it. The arcs represent all possible moves of the assigned player from the originating configurations. The endpoints are those configurations in which the game terminates. The payoff values are from the set $\{1, 0, -1\}$ where 1 means win, -1 represents loss, and 0 shows that the game ends with a tie.

Theorem 7.1 *All games represented by finite trees have at least one equilibrium.*

Proof. We present the proof of this result here, since it suggests a practical algorithm to find equilibria. The proof goes by induction with respect to the number of nodes of the game tree. If the game has only one node, then clearly it is the only equilibrium.

Assume next that the theorem holds for any tree with less than n nodes ($n \geq 2$), and consider a game T_0 with n nodes. Let R be the root of the tree and let $r_1, r_2, \ldots, r_m$ ($m < n$) be the nodes connected to R by an arc. If $T_1, T_2, \ldots, T_m$ denote the disjoint subtrees of T_0 with roots $r_1, r_2, \ldots, r_m$, then each subtree has less than n nodes, so each of them has an equilibrium. Assume that player $\mathcal{P}_k$ is assigned to R. Let $e_1, e_2, \ldots, e_m$ be the equilibrium payoffs of player $\mathcal{P}_k$ on the subtrees $T_1, T_2, \ldots, T_m$ and let $e_j = \max\{e_1, e_2, \ldots, e_m\}$. Then player $\mathcal{P}_k$ will move to node r_j from the root, and then the equilibrium continues with the equilibrium obtained on the subtree T_j. We note that not all equilibria can be obtained by this method, however the payoff vectors of all equilibria, which can obtained by this method, are identical. ■

We note that not all equilibria can be obtained by this method, however the payoff vectors of all equilibria, which can be obtained by this method, are identical.

The proof of the theorem suggests a dynamic programming-type algorithm which is called ***backward induction***. It can be extended to the more general case when the tree has chance nodes from which the continuations of the game are random according to given discrete distributions.

The solution algorithm can be formally presented as follows. Assume that the nodes are numbered so each arc connects nodes i and j only for $i < j$. The root has to get the smallest number 1, and the largest number n is given to one of the terminal nodes. For each node i

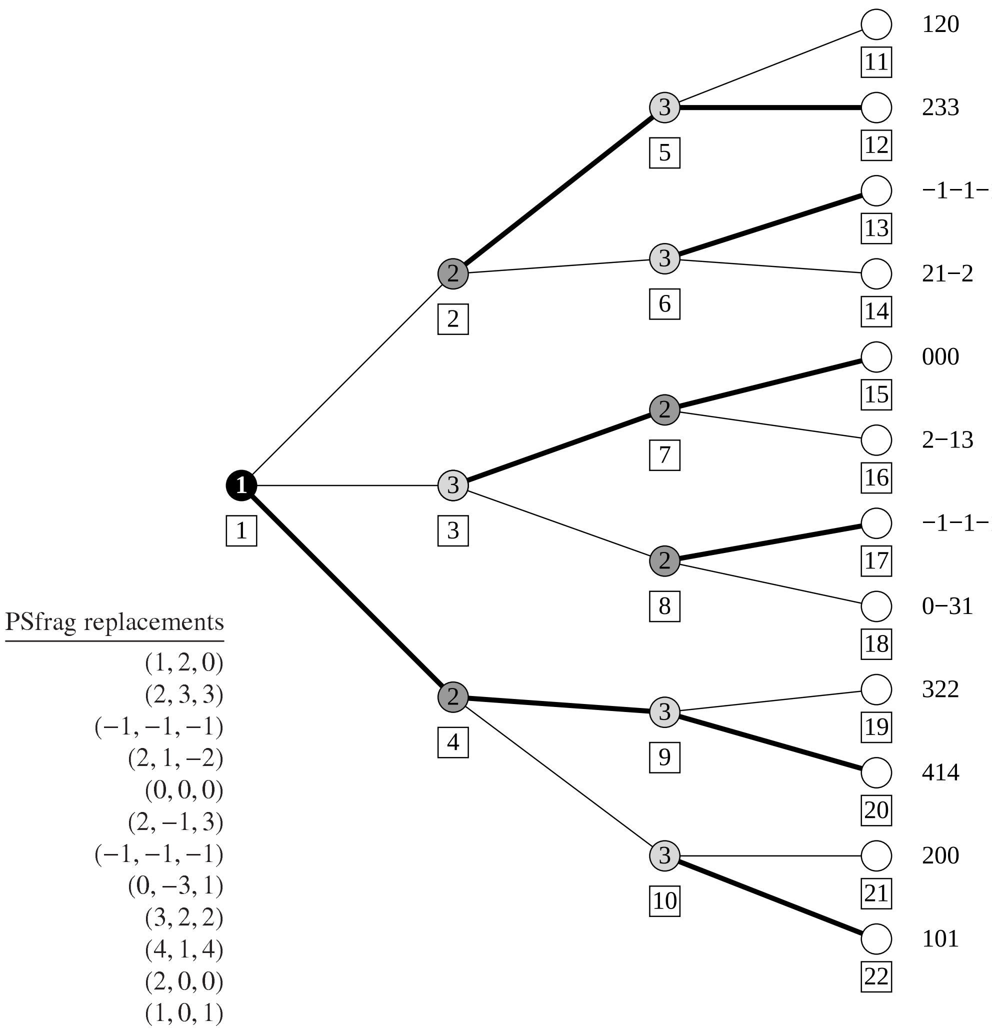


Figure 7.3. Finite tree of Example 10.3.

let $J(i)$ denote the set of all nodes j such that there is an arc from i to j. For each terminal node i, $J(i)$ is empty, and let $\mathbf{p}^{(i)} = (p_1^{(i)}, \ldots, p_N^{(i)})$ denote the payoff vector associated to this node. And finally we will denote player assigned to node i by K_i for all i. The algorithm starts at the last node n and moves backward in the order $n, n-1, n-2, \ldots, 2$ and 1. Node n is an endpoint, so vector $\mathbf{p}^{(n)}$ has been already assigned. If in the process the next node i is an endpoint, then $\mathbf{p}^{(i)}$ is already given, otherwise we find the largest among the values $p_{K_i}^{(j)}$, $j \in J(i)$. Assume that the maximal value occurs at node j_i, then we assign $\mathbf{p}^{(i)} = \mathbf{p}^{(j_i)}$ to node i, and move to node $i-1$. After all payoff vectors $\mathbf{p}^{(n)}$, $\mathbf{p}^{(n-1)}$, $\ldots, \mathbf{p}^{(2)}$ and $\mathbf{p}^{(1)}$ are determined, then vector $\mathbf{p}^{(1)}$ gives the equilibrium payoffs and the equilibrium path is obtained by nodes:

$$1 \to i_1 = j_1 \to i_2 = j_{i_1} \to i_3 = j_{i_2} \to \ldots,$$

until an endpoint is reached, when the equilibrium path terminates.

At each node the number of comparisons equals the number of arcs starting at that node minus 1. Therefore the total number of comparisons in the algorithm is the total number of arcs minus the number of nodes.

This algorithm can be formally given as follows:

for $i \leftarrow n$ **to** 1
  do $p_{K_i}^{(j_i)} \leftarrow \max\{p_{K_i}^{(l)}, l \in J(i)\}$
  $\mathbf{p}^{(i)} \leftarrow \mathbf{p}^{(j_i)}$
print sequence $1, i_1(= j_1), i_2(= j_{i_1}), i_3(= j_{i_2}), \ldots$
until an endpoint is reached

Example 7.3 Figure 7.3 shows a finite tree. In the circle at each nonterminal node we indicate the player assigned to that node. The payoff vectors are also shown at all terminal nodes. We have three players, so the payoff vectors have three elements.

First we number the nodes such that the beginning of each arc has a smaller number than its endpoint. We indicated these numbers in a box under each node. All nodes i for $i \geq 11$ are terminal nodes, as we start the backward induction with node 10. Since player $\mathcal{P}_3$ is assigned to this node we have to compare the third components of the payoff vectors $(2, 0, 0)$ and $(1, 0, 1)$ associated to the endpoints of the two arcs originating from node 10. Since $1 > 0$, player $\mathcal{P}_3$ will select the arc to node 22 as his best choice. Hence $j_{10} = 22$, and $\mathbf{p}^{(10)} = \mathbf{p}^{(22)} = (1, 0, 1)$. Then we check node 9. By comparing the third components of vectors $\mathbf{p}^{(19)}$ and $\mathbf{p}^{(20)}$ it is clear that player $\mathcal{P}_3$ will select node 20, so $j_9 = 20$, and $\mathbf{p}^{(9)} = \mathbf{p}^{(20)} = (4, 1, 4)$. In the graph we also indicated the choices of the players by thicker arcs. Continuing the procedure in the same way for nodes $8, 7, \ldots, 1$ we finally obtain the payoff vector $\mathbf{p}^{(1)} = (4, 1, 4)$ and equilibrium path $1 \to 4 \to 9 \to 20$.

Exercises

7.1-1 An entrepreneur (E) enters to a market, which is controlled by a chain store (C). Their competition is a two-person game. The strategies of the chain store are soft (S), when it allows the competitor to operate or tough (T), when it tries to drive out the competitor. The strategies of the entrepreneur are staying in (I) or leaving (L) the market. The payoff tables of the two player are assumed to be

	I	L
S	2	5
T	0	5

payoffs of C

	I	L
S	2	1
T	0	1

payoffs of E

Find the equilibrium.

7.1-2 A salesman sells an equipment to a buyer, which has 3 parts, under the following conditions. If all parts are good, then the customer pays $\$\alpha$ to the salesman, otherwise the salesman has to pay $\$\beta$ to the customer. Before selling the equipment, the salesman is able to check any one or more of the parts, but checking any one costs him $\$\gamma$. Consider a two-

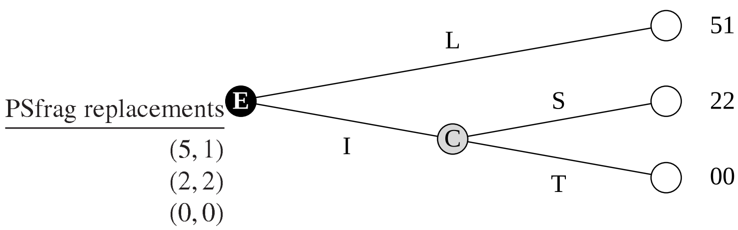


Figure 7.4. Tree for Exercise 7.1-5.

person game in which player $\mathcal{P}_1$ is the salesman with strategies $0, 1, 2, 3$ (how many parts he checks before selling the equipment), and player $\mathcal{P}_2$ is the equipment with strategies $0, 1, 2, 3$ (how many parts are defective). Show that the payoff matrix of player $\mathcal{P}_1$ is given as below when we assume that the different parts can be defective with equal probability.

		player $\mathcal{P}_2$			
		0	1	2	3
	0	α	$-\beta$	$-\beta$	$-\beta$
player $\mathcal{P}_1$	1	$\alpha-\gamma$	$-\frac{2}{3}\beta-\gamma$	$-\frac{1}{3}\beta-\gamma$	$-\gamma$
	2	$\alpha-2\gamma$	$-\frac{1}{3}\beta-\frac{5}{3}\gamma$	$-\frac{4}{3}\gamma$	$-\gamma$
	3	$\alpha-3\gamma$	-2γ	$-\frac{4}{3}\gamma$	$-\gamma$

7.1-3 Assume that in the previous problem the payoff of the second player is the negative of the payoff of the salesman. Give a complete description of the number of equilibria as a function of the parameter values α, β, γ. Determine the equilibria in all cases.
7.1-4 Assume that the payoff function of the equipment is its value (V if all parts are good, and zero otherwise) in the previous exercise. Is there an equilibrium point?
7.1-5 Exercise 7.1-1. can be represented by the tree shown in Figure 7.4.
Find the equilibrium with backward induction.
7.1-6 Show that in the one-player case backward induction reduces to the classical dynamic programming method.
7.1-7 Assume that in the tree of a game some nodes are so called "chance nodes" from which the game continuous with given probabilities assigned to the possible next nodes. Show the existence of the equilibrium for this more general case.
7.1-8 Consider the tree given in Figure 7.3, and double the payoff values of player $\mathcal{P}_1$, change the sign of the payoff values of player $\mathcal{P}_2$, and do not change those for Player $\mathcal{P}_3$. Find the equilibrium of this new game.

7.2. Continuous Games

If the strategy sets S_k are connected subsets of finite dimensional Euclidean Spaces and the payoff functions are continuous, then the game is considered continuous.

7.2.1. Fixed-Point Methods Based on Best Responses

It is very intuitive and usefull from algorithmic point of view to reformulate the equilibrium concept as follows. For all players $\mathcal{P}_k$ and $\mathbf{s} = (s_1, s_2, \ldots, s_N) \in S = S_1 \times S_2 \times \cdots \times S_N$ define the mapping:

$$B_k(\mathbf{s}) = \{s_k \in S_k | f_k(s_1, s_2, \ldots, s_{k-1}, s_k, s_{k+1}, \ldots, s_N) \\ = \max_{t_k \in S_k} f_k(s_1, s_2, \ldots, s_{k-1}, t_k, s_{k+1}, \ldots, s_N)\}, \tag{7.3}$$

which is the set of the best choices of player $\mathcal{P}_k$ with given strategies $s_1, s_2, \ldots, s_{k-1}, s_{k+1}, \ldots, s_N$ of the other players. Note that $B_k(\mathbf{s})$ does not depend on s_k, it depends only on all other strategies $\mathbf{s}_l$, $k \neq l$. There is no guarantee that maximum exists for all $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_N$. Let $\sum \subseteq S$ be the subset of S such that $B_k(\mathbf{s})$ exists for all k and $\mathbf{s} \in \sum$. A simultaneous strategy vector $\mathbf{s}^\star = (s_1^\star, s_2^\star, \ldots, s_N^\star)$ is an equilibrium if and only if $\mathbf{s}^\star \in \sum$, and $s_k^\star \in B_k(\mathbf{s}^\star)$ for all k. By introducing the ***best reply mapping,*** $\mathbf{B}_k(\mathbf{s}) = (B_1(\mathbf{s}), \ldots, B_N(\mathbf{s}))$ we can further simplify the above reformulation:

Theorem 7.2 *Vector $\mathbf{s}^\star$ is equilibrium if and only if $\mathbf{s}^\star \in \sum$ and $\mathbf{s}^\star \in \mathbf{B}(\mathbf{s}^\star)$.*

Hence we have shown that the equilibrium-problem of N-person games is equivalent to find fixed points of certain point-to-set mappings.

The most frequently used existence theorems of equilibria are based on fixed point theorems such as the theorems of Brouwer, Kakutani, Banach, Tarski etc. Any algorithm for finding fixed points can be successfully applied for computing equilibria.

The most popular existence result is a straightforward application of the Kakutani-fixed point theorem.

Theorem 7.3 *Assume that in an N-person game*

1. *the strategy sets S_k are nonempty, closed, bounded, convex subsets of finite dimensional Euclidean spaces;*
   for all k,
2. *the payoff function f_k are continuous on S;*
3. *f_k is concave in s_k with all fixed $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_N$.*

Then there is at least one equilibrium.

Example 7.4 Consider a 2-person game, $N = 2$, with strategy sets $S_1 = S_2 = [0, 1]$, and payoff functions $f_1(s_1, s_2) = s_1 s_2 - 2s_1^2 + 5$, and $f_2(s_1, s_2) = s_1 s_2 - 2s_2^2 + s_2 + 3$. We will first find the best responses of both players. Both payoff functions are concave parabolas in their variables with vertices:

$$s_1 = \frac{s_2}{4} \quad \text{and} \quad s_2 = \frac{s_1 + 1}{4}.$$

For all $s_2 \in [0, 1]$ and $s_1 \in [0, 1]$ these values are clearly feasible strategies, so

$$B_1(\mathbf{s}) = \frac{s_2}{4} \quad \text{and} \quad B_2(\mathbf{s}) = \frac{s_1 + 1}{4}.$$

So $(s_1^\star, s_2^\star)$ is equilibrium if and only if it satisfies equations:

$$s_1^\star = \frac{s_2^\star}{4} \quad \text{and} \quad s_2^\star = \frac{s_1^\star + 1}{4}.$$

It is easy to see that the unique solution is:

$$s_1^\star = \frac{1}{15} \quad \text{and} \quad s_2^\star = \frac{4}{15},$$

which is therefore the unique equilibrium of the game.

Example 7.5 Consider a portion of a sea-channel, assume it is the unit interval $[0, 1]$. Player $\mathcal{P}_2$ is a submarine hiding in location $s_2 \in [0, 1]$, player $\mathcal{P}_1$ is an airplane dropping a bomb at certain location $s_1 \in [0, 1]$ resulting in a damage $\alpha e^{-\beta(s_1-s_2)^2}$ to the submarine. Hence a special two-person game is defined in which $S_1 = S_2 = [0, 1]$, $f_1(s_1, s_2) = \alpha e^{-\beta(s_1-s_2)^2}$ and $f_2(s_1, s_2) = -f_1(s_1, s_2)$. With fixed s_2, $f_1(s_1, s_2)$ is maximal if $s_1 = s_2$, therefore the best response of player $\mathcal{P}_1$ is $B_1(\mathbf{s}) = s_2$. Player $\mathcal{P}_2$ wants to minimize f_1 which occurs if $|s_1 - s_2|$ is as large as possible, which implies that

$$B_2(\mathbf{s}) = \begin{cases} 1, & \text{if } s_1 < 1/2, \\ 0, & \text{if } s_1 > 1/2, \\ \{0, 1\}, & \text{if } s_1 = 1/2. \end{cases}$$

Clearly, there is no $(s_1, s_2) \in [0, 1] \times [0, 1]$ such that $s_1 = B_1(\mathbf{s})$ and $s_2 \in B_2(\mathbf{s})$, consequently no equilibrium exists.

7.2.2. Applying Fan's Inequality

Define the ***aggregation function*** $H : S \times S \to \mathbb{R}$ as:

$$H_{\mathbf{r}}(\mathbf{s}, \mathbf{z}) = \sum_{k=1}^{N} r_k f_k(s_1, \ldots, s_{k-1}, z_k, s_{k+1}, \ldots, s_N) \tag{7.4}$$

for all $\mathbf{s} = (s_1, \ldots, s_N)$ and $\mathbf{z} = (z_1, \ldots, z_N)$ from S and some $\mathbf{r} = (r_1, r_2, \ldots, r_N) > \mathbf{0}$.

Theorem 7.4 *Vector $\mathbf{s}^\star \in S$ is an equilibrium if and only if*

$$H_{\mathbf{r}}(\mathbf{s}^\star, \mathbf{z}) \le H_{\mathbf{r}}(\mathbf{s}^\star, \mathbf{s}^\star) \tag{7.5}$$

for all $\mathbf{z} \in S$.

Proof. Assume first that $\mathbf{s}^\star$ is an equilibrium, then inequality (7.1) holds for all k and $s_k \in S_k$. Adding the r_k-multiples of these relations for $k = 1, 2, \ldots, N$ we immediately have (7.5).

Assume next that (7.5) holds for all $\mathbf{z} \in S$. Select any k and $s_k \in S_k$, define $\mathbf{z} = (s_1^\star, \ldots, s_{k-1}^\star, s_k, s_{k+1}^\star, \ldots, s_N^\star)$, and apply inequality (7.5). All but the k-th terms cancel and the remaining term shows that inequality (7.1) holds. Hence $\mathbf{s}^\star$ is an equilibrium. ■

Introduce function $\phi(\mathbf{s}, \mathbf{z}) = H_{\mathbf{r}}(\mathbf{s}, \mathbf{z}) - H_{\mathbf{r}}(\mathbf{s}, \mathbf{s})$, then clearly $\mathbf{s}^\star \in S$ is an equilibrium if and only if

$$\phi(\mathbf{s}^\star, \mathbf{z}) \le 0 \quad \text{for all} \quad \mathbf{z} \in S. \tag{7.6}$$

Relation (7.6) is known as ***Fan's inequality*** . It can be rewritten as a variational inequality (see section 7.2.9 later), or as a fixed point problem. We show here the second approach. For all $\mathbf{s} \in S$ define

$$\Phi(\mathbf{s}) = \{\mathbf{z} | \mathbf{z} \in S, \ \phi(\mathbf{s}, \mathbf{z}) = \max_{\mathbf{t} \in S} \phi(\mathbf{s}, \mathbf{t})\}. \tag{7.7}$$

Since $\phi(\mathbf{s}, \mathbf{s}) = 0$ for all $\mathbf{s} \in S$, relation (7.6) holds if and only if $\mathbf{s}^\star \in \Phi(\mathbf{s}^\star)$, that is $\mathbf{s}^\star$ is a fixed-point of mapping $\Phi : S \to 2^S$. Therefore any method to find fixed point is applicable for computing equilibria.

The computation cost depends on the type and size of the fixed point problem and also on the selected method.

Example 7.6 Consider again the problem of Example 7.4.. In this case

$$f_1(z_1, s_2) = z_1 s_2 - 2z_1^2 + 5,$$

$$f_2(s_1, z_2) = s_1 z_2 - 2z_2^2 + z_2 + 3,$$

so the aggregate function has the form with $r_1 = r_2 = 1$:

$$H_{\mathbf{r}}(\mathbf{s}, \mathbf{z}) = z_1 s_2 - 2z_1^2 + s_1 z_2 - 2z_2^2 + z_2 + 8.$$

Therefore

$$H_{\mathbf{r}}(\mathbf{s}, \mathbf{s}) = 2s_1 s_2 - 2s_1^2 - 2s_2^2 + s_2 + 8,$$

and

$$\phi(\mathbf{s}, \mathbf{z}) = z_1 s_2 - 2z_1^2 + s_1 z_2 - 2z_2^2 + z_2 - 2s_1 s_2 + 2s_1^2 + 2s_2^2 - s_2.$$

Notice that this function is strictly concave in z_1 and z_2, and is separable in these variables. At the stationary points:

$$\frac{\partial \phi}{\partial z_1} = s_2 - 4z_1 = 0$$

$$\frac{\partial \phi}{\partial z_2} = s_1 - 4z_2 + 1 = 0$$

implying that at the optimum

$$z_1 = \frac{s_2}{4} \quad \text{and} \quad z_2 = \frac{s_1 + 1}{4} \ ,$$

since both right hand sides are feasible. At the fixed point:

$$s_1 = \frac{s_2}{4} \quad \text{and} \quad s_2 = \frac{s_1 + 1}{4},$$

giving the unique solution:

$$s_1 = \frac{1}{15} \quad \text{and} \quad s_2 = \frac{4}{15}.$$

7.2.3. Solving the Kuhn–Tucker Conditions

Assume that for all k,

$$S_k = \{s_k | \mathbf{g}_k(s_k) \geq \mathbf{0}\},$$

where $\mathbf{g}_k : \mathbb{R}^{n_k} \to \mathbb{R}^{m_k}$ is a vector variable vector valued function which is continuously differentiable in an open set O_k containing S_k. Assume furthermore that for all k, the payoff function f_k is continuously differentiable in s_k on O_k with any fixed $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_N$.

If $\mathbf{s}^\star = (s_1^\star, \ldots, s_N^\star)$ is an equilibrium, then for all k, $s_k^\star$ is the optimal solution of problem:

$$\begin{array}{ll} \text{maximize} & f_k(s_1^\star, \ldots, s_{k-1}^\star, s_k, s_{k+1}^\star, \ldots, s_N^\star) \\ \text{sugject to} & \mathbf{g}_k(s_k) \geq \mathbf{0}. \end{array} \tag{7.8}$$

By assuming that at s_k the Kuhn–Tucker regularity condition is satisfied, the solution has to satisfy the Kuhn–Tucker necessary condition:

$$\begin{aligned} \mathbf{u}_k &\geq \mathbf{0} \\ \mathbf{g}_k(s_k) &\geq \mathbf{0} \\ \nabla_k f_k(\mathbf{s}) + \mathbf{u}_k^T \nabla_k \mathbf{g}_k(s_k) &= \mathbf{0}^T \\ \mathbf{u}_k^T \mathbf{g}_k(s_k) &= 0, \end{aligned} \tag{7.9}$$

where $\mathbf{u}_k$ is an m_k-element column vector, $\mathbf{u}_k^T$ is its transpose, $\nabla_k f_k$ is the gradient of f_k (as a row vector) with respect to s_k and $\nabla_k \mathbf{g}_k$ is the Jacobian of function g_k.

Theorem 7.5 *If $\mathbf{s}^\star$ is an equilibrium, then there are vectors $\mathbf{u}_k^\star$ such that relations (7.9) are satisfied.*

Relations (7.9) for $k = 1, 2, \ldots, N$ give a (usually large) system of equations and inequalities for the unknowns s_k and $\mathbf{u}_k$ ($k = 1, 2, \ldots, N$). Any equilibrium (if exists) has to be among the solutions. If in addition for all k, all components of $\mathbf{g}_k$ are concave, and f_k is concave in s_k, then the Kuhn–Tucker conditions are also sufficient, and therefore all solutions of (7.9) are equilibria.

The computation cost in solving system (7.9) depends on its type and the chosen method.

Example 7.7 Consider again the two-person game of the previous example. Clearly,

$$S_1 = \{s_1 | s_1 \geq 0,\ 1 - s_1 \geq 0\},$$

$$S_2 = \{s_2 | s_2 \geq 0,\ 1 - s_2 \geq 0\},$$

so we have

$$\mathbf{g}_1(s_1) = \begin{pmatrix} s_1 \\ 1 - s_1 \end{pmatrix} \text{ and } \mathbf{g}_2(s_2) = \begin{pmatrix} s_2 \\ 1 - s_2 \end{pmatrix}.$$

Simple differentiation shows that

$$\nabla_1 \mathbf{g}_1(s_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \nabla_2 \mathbf{g}_2(s_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\nabla_1 f_1(s_1, s_2) = s_2 - 4s_1, \quad \nabla_2 f_2(s_1, s_2) = s_1 - 4s_2 + 1,$$

therefore the Kuhn–Tucker conditions can be written as follows:

$$\begin{aligned} u_1^{(1)}, u_2^{(1)} &\geq 0 \\ s_1 &\geq 0 \\ s_1 &\leq 1 \\ s_2 - 4s_1 + u_1^{(1)} - u_2^{(1)} &= 0 \\ u_1^{(1)} s_1 + u_2^{(1)}(1 - s_1) &= 0 \\ u_1^{(2)}, u_2^{(2)} &\geq 0 \\ s_2 &\geq 0 \\ s_2 &\leq 1 \\ s_1 - 4s_2 + 1 + u_1^{(2)} - u_2^{(2)} &= 0 \\ u_1^{(2)} s_2 + u_2^{(2)}(1 - s_2) &= 0. \end{aligned}$$

Notice that f_1 is concave in s_1, f_2 is concave in s_2, and all constraints are linear, therefore all solutions of this equality-inequality system are really equilibria. By systematically examining the combination of cases

$$s_1 = 0, \quad 0 < s_1 < 1, \quad s_1 = 1,$$

and

$$s_2 = 0, \quad 0 < s_2 < 1, \quad s_2 = 1,$$

it is easy to see that there is a unique solution

$$u_1^{(1)} = u_1^{(2)} = u_2^{(1)} = u_2^{(2)} = 0, \quad s_1 = \frac{1}{15}, \quad s_2 = \frac{4}{15}.$$

By introducing slack and surplus variables the Kuhn–Tucker conditions can be rewritten as a system of equations with some nonnegative variables. The nonnegativity conditions can be formally eliminated by considering them as squares of some new variables, so the result becomes a system of (usually) nonlinear equations without additional constraints. There is a large set of numerical methods for solving such systems.

7.2.4. Reduction to Optimization Problems

Assume that all conditions of the previous section hold. Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{N} \mathbf{u}_k^T \mathbf{g}_k(s_k) \\ \text{subjective to} & \mathbf{u}_k \geq \mathbf{0} \\ & \mathbf{g}_k(s_k) \geq \mathbf{0} \\ & \nabla_k f_k(\mathbf{s}) + \mathbf{u}_k^T \nabla_k \mathbf{g}_k(s_k) = 0. \end{array} \tag{7.10}$$

The two first constraints imply that the objective function is nonnegative, so is the minimal value of it. Therefore system (7.9) has feasible solution if and only if the optimal value of the objective function of problem (7.10) is zero, and in this case any optimal solution satisfies relations (7.9).

Theorem 7.6 *The N-person game has equilibrium only if the optimal value of the objective function is zero. Then any equilibrium is optimal solution of problem* (7.10)*. If in addition all components of $\mathbf{g}_k$ are concave and f_k is concave in s_k for all k, then any optimal solution of problem* (7.10) *is equilibrium.*

Hence the equilibrium problem of the N-person game has been reduced to finding the optimal solutions of this (usually nonlinear) optimization problem. Any nonlinear programming method can be used to solve the problem.

The computation cost in solving the optimization problem (7.10) depends on its type and the chosen method. For example, if (7.10) is an LP, and solved by the simplex method, then the maximum number of operations is exponential. However in particular cases the procedure terminates with much less operations.

Example 7.8 In the case of the previous problem the optimization problem has the following form:

$$\begin{aligned}
\text{minimize} \quad & u_1^{(1)} s_1 + u_2^{(1)}(1 - s_1) + u_1^{(2)} s_2 + u_2^{(2)}(1 - s_2) \\
\text{subject to} \quad & u_1^{(1)}, u_1^{(2)}, u_2^{(1)}, u_2^{(2)} \geq 0 \\
& s_1 \geq 0 \\
& s_1 \leq 1 \\
& s_2 \geq 0 \\
& s_2 \leq 1 \\
& s_2 - 4s_1 + u_1^{(1)} - u_2^{(1)} = 0 \\
& s_1 - 4s_2 + 1 + u_1^{(2)} - u_2^{(2)} = 0.
\end{aligned}$$

Notice that the solution $u_1^{(1)} = u_1^{(2)} = u_2^{(1)} = u_2^{(2)} = 0$, $s_1 = 1/15$ and $s_2 = 4/15$ is feasible with zero objective function value, so it is also optimal. Hence it is a solution of system (7.9) and consequently an equilibrium.

Mixed Extension of Finite Games

We have seen earlier that a finite game does not necessary have equilibrium. Even if it does, in the case of repeating the game many times the players wish to introduce some randomness into their actions in order to make the other players confused and to seek an equilibrium in the stochastic sense. This idea can be modeled by introducing probability distributions as the strategies of the players and the expected payoff values as their new payoff functions.

Keeping the notation of section 7.1. assume that we have N players, the finite strategy set of player $\mathcal{P}_k$ is $S_k = \{s_k^{(1)}, \ldots, s_k^{(n_k)}\}$. In the mixed extension of this finite game each player selects a discrete probability distribution on its strategy set and in each realization of the game an element of S_k is chosen according to the selected distribution. Hence the new strategy set of player $\mathcal{P}_k$ is

$$\overline{S}_k = \{\mathbf{x}_k | \mathbf{x}_k = (x_k^{(1)}, \ldots, x_k^{(n_k)}),\ \sum_{i=1}^{n_k} x_k^{(i)} = 1,\ x_k^{(i)} \geq 0 \text{ for all } i\}, \tag{7.11}$$

which is the set of all n_k-element probability vectors. The new payoff function of this player is the expected value:

$$\overline{f}_k(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \ldots \sum_{i_N=1}^{n_N} f_k(s_1^{(i_1)}, s_2^{(i_2)}, \ldots, s_N^{(i_N)}) x_1^{(i_1)} x_2^{(i_2)} \ldots x_N^{(i_N)}. \tag{7.12}$$

Notice that the original "pure" strategies $s_k^{(i)}$ can be obtained by selecting $\mathbf{x}_k$ as the k-th basis vector. This is a continuous game and as a consequence of Theorem 7.3 it has at least one equilibrium. Hence if a finite game is without an equilibrium, its mixed extension has always at least one equilibrium, which can be obtained by using the methods outlined in the previous sections.

Example 7.9 Consider the two-person case in which $N = 2$, and as in section 7.1 introduce matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ with (i, j) elements $f_1(s_1^{(i)}, s_2^{(j)})$ and $f_2(s_1^{(i)}, s_2^{(j)})$. In this special case

$$\overline{f}_k(\mathbf{x}_1, \mathbf{x}_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij}^{(k)} x_i^{(1)} x_j^{(2)} = \mathbf{x}_1^T \mathbf{A}^{(k)} \mathbf{x}_2 \,. \tag{7.13}$$

The constraints of $\overline{S}_k$ can be rewritten as:

$$\begin{aligned} x_k^{(i)} &\geq 0 \qquad (i = 1, 2, \ldots, n_k), \\ -1 + \sum_{i=1}^{n_k} x_k^{(i)} &\geq 0, \\ 1 - \sum_{i=1}^{n_k} x_k^{(i)} &\geq 0. \end{aligned}$$

so we may select

$$\mathbf{g}_k(\mathbf{x}_k) = \begin{pmatrix} x_k^{(1)} \\ \vdots \\ x_k^{(n_k)} \\ \sum_{i=1}^{n_k} x_k^{(i)} - 1 \\ -\sum_{i=1}^{n_k} x_k^{(i)} + 1 \end{pmatrix}. \tag{7.14}$$

The optimization problem (7.10) now reduces to the following:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{2}[\sum_{i=1}^{n_k} u_k^{(i)} x_k^{(i)} + u_k^{(n_k+1)}(\sum_{j=1}^{n_k} x_k^{(j)} - 1) + u_k^{(n_k+2)}(-\sum_{j=1}^{n_k} x_k^{(j)} + 1)] \\ \text{subject to} & u_k^{(i)} \geq 0 \quad (1 \leq i \leq n_k + 2) \\ & x_k^{(i)} \geq 0 \quad (1 \leq i \leq n_k) \\ & \mathbf{1}^T \mathbf{x}_k = 1 \\ & \mathbf{x}_2^T(\mathbf{A}^{(1)})^T + \mathbf{v}_1^T + (u_1^{(n_1+1)} - u_1^{(n_1+2)})\mathbf{1}_1^T = \mathbf{0}_1^T \\ & \mathbf{x}_1^T(\mathbf{A}^{(2)}) + \mathbf{v}_2^T + (u_2^{(n_2+1)} - u_2^{(n_2+2)})\mathbf{1}_2^T = \mathbf{0}_2^T, \end{array} \tag{7.15}$$

where $\mathbf{v}_k^T = (u_k^{(1)}, \ldots . u_k^{(n_k)})$, $\mathbf{1}_k^T = (1^{(1)}, \ldots, 1^{(n_k)})$ and $\mathbf{0}_k^T = (0^{(1)}, \ldots, 0^{(n_k)})$, $k = 1, 2$.

Notice this is a quadratic optimization problem. Computation cost depends on the selected method. Observe that the problem is usually nonconvex, so there is the possibility of stopping at a local optimum.

Bimatrix games

Mixed extensions of two-person finite games are called ***bimatrix games*** . They were already examined in Example 7.9.. For notational convenience introduce the simplifying notation:

$$\mathbf{A} = \mathbf{A}^{(1)}, \mathbf{B} = \mathbf{A}^{(2)}, \mathbf{x} = \mathbf{x}_1, \mathbf{y} = \mathbf{x}_2, m = n_1 \quad \text{and} \quad n = n_2.$$

We will show that problem (7.15) can be rewritten as quadratic programming problem with linear constraints.

Consider the objective function first. Let

$$\alpha = u_1^{(m+2)} - u_1^{(m+1)}, \quad \text{and} \quad \beta = u_2^{(n+2)} - u_2^{(n+1)},$$

then the objective function can be rewritten as follows:

$$\mathbf{v}_1^T \mathbf{x} + \mathbf{v}_2^T \mathbf{y} - \alpha(\mathbf{1}_m^T \mathbf{x} - 1) - \beta(\mathbf{1}_n^T \mathbf{y} - 1). \tag{7.16}$$

The last two constraints also simplify:

$$\begin{aligned} \mathbf{y}^T \mathbf{A}^T + \mathbf{v}_1^T - \alpha \mathbf{1}_m^T &= \mathbf{0}_m^T, \\ \mathbf{x}^T \mathbf{B} + \mathbf{v}_2^T - \beta \mathbf{1}_n^T &= \mathbf{0}_n^T, \end{aligned}$$

implying that

$$\mathbf{v}_1^T = \alpha \mathbf{1}_m^T - \mathbf{y}^T \mathbf{A}^T \quad \text{and} \quad \mathbf{v}_2^T = \beta \mathbf{1}_n^T - \mathbf{x}^T \mathbf{B}, \tag{7.17}$$

so we may rewrite the objective function again:

$$(\alpha \mathbf{1}_m^T - \mathbf{y}^T \mathbf{A}^T)\mathbf{x} + (\beta \mathbf{1}_n^T - \mathbf{x}^T \mathbf{B})\mathbf{y} - \alpha(\mathbf{1}_m^T \mathbf{x} - 1) - \beta(\mathbf{1}_n^T \mathbf{y} - 1) = \alpha + \beta - \mathbf{x}^T(\mathbf{A} + \mathbf{B})\mathbf{y},$$

since

$$\mathbf{1}_m^T \mathbf{x} = \mathbf{1}_n^T \mathbf{y} = 1.$$

Hence we have the following quadratic programming problem :

$$\begin{array}{ll} \text{maximize} & \mathbf{x}^T(\mathbf{A} + \mathbf{B})\mathbf{y} - \alpha - \beta \\ \text{subject to} & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{1}_m^T \mathbf{x} = 1 \\ & \mathbf{1}_n^T \mathbf{y} = 1 \\ & \mathbf{A}\mathbf{y} \leq \alpha \mathbf{1}_m \\ & \mathbf{B}^T \mathbf{x} \leq \beta \mathbf{1}_n, \end{array} \tag{7.18}$$

where the last two conditions are obtained from (7.17) and the nonnegativity of vectors $\mathbf{v}_1$, $\mathbf{v}_2$.

Theorem 7.7 *Vectors $\mathbf{x}^\star$ and $\mathbf{y}^\star$ are equilibria of the bimatrix game if and only if with some $\alpha^\star$ and $\beta^\star$, $(\mathbf{x}^\star, \mathbf{y}^\star, \alpha^\star, \beta^\star)$ is optimal solution of problem (7.18). The optimal value of the objective function is zero.*

This is a quadratic programming problem. Computation cost depends on the selected method. Since it is usually nonconvex, the algorithm might terminate at local optimum. We know that at the global optimum the objective function must be zero, which can be used for optimality check.

Example 7.10 Select

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

so problem (7.18) has the form:

$$\begin{array}{ll} \text{maximize} & 3x_1y_1 - 2x_1y_2 - 2x_2y_1 + 3x_2y_2 - \alpha - \beta \\ \text{subject to} & x_1, x_2, y_1, y_2 \geq 0 \\ & x_1 + x_2 = 1 \\ & y_1 + y_2 = 1 \\ & 2y_1 - y_2 \leq \alpha \\ & -y_1 + y_2 \leq \alpha \\ & x_1 - x_2 \leq \beta \\ & -x_1 + 2x_2 \leq \beta, \end{array}$$

where $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$. We also know from Theorem 7.7 that the optimal objective function value is zero, therefore any feasible solution with zero objective function value is necessarily optimal. It is easy to see that the solutions

$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{y} = \begin{pmatrix}0\\1\end{pmatrix},\ \alpha = 2, \beta = 1,$$
$$\mathbf{x} = \begin{pmatrix}0\\1\end{pmatrix},\ \mathbf{y} = \begin{pmatrix}1\\0\end{pmatrix},\ \alpha = 1,\ \beta = 2,$$
$$\mathbf{x} = \begin{pmatrix}0.6\\0.4\end{pmatrix},\ \mathbf{y} = \begin{pmatrix}0.4\\0.6\end{pmatrix},\ \alpha = 0.2,\ \beta = 0.2$$

are all optimal, so they provide equilibria.

One might apply relations (7.9) to find equilibria by solving the equality-inequality system instead of solving an optimization problem. In the case of bimatrix games problem (7.9) simplifies as

$$\begin{array}{rcl} \mathbf{x}^T\mathbf{A}\mathbf{y} & = & \alpha \\ \mathbf{x}^T\mathbf{B}\mathbf{y} & = & \beta \\ \mathbf{A}\mathbf{y} & \leq & \alpha\mathbf{1}_m \\ \mathbf{B}^T\mathbf{x} & \leq & \beta\mathbf{1}_n \\ \mathbf{x} & \geq & \mathbf{0}_m \\ \mathbf{y} & \geq & \mathbf{0}_n \\ \mathbf{1}_m^T\mathbf{x} = \mathbf{1}_n^T\mathbf{y} & = & 1, \end{array} \tag{7.19}$$

which can be proved along the lines of the derivation of the quadratic optimization problem.

The computation cost of the solution of system (7.19) depends on the particular method being selected.

Example 7.11 Consider again the bimatrix game of the previous example. Substitute the first and second constraints $\alpha = \mathbf{x}^T\mathbf{A}\mathbf{y}$ and $\beta = \mathbf{x}^T\mathbf{B}\mathbf{y}$ into the third and fourth condition to have

$$\begin{array}{rcl} 2y_1 - y_2 & \leq & 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 \\ -y_1 + y_2 & \leq & 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 \\ x_1 - x_2 & \leq & x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 \\ -x_1 + 2x_2 & \leq & x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 \\ x_1, x_2, y_1, y_2 & \geq & 0 \\ x_1 + x_2 = y_1 + y_2 & = & 1. \end{array}$$

It is easy to see that the solutions given in the previous example solve this system, so they are equilibria.

We can also rewrite the equilibrium problem of bimatrix games as an equality-inequality system with mixed variables. Assume first that all elements of $\mathbf{A}$ and $\mathbf{B}$ are between 0 and 1. This condition is not really restrictive, since by using linear transformations

$$\overline{\mathbf{A}} = a_1\mathbf{A} + b_1\mathbf{1} \quad \text{and} \quad \overline{\mathbf{B}} = a_2\mathbf{B} + b_2\mathbf{1},$$

where $a_1, a_2 > 0$, and $\mathbf{1}$ is the matrix all elements of which equal 1, the equilibria remain the same and all matrix elements can be transformed into interval $[0, 1]$.

Theorem 7.8 *Vectors* $\mathbf{x}$, $\mathbf{y}$ *are an equilibrium if and only if there are real numbers* α, β *and zero-one vectors* $\mathbf{u}$*, and* $\mathbf{v}$ *such that*

$$\begin{array}{ccccccc} 0 & \leq & \alpha\mathbf{1}_m - \mathbf{A}\mathbf{y} & \leq & \mathbf{1}_m - \mathbf{u} & \leq & \mathbf{1}_m - \mathbf{x} \\ 0 & \leq & \beta\mathbf{1}_n - \mathbf{B}^T\mathbf{x} & \leq & \mathbf{1}_n - \mathbf{v} & \leq & \mathbf{1}_n - \mathbf{y} \\ & & & & \mathbf{x} & \geq & \mathbf{0}_m \\ & & & & \mathbf{y} & \geq & \mathbf{0}_n \\ & & \mathbf{1}_m^T\mathbf{x} & = & \mathbf{1}_n^T\mathbf{y} & = & 1, \end{array} \tag{7.20}$$

where $\mathbf{1}$ denotes the vector with all unit elements.

Proof. Assume first that $\mathbf{x}$, $\mathbf{y}$ is an equilibrium, then with some α and β, (7.19) is satisfied. Define

$$u_i = \begin{cases} 1, & \text{if } x_i > 0, \\ 0, & \text{if } x_i = 0, \end{cases} \quad \text{and} \quad v_j = \begin{cases} 1, & \text{if } y_j > 0, \\ 0, & \text{if } y_j = 0. \end{cases}$$

Since all elements of $\mathbf{A}$ and $\mathbf{B}$ are between 0 and 1, the values $\alpha = \mathbf{x}^T\mathbf{A}\mathbf{y}$ and $\beta = \mathbf{x}^T\mathbf{B}\mathbf{y}$ are also between 0 and 1. Notice that

$$0 = \mathbf{x}^T(\alpha\mathbf{1}_m - \mathbf{A}\mathbf{y}) = \mathbf{y}^T(\beta\mathbf{1}_n - \mathbf{B}^T\mathbf{x}),$$

which implies that (7.20) holds.

Assume next that (7.20) is satisfied. Then

$$\mathbf{0} \leq \mathbf{x} \leq \mathbf{u} \leq \mathbf{1}_m \quad \text{and} \quad \mathbf{0} \leq \mathbf{y} \leq \mathbf{v} \leq \mathbf{1}_n.$$

If $u_i = 1$, then $\alpha - \mathbf{e}_i^T\mathbf{A}\mathbf{y} = 0$, (where $\mathbf{e}_i$ is the i-th basis vector), and if $u_i = 0$, then $x_i = 0$. Therefore

$$\mathbf{x}^T(\alpha\mathbf{1}_m - \mathbf{A}\mathbf{y}) = 0,$$

implying that $\alpha = \mathbf{x}^T\mathbf{A}\mathbf{y}$. We can similarly show that $\beta = \mathbf{x}^T\mathbf{B}\mathbf{y}$. Thus (7.19) is satisfied, so, $\mathbf{x}, \mathbf{y}$ is an equilibrium. ∎

The computation cost of the solution of system (7.20) depends on the particular method being seleced.

Example 7.12 In the case of the bimatrix game introduced earlier in Example 7.10. we have the following:

$$\begin{array}{ccccccc} 0 & \leq & \alpha - 2y_1 + y_2 & \leq & 1 - u_1 & \leq & 1 - x_1 \\ 0 & \leq & \alpha + y_1 - y_2 & \leq & 1 - u_2 & \leq & 1 - x_2 \\ 0 & \leq & \beta - x_1 + x_2 & \leq & 1 - v_1 & \leq & 1 - y_1 \\ 0 & \leq & \beta + x_1 - 2x_2 & \leq & 1 - v_2 & \leq & 1 - y_2 \\ & & x_1 + x_2 & = & y_1 + y_2 & = & 1 \\ & & & & x_1, x_2, y_1, y_2 & \geq & 0 \\ & & & & u_1, u_2, v_1, v_2 & \in & \{0, 1\}. \end{array}$$

Notice that all three solutions given in Example 7.10. satisfy these relations with

$$\mathbf{u} = (1, 0), \quad \mathbf{v} = (0, 1)$$

$$\mathbf{u} = (0, 1), \quad \mathbf{v} = (1, 0)$$

and

$$\mathbf{u} = (1, 1), \quad \mathbf{v} = (1, 1),$$

respectively.

Matrix games
In the special case of $\mathbf{B} = -\mathbf{A}$, bimatrix games are called ***matrix games*** and they are represented by matrix $\mathbf{A}$. Sometimes we refer to the game as matrix $\mathbf{A}$ game. Since $\mathbf{A} + \mathbf{B} = \mathbf{0}$, the quadratic optimization problem (7.18) becomes linear:

$$\begin{array}{ll} \text{minimize} & \alpha + \beta \\ \text{subject to} & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{1}_m \mathbf{x} = 1 \\ & \mathbf{1}_n \mathbf{y} = 1 \\ & \mathbf{A}\mathbf{y} \leq \alpha \mathbf{1}_m \\ & \mathbf{A}^T \mathbf{x} \geq -\beta \mathbf{1}_n. \end{array} \tag{7.21}$$

From this formulation we see that the set of the equilibrium strategies is a convex polyhedron. Notice that variables $(\mathbf{x}, \beta)$ and $(\mathbf{y}, \alpha)$ can be separated, so we have the following result.

Theorem 7.9 *Vectors $\mathbf{x}^\star$ and $\mathbf{y}^\star$ give an equilibrium of the matrix game if and only if with some $\alpha^\star$ and $\beta^\star$, $(\mathbf{x}^\star, \beta^\star)$ and $(\mathbf{y}^\star, \alpha^\star)$ are optimal solutions of the linear programming problems:*

$$\begin{array}{ll} \textit{minimize} & \alpha \\ \textit{subject to} & \mathbf{y} \geq \mathbf{0}_n \\ & \mathbf{1}_n^T \mathbf{y} = 1 \\ & \mathbf{A}\mathbf{y} \leq \alpha \mathbf{1}_m \end{array} \qquad \begin{array}{ll} \textit{minimize} & \beta \\ \textit{subject to} & \mathbf{x} \geq \mathbf{0}_m \\ & \mathbf{1}_m^T \mathbf{x} = 1 \\ & \mathbf{A}^T \mathbf{x} \geq -\beta \mathbf{1}_n. \end{array} \tag{7.22}$$

Notice that at the optimum, $\alpha + \beta = 0$. The optimal α value is called the ***value of the matrix game*** .

Solving problem (7.22) requires exponential number of operations if the simplex method is chosen. With polynomial algorithm (such as the interior point method) the number of operations is only polynomial.

Example 7.13 Consider the matrix game with matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & 3 & 3 \end{pmatrix}.$$

In this case problems (7.22) have the from:

$$\begin{array}{llcll} \text{minimize} & \alpha & \text{and} & \text{minimize} & \beta \\ \text{subject to} & y_1, y_2, y_3 \geq 0 & & \text{subject to} & x_1, x_2, x_3 \geq 0 \\ & y_1 + y_2 + y_3 = 1 & & & x_1 + x_2 + x_3 = 1 \\ & 2y_1 + y_2 - \alpha \leq 0 & & & 2x_1 + 2x_2 - x_3 + \beta \geq 0 \\ & 2y_1 + 3y_3 - \alpha \leq 0 & & & x_1 + 3x_3 + \beta \geq 0 \\ & -y_1 + 3y_2 + 3y_3 - \alpha \leq 0 & & & 3x_2 + 3x_3 + \beta \geq 0. \end{array}$$

The application of the simplex method shows that the optimal solutions are $\alpha = 9/7$, $\mathbf{y} = (3/7, 3/7, 1/7)^T$, $\beta = -9/7$, and $\mathbf{x} = (4/7, 4/21, 5/21)^T$.

We can also obtain the equilibrium by finding feasible solutions of a certain set of linear constraints. Since at the optimum of problem (7.21), $\alpha + \beta = 0$, vectors $\mathbf{x}$, $\mathbf{y}$ and scalers α

and β are optimal solutions if and only if

$$\begin{array}{rcl} \mathbf{x}, \mathbf{y} & \geq & \mathbf{0} \\ \mathbf{1}_m^T \mathbf{x} & = & 1 \\ \mathbf{1}_n^T \mathbf{y} & = & 1 \\ \mathbf{A}\mathbf{y} & \leq & \alpha \mathbf{1}_m \\ \mathbf{A}^T \mathbf{x} & \geq & \alpha \mathbf{1}_n . \end{array} \tag{7.23}$$

The first phase of the simplex method has to be used to solve system (7.23), where the number of operations might be experimental. However in most practical examples much less operations are needed.

Example 7.14 Consider again the matrix game of the previous example. In this case system (7.23) has the following form:

$$\begin{array}{rcl} x_1, x_2, x_3, y_1, y_2, y_3 & \geq & 0 \\ x_1 + x_2 + x_3 = y_1 + y_2 + y_3 & = & 1 \\ 2y_1 + y_2 & \leq & \alpha \\ 2y_1 + 3y_3 & \leq & \alpha \\ -y_1 + 3y_2 + 3y_3 & \leq & \alpha \\ 2x_1 + 2x_2 - x_3 & \geq & \alpha \\ x_1 + 3x_3 & \geq & \alpha \\ 3x_2 + 3x_3 & \geq & \alpha . \end{array}$$

It is easy to see that $\alpha = 9/7$, $\mathbf{x} = (4/7, 4/21, 5/21)^T$, $\mathbf{y} = (3/7, 3/7, 1/7)^T$ satisfy these relations, so $\mathbf{x}$, $\mathbf{y}$ is an equilibrium.

7.2.5. Method of Fictitious Play

Consider now a matrix game with matrix $\mathbf{A}$. The main idea of this method is that at each step both players determine their best pure strategy choices against the average strategies of the other player of all previous steps. Formally the method can be described as follows.

Let $\mathbf{x}_1$ be the initial (mixed) strategy of player $\mathcal{P}_1$. Select $\mathbf{y}_1 = \mathbf{e}_{j_1}$ (the j_1st basis vector) such that

$$\mathbf{x}_1^T \mathbf{A} \mathbf{e}_{j_1} = \min_j \{\mathbf{x}_1^T \mathbf{A} \mathbf{e}_j\}. \tag{7.24}$$

In any further step $k \geq 2$, let

$$\overline{\mathbf{y}}_{k-1} = \frac{1}{k-1}((k-2)\overline{\mathbf{y}}_{k-2} + \mathbf{y}_{k-1}), \tag{7.25}$$

and select $\mathbf{x}_k = \mathbf{e}_{i_k}$ so that

$$\mathbf{e}_{i_k}^T \mathbf{A} \overline{\mathbf{y}}_{k-1} = \max_i \{\mathbf{e}_i^T \mathbf{A} \overline{\mathbf{y}}_{k-1}\}. \tag{7.26}$$

Let then

$$\overline{\mathbf{x}}_k = \frac{1}{k}((k-1)\overline{\mathbf{x}}_{k-1} + \mathbf{x}_k), \tag{7.27}$$

and select $\mathbf{y}_k = \mathbf{e}_{j_k}$ so that

$$\overline{\mathbf{x}}_k^T \mathbf{A} \mathbf{e}_{j_k} = \min_j \{\overline{\mathbf{x}}_k^T \mathbf{A} \mathbf{e}_j\}. \tag{7.28}$$

By repeating the general step for $k = 2, 3, \ldots$ two sequences are generated: $\{\overline{\mathbf{x}}_k\}$, and $\{\overline{\mathbf{y}}_k\}$. We have the following result:

Theorem 7.10 *Any cluster point of these sequences is an equilibrium of the matrix game. Since all $\overline{\mathbf{x}}_k$ and $\overline{\mathbf{y}}_k$ are probability vectors, they are bounded. Therefore there is at least one cluster point.*

Assume, matrix $\mathbf{A}$ is $m \times n$. In (7.24) we need mn multiplications. In (7.25) and (7.27) $m + n$ multiplications and divisions. In (7.26) and (7.28) mn multiplications. If we make L iteration steps, then the total number of multiplications and divisions is:

$$mn + L[2(m+n) + 2mn] = \ominus(Lmn).$$

The formal algorithm is as follows:

```
1  k ← 1
2  define j_1 such that x_1^T A e_{j_1} = min_j{x_1^T A e_j}
3  y_1 ← e_{j_1}
4  k ← k + 1
5  ȳ_{k-1} ← 1/(k-1) ((k - 2) ȳ_{k-2} + y_{k-1})
6  define i_k such that e_{i_k}^T A ȳ_{k-1} = max_i{e_i^T A ȳ_{k-1}}
7  x_k ← e_{i_k}
8  x̄_k ← 1/k ((k - 1) x̄_{k-1} + x_k)
9  define j_k such that x̄_k^T A e_{j_k} = min_j{x̄_k^T A e_j}
10 y_k ← e_{j_k}
11 if ||x̄_k - x̄_{k-1}|| < ε and ||ȳ_{k-1} - x̄_{k-2}|| < ε
12     then (x̄_k, ȳ_{k-1}) is equilibrium
13 else go back to 4
```

Here $\varepsilon > 0$ is a user selected error tolerance.

Example 7.15 We applied the above method for the matrix game of the previous example and started the procedure with $\mathbf{x}_1 = (1, 0, 0)^T$. After 100 steps we obtained $\overline{\mathbf{x}}_{101} = (0.446, 0.287, 0.267)^T$ and $\overline{\mathbf{y}}_{101} = (0.386, 0.436, 0.178)^T$. Comparing it to the true values of the equilibrium strategies we see that the error is below 0.126, showing the very slow convergence of the method.

7.2.6. Symmetric Matrix Games

A matrix game with skew-symmetric matrix is called symmetric. In this case $\mathbf{A}^T = -\mathbf{A}$ and the two linear programming problems are identical. Therefore at the optimum $\alpha = \beta = 0$, and the equilibrium strategies of the two players are the same. Hence we have the following result:

Theorem 7.11 *A vector* $\mathbf{x}^\star$ *is equilibrium of the symmetric matrix game if and only if*

$$\begin{array}{rcl} \mathbf{x} & \geq & \mathbf{0} \\ \mathbf{1}^T\mathbf{x} & = & 1 \\ \mathbf{A}\mathbf{x} & \leq & \mathbf{0}. \end{array} \tag{7.29}$$

Solving system (7.29) the first phase of the simplex method is needed, the number of operations is exponential in the worst case but in practical case usually much less.

Example 7.16 Consider the symmetric matrix game with matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this case relations (7.29) simplify as follows:

$$\begin{array}{rcl} x_1, x_2 & \geq & 0 \\ x_1 + x_2 & = & 1 \\ x_2 & \leq & 0 \\ -x_1 & \leq & 0. \end{array}$$

Clearly the only solution is $x_1 = 1$ and $x_2 = 0$, that is the first pure strategy.

We will see in the next subsection that linear programming problems are equivalent to symmetric matrix games so any method for solving such games can be applied to solve linear programming problems, so they serve as alternative methodology to the simplex method. As we will see next, symmetry is not a strong assumption, since any matrix game is equivalent to a symmetric matrix game.

Consider therefore a matrix game with matrix $\mathbf{A}$, and construct the skew-symmetric matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{A} & -\mathbf{1}_m \\ -\mathbf{A}^T & \mathbf{0}_{n\times n} & \mathbf{1}_n \\ \mathbf{1}_m^T & -\mathbf{1}_n^T & 0 \end{pmatrix},$$

where all components of vector $\mathbf{1}$ equal 1. Matrix games $\mathbf{A}$ and $\mathbf{P}$ are equivalent in the following sense. Assume that $\mathbf{A} > \mathbf{0}$, which is not a restriction, since by adding the same constant to all element of $\mathbf{A}$ they become positive without changing equilibria.

Theorem 7.12

1. *If* $\mathbf{z} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \lambda \end{pmatrix}$ *is an equilibrium strategy of matrix game* $\mathbf{P}$ *then with* $\mathbf{a} = (1-\lambda)/2$, $\mathbf{x} = (1/a)\mathbf{u}$ *and* $\mathbf{y} = (1/a)\mathbf{v}$ *is an equilibrium of matrix game* $\mathbf{A}$ *with value* $v = \lambda/a$;
2. *If* $\mathbf{x}, \mathbf{y}$ *is an equilibrium of matrix game* $\mathbf{A}$ *and* v *is the value of the game, then*

$$\mathbf{z} = \frac{1}{2+v}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ v \end{pmatrix}$$

is equilibrium strategy of matrix game $\mathbf{P}$.

Proof. Assume first that $\mathbf{z}$ is an equilibrium strategy of game $\mathbf{P}$, then $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$, $\mathbf{Pz} \leq \mathbf{0}$, so

$$\begin{array}{rcl} \mathbf{Av} - \lambda \mathbf{1}_m & \leq & \mathbf{0} \\ -\mathbf{A}^T\mathbf{u} + \lambda \mathbf{1}_n & \leq & \mathbf{0} \\ \mathbf{1}_m^T\mathbf{u} - \mathbf{1}_n^T\mathbf{v} & \leq & 0. \end{array} \tag{7.30}$$

First we show that $0 < \lambda < 1$, that is $a \neq 0$. If $\lambda = 1$, then (since $\mathbf{z}$ is a probability vector) $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$, contradicting the second inequality of (7.30). If $\lambda = 0$, then $\mathbf{1}_m^T\mathbf{u} + \mathbf{1}_n^T\mathbf{v} = 1$, and by the third inequality of (7.30), $\mathbf{v}$ must have at least one positive component which makes the first inequality impossible.

Next we show that $\mathbf{1}^T\mathbf{u} = \mathbf{1}^T\mathbf{v}$. From (7.30) we have

$$\begin{array}{rcl} \mathbf{u}^T\mathbf{Av} - \lambda \mathbf{u}^T\mathbf{1}_m & \leq & 0, \\ -\mathbf{v}^T\mathbf{A}^T\mathbf{u} + \lambda \mathbf{u}^T\mathbf{1}_n & \leq & 0 \end{array}$$

and by adding these inequalities we see that

$$\mathbf{v}^T\mathbf{1}_n - \mathbf{u}^T\mathbf{1}_m \leq 0,$$

and combining this relation with the third inequality of (7.30) we see that $\mathbf{1}_m^T\mathbf{u} - \mathbf{1}_n^T\mathbf{v} = 0$.

Select $a = (1 - \lambda)/2 \neq 0$, then $\mathbf{1}_m^T\mathbf{u} = \mathbf{1}_n^T\mathbf{v} = a$, so both $\mathbf{x} = \mathbf{u}/a$, and $\mathbf{y} = \mathbf{v}/a$ are probability vectors, furthermore from (7.30),

$$\begin{array}{rcccl} \mathbf{A}^T\mathbf{x} & = & \frac{1}{a}\mathbf{A}^T\mathbf{u} & \geq & \frac{\lambda}{a}\mathbf{1}_n, \\ \mathbf{Ay} & = & \frac{1}{a}\mathbf{Av} & \leq & \frac{\lambda}{a}\mathbf{1}_m. \end{array}$$

So by selecting $\alpha = \lambda/a$ and $\beta = -\lambda/a$, $\mathbf{x}$ and $\mathbf{y}$ are feasible solutions of the pair (7.22) of linear programming problems with $\alpha + \beta = 0$, therefore $\mathbf{x}, \mathbf{y}$ is an equilibrium of matrix game $\mathbf{A}$. Part 2. can be proved in a similar way, the details are not given here. ■

7.2.7. Linear Programming and Matrix Games

In this section we will show that linear programming problems can be solved by finding the equilibrium strategies of symmetric matrix games and hence, any method for finding the equilibria of symmetric matrix games can be applied instead of the simplex method.

Consider the primal-dual linear programming problem pair:

$$\begin{array}{lllll} \text{maximize} & \mathbf{c}^T\mathbf{x} & \text{and} & \text{minimize} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{x} \geq \mathbf{0} & & \text{subject to} & \mathbf{y} \geq \mathbf{0} \\ & \mathbf{Ax} \leq \mathbf{b} & & & \mathbf{A}^T\mathbf{y} \geq \mathbf{c}. \end{array} \tag{7.31}$$

Construct the skew-symmetric matrix:

$$\mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{A} & -\mathbf{b} \\ -\mathbf{A}^T & \mathbf{0} & \mathbf{c} \\ \mathbf{b}^T & -\mathbf{c}^T & 0 \end{pmatrix}.$$

Theorem 7.13 *Assume* $\mathbf{z} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \lambda \end{pmatrix}$ *is an equilibrium strategy of the symmetric matrix game*

P *with* $\lambda > 0$. *Then*

$$\mathbf{x} = \frac{1}{\lambda}\mathbf{v} \quad \textit{and} \quad \mathbf{y} = \frac{1}{\lambda}\mathbf{u}$$

are optimal solutions of the primal and dual problems, respectively.

Proof. If $\mathbf{z}$ is an equilibrium strategy, then $\mathbf{Pz} \le \mathbf{0}$, that is,

$$\begin{aligned} \mathbf{Av} - \lambda\mathbf{b} &\le \mathbf{0} \\ -\mathbf{A}^T\mathbf{u} + \lambda\mathbf{c} &\le \mathbf{0} \\ \mathbf{b}^T\mathbf{u} - \mathbf{c}^T\mathbf{v} &\le 0. \end{aligned} \tag{7.32}$$

Since $\mathbf{z} \ge \mathbf{0}$ and $\lambda > 0$, both vectors $\mathbf{x} = (1/\lambda)\mathbf{v}$, and $\mathbf{y} = (1/\lambda)\mathbf{u}$ are nonnegative, and by dividing the first two relations of (7.32) by λ,

$$\mathbf{Ax} \le \mathbf{b} \quad \text{and} \quad \mathbf{A}^T\mathbf{y} \ge \mathbf{c},$$

showing that $\mathbf{x}$ and $\mathbf{y}$ are feasible for the primal and dual, respectively. From the last condition of (7.32) we have

$$\mathbf{b}^T\mathbf{y} \le \mathbf{c}^T\mathbf{x}.$$

However

$$\mathbf{b}^T\mathbf{y} \ge (\mathbf{x}^T\mathbf{A}^T)\mathbf{y} = \mathbf{x}^T(\mathbf{A}^T\mathbf{y}) \ge \mathbf{x}^T\mathbf{c} = \mathbf{c}^T\mathbf{x},$$

consequently, $\mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x}$, showing that the primal and dual objective functions are equal. The duality theorem implies the optimality of $\mathbf{x}$ and $\mathbf{y}$. ■

Example 7.17 Consider the linear programming problem:

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & x_1 \ge 0 \\ & -x_1 + x_2 \ge 1 \\ & 5x_1 + 7x_2 \le 25. \end{array}$$

First we have to rewrite the problem as a primal problem. Introduce the new variables:

$$x_2^+ = \begin{cases} x_2, & \text{if } x_2 \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$x_2^- = \begin{cases} -x_2, & \text{if } x_2 < 0, \\ 0 & \text{otherwise.} \end{cases}$$

and multiply the $\ge$-type constraint by -1. Then the problem becomes the following:

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2^+ - 2x_2^- \\ \text{subject to} & x_1, x_2^+, x_2^- \ge 0 \\ & x_1 - x_2^+ + x_2^- \le -1 \\ & 5x_1 + 7x_2^+ - 7x_2^- \le 25. \end{array}$$

Hence

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 5 & 7 & -7 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 25 \end{pmatrix}, \quad \mathbf{c}^T = (1, 2, -2),$$

and so matrix **P** becomes:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & \vdots & 1 & -1 & 1 & \vdots & 1 \\ 0 & 0 & \vdots & 5 & 7 & -7 & \vdots & -25 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -5 & \vdots & 0 & 0 & 0 & \vdots & 1 \\ 1 & -7 & \vdots & 0 & 0 & 0 & \vdots & 2 \\ -1 & 7 & \vdots & 0 & 0 & 0 & \vdots & -2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 25 & \vdots & -1 & -2 & 2 & \vdots & 0 \end{pmatrix} .$$

7.2.8. The Method of Von Neumann

The fictitious play method is an iteration algorithm in which at each step the players adjust their strategies based on the opponent's strategies. This method can therefore be considered as the realization of a discrete system where the strategy selections of the players are the state variables. For symmetric matrix games John von Neumann introduced a continuous systems approach when the players continuously adjust their strategies. This method can be applied to general matrix games, since–as we have seen earlier–any matrix game is equivalent to a symmetric matrix game. The method can also be used to solve linear programming problems as we have seen earlier that any primal-dual pair can be reduced to the solution of a symmetric matrix game.

Let now **P** be a skew-symmetric $n \times n$ matrix. The strategy of player $\mathcal{P}_2$, $\mathbf{y}(t)$ is considered as the function of time $t \geq 0$. Before formulating the dynamism of the system, introduce the following notation:

$$\begin{array}{llllrcl} u_i : & \mathbb{R}^n & \to & \mathbb{R}, & u_i(\mathbf{y}(t)) & = & \mathbf{e}_i^T \mathbf{P} \mathbf{y}(t) \quad (i = 1, 2, \ldots, n) , \\ \phi : & \mathbb{R} & \to & \mathbb{R}, & \phi(u_i) & = & \max\{0, u_i\} , \\ \Phi : & \mathbb{R}^n & \to & \mathbb{R}, & \Phi(\mathbf{y}(t)) & = & \sum_{i=1}^n \phi(u_i(\mathbf{y}(t))) . \end{array} \tag{7.33}$$

For arbitrary probability vector $\mathbf{y}_0$ solve the following nonlinear initial-value problem:

$$y_j'(t) = \phi(u_j(\mathbf{y}(t))) - \Phi(\mathbf{y}(t)) y_j(t), \quad y_j(0) = y_{j0} \quad (1 \leq j \leq n) . \tag{7.34}$$

Since the right-hand side is continuous, there is at least one solution. The right hand side of the equation can be interpreted as follows. Assume that $\phi(u_j(\mathbf{y}(t))) > 0$. If player $\mathcal{P}_2$ selects strategy $\mathbf{y}(t)$, then player $\mathcal{P}_1$ is able to obtain a positive payoff by choosing the pure strategy $\mathbf{e}_j$, which results in a negative payoff for player $\mathcal{P}_2$. However if player $\mathcal{P}_2$ increases $y_j(t)$ to one by choosing the same strategy $\mathbf{e}_j$ its payoff $\mathbf{e}_j^T \mathbf{P} \mathbf{e}_j$ becomes zero, so it increases. Hence it is the interest of player $\mathcal{P}_2$ to increase $y_j(t)$. This is exactly what the first term represents. The second term is needed to ensure that $\mathbf{y}(t)$ remains a probability vector for all $t \geq 0$.

The computation of the right hand side of equations (7.34) for all t requires $n^2 + n$ multiplications. The total computation cost depends on the length of solution interval, on the selected step size, and on the choice of the differential equation solver.

Theorem 7.14 *Assume that $t_1, t_2, \ldots$ is a positive strictly increasing sequence converging to ∞, then any cluster point of the sequence $\{\mathbf{y}(t_k)\}$ is equilibrium strategy, furthermore there is a constant c such that*

$$\mathbf{e}_i^T \mathbf{P}\mathbf{y}(t_k) \le \frac{\sqrt{n}}{c + t_k} \quad (i = 1, 2, \ldots, n) \ . \tag{7.35}$$

Proof. First we have to show that $\mathbf{y}(t)$ is a probability vector for all $t \ge 0$. Assume that with some j and $t_1 > 0$, $y_j(t_1) < 0$. Define

$$t_0 = \sup\{t | 0 < t < t_1, y_j(t) \ge 0\} \ .$$

Since $y_j(t)$ is continuous and $y_j(0) \ge 0$, clearly $y_j(t_0) = 0$, and for all $\tau \in (t_0, t_1)$, $y_j(\tau) < 0$. Then for all $\tau \in (t_0, t_1]$,

$$y_j'(\tau) = \phi(u_j(\mathbf{y}(\tau))) - \Phi(\mathbf{y}(\tau)) y_j(\tau) \ge 0,$$

and the Lagrange mean-value theorem implies that with some $\tau \in (t_0, t_1)$,

$$y_j(t_1) = y_j(t_0) + y_j'(\tau)(t_1 - t_0) \ge 0 \ ,$$

which is a contradiction. Hence $y_j(t)$ is nonnegative. Next we show that $\sum_{j=1}^n y_j(t) = 1$ for all t. Let $f(t) = 1 - \sum_{j=1}^n y_j(t)$, then

$$f'(t) = -\sum_{j=1}^n y_j'(t) = -\sum_{j=1}^n \phi(u_j(\mathbf{y}(t))) + \Phi(\mathbf{y}(t))(\sum_{j=1}^n y_j(t)) = -\Phi(\mathbf{y}(t))(1 - \sum_{j=1}^n y_j(t)) \ ,$$

so $f(t)$ satisfies the homogeneous equation

$$f'(t) = -\Phi(\mathbf{y}(t)) f(t)$$

with the initial condition $f(0) = 1 - \sum_{j=1}^n y_{j0} = 0$. Hence for all $t \ge 0$, $f(t) = 0$, showing that $\mathbf{y}(t)$ is a probability vector.

Assume that for some t, $\phi(u_i(\mathbf{y}(t))) > 0$. Then

$$\begin{aligned} \frac{d}{dt}\phi(u_i(\mathbf{y}(t))) &= \sum_{j=1}^n p_{ij} y_j'(t) = \sum_{j=1}^n p_{ij}[\phi(u_j(\mathbf{y}(t))) - \Phi(\mathbf{y}(t)) y_j(t)] \\ &= \sum_{j=1}^n p_{ij}\phi(u_j(\mathbf{y}(t))) - \Phi(\mathbf{y}(t))\phi(u_i(\mathbf{y}(t))). \end{aligned} \tag{7.36}$$

By multiplying both sides by $\phi(u_i(\mathbf{y}(t)))$ and adding the resulted equations for $i = 1, 2, \ldots, n$ we have:

$$\begin{aligned} \sum_{i=1}^n \phi(u_i(\mathbf{y}(t))) \frac{d}{dt}\phi(u_i(\mathbf{y}(t))) &= \sum_{i=1}^n \sum_{j=1}^n p_{ij}\phi(u_i(\mathbf{y}(t)))\phi(u_j(\mathbf{y}(t))) \\ &\quad -\Phi(\mathbf{y}(t))(\sum_{i=1}^n \phi^2(u_i(\mathbf{y}(t)))). \end{aligned} \tag{7.37}$$

The first term is zero, since $\mathbf{P}$ is skew-symmetric. Notice that this equation remains valid even as $\phi(u_i(\mathbf{y}(t))) = 0$ except the break-points (where the derivative of $\phi(u_i(\mathbf{y}(t)))$ does not exist) since (7.36) remains true.

Assume next that with a positive t, $\Phi(\mathbf{y}(t)) = 0$. Then for all i, $\phi(u_i(\mathbf{y}(t))) = 0$. Since equation (7.37) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\Psi(\mathbf{y}(t)) = -\Phi(\mathbf{y}(t))\Psi(\mathbf{y}(t)) \tag{7.38}$$

with

$$\Psi : \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad \Psi(\mathbf{y}(t)) = \sum_{i=1}^{n} \phi^2(u_i(\mathbf{y}(t))) \ ,$$

we see that $\Psi(\mathbf{y}(t))$ satisfies a homogeneous equation with zero initial solution at t, so the solution remains zero for all $\tau \geq t$. Therefore $\phi(u_i(\mathbf{y}(\tau))) = 0$ showing that $\mathbf{P}\mathbf{y}(\tau) \leq \mathbf{0}$, that is, $\mathbf{y}(\tau)$ is equilibrium strategy.

If $\Phi(\mathbf{y}(\mathbf{t})) > 0$ for all $t \geq 0$, then $\Psi(\mathbf{y}(t)) > 0$, and clearly

$$\frac{1}{2}\frac{d}{dt}\Psi(\mathbf{y}(t)) \leq -\sqrt{\Psi(\mathbf{y}(t))}\Psi(\mathbf{y}(t)) \ ,$$

that is

$$\frac{1}{2}\frac{d}{dt}\Psi(\mathbf{y}(t))(\Psi(\mathbf{y}(t)))^{-\frac{3}{2}} \leq -1 \ .$$

Integrate both sides in interval $[0, t]$ to have

$$-\Psi(\mathbf{y}(t))^{-(1/2)} + c \leq -t \ ,$$

with $c = (\Psi(\mathbf{y}(0)))^{-(1/2)}$, which implies that

$$(\Psi(\mathbf{y}(t)))^{1/2} \leq \frac{1}{c+t} \ . \tag{7.39}$$

By using the Cauchy–Schwartz inequality we get

$$\mathbf{e}_i^T\mathbf{P}\mathbf{y}(t) = u_i(\mathbf{y}(t)) \leq \phi(u_i(\mathbf{y}(t))) \leq \Phi(\mathbf{y}(t)) \leq \sqrt{n\Psi(\mathbf{y}(t))} \leq \frac{\sqrt{n}}{c+t} \ , \tag{7.40}$$

which is valid even at the break points because of the continuity of functions u_i. And finally, take a sequence $\{\mathbf{y}(t_k)\}$ with t_k increasingly converging to ∞. The sequence is bounded (being probability vectors), so there is at least one cluster point $\mathbf{y}^\star$. From (7.40), by letting $t_k \to \infty$ we have that $\mathbf{P}\mathbf{y}^\star \leq \mathbf{0}$ showing that $\mathbf{y}^\star$ is an equilibrium strategy. ■

Example 7.18 Consider the matrix game with matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & 3 & 3 \end{pmatrix} ,$$

which was the subject of our earlier Example 7.13. In order to apply the method of von Neumann we have to find first an equivalent symmetric matrix game. The application of the method given in Theorem 7.12. requires that the matrix has to be positive. Without changing the equilibria we can add

2 to all matrix elements to have

$$\mathbf{A}_{new} = \begin{pmatrix} 4 & 3 & 2 \\ 4 & 2 & 5 \\ 1 & 5 & 5 \end{pmatrix},$$

and by using the method we get the skew-symmetric matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & \vdots & 4 & 3 & 2 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 4 & 2 & 5 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 1 & 5 & 5 & \vdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -4 & -4 & -1 & \vdots & 0 & 0 & 0 & \vdots & 1 \\ -3 & -2 & -5 & \vdots & 0 & 0 & 0 & \vdots & 1 \\ -2 & -5 & -5 & \vdots & 0 & 0 & 0 & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \vdots & -1 & -1 & -1 & \vdots & 0 \end{pmatrix}.$$

The differential equations (7.34) were solved by using the 4th order Runge–Kutta method in the interval $[0, 100]$ with the step size $h = 0.01$ and initial vector $\mathbf{y}(0) = (1, 0, \ldots, 0)^T$. From $\mathbf{y}(100)$ we get the approximations

$$\mathbf{x} \approx (0.563619, 0.232359, 0.241988),$$

$$\mathbf{y} \approx (0.485258, 0.361633, 0.115144)$$

of the equilibrium strategies of the original game. Comparing these values to the exact values:

$$\mathbf{x} = \left(\frac{4}{7}, \frac{4}{21}, \frac{5}{21}\right) \quad \text{and} \quad \mathbf{y} = \left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right)$$

we see that the maximum error is about 0.067.

7.2.9. Diagonally Strictly Concave Games

Consider an N-person continuous game and assume that all conditions presented at the beginning of Section 7.2.3 are satisfied. In addition, assume that for all k, S_k is bounded, all components of g_k are concave and f_k is concave in s_k with any fixed $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_N$. Under these conditions there is at least one equilibrium (Theorem 7.3). The uniqueness of the equilibrium is not true in general, even if all f_k are strictly concave in s_k. Such an example is shown next.

Example 7.19 Consider a two-person game with $S_1 = S_2 = [0, 1]$ and $f_1(s_1, s_2) = f_2(s_1, s_2) = 1-(s_1-s_2)^2$. Clearly both payoff functions are strictly concave and there are infinitely many equilibria: $s_1^\star = s_2^\star \in [0, 1]$.

Select an arbitrary nonnegative vector $\mathbf{r} \in \mathbb{R}^N$ and define function

$$\mathbf{h} : \mathbb{R}^M \to \mathbb{R}^M, \quad \mathbf{h}(\mathbf{s}, \mathbf{r}) = \begin{pmatrix} r_1 \nabla_1 f_1(\mathbf{s})^T \\ r_2 \nabla_2 f_2(\mathbf{s})^T \\ \vdots \\ r_N \nabla_N f_N(\mathbf{s})^T \end{pmatrix}, \tag{7.41}$$

where $M = \sum_{k=1}^N n_k$, and $\nabla_k f_k$ is the gradient (as a row vector) of f_k with respect to s_k. The game is said to be ***diagonally strictly concave*** if for all $\mathbf{s}^{(1)} \neq \mathbf{s}^{(2)}$, $\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in S$ and for some $\mathbf{r} \geq \mathbf{0}$,

$$(\mathbf{s}^{(1)} - \mathbf{s}^{(2)})^T (\mathbf{h}(\mathbf{s}^{(1)}, \mathbf{r}) - \mathbf{h}(\mathbf{s}^{(2)}, \mathbf{r})) < 0. \tag{7.42}$$

Theorem 7.15 *Under the above conditions the game has exactly one equilibrium.*

Proof. The existence of the equilibrium follows from Theorem 7.3. In proving uniqueness assume that $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$ are both equilibria, and both satisfy relations (7.9). Therefore for $l = 1, 2$,

$$\begin{aligned} \mathbf{u}_k^{(l)T} \mathbf{g}_k(s_k^{(l)}) &= 0 \\ \nabla_k f_k(\mathbf{s}^{(l)}) + \mathbf{u}_k^{(l)T} \nabla_k \mathbf{g}_k(s_k^{(l)}) &= \mathbf{0}^T, \end{aligned}$$

and the second equation can be rewritten as

$$\nabla_k f_k(\mathbf{s}^{(l)}) + \sum_{j=1}^{m_k} u_{kj}^{(l)} \nabla_k g_{kj}(s_k^{(l)}) = 0 , \tag{7.43}$$

where $u_{kj}^{(l)}$ and g_{kj} are the jth components of $\mathbf{u}_k^{(l)}$ and g_k, respectively. Multiplying (7.43) by $(r_k(s_k^{(2)} - s_k^{(1)})^T)$ for $l = 1$ and by $r_k(s_k^{(1)} - s_k^{(2)})^T$ for $l = 2$ and adding the resulted equalities for $k = 1, 2, \ldots, N$ we have

$$\begin{aligned} 0 = \{&(\mathbf{s}^{(2)} - \mathbf{s}^{(1)})^T \mathbf{h}(\mathbf{s}^{(1)}, \mathbf{r}) + (\mathbf{s}^{(1)} - \mathbf{s}^{(2)})^T \mathbf{h}(\mathbf{s}^{(2)}, \mathbf{r})\} \\ + \sum_{k=1}^N \sum_{j=1}^{m_k} r_k [u_{kj}^{(1)} (s_k^{(2)} - s_k^{(1)})^T \nabla_k g_{kj}(s_k^{(1)}) &+ u_{kj}^{(2)} (s_k^{(1)} - s_k^{(2)})^T \nabla_k g_{kj}(s_k^{(2)})] . \end{aligned} \tag{7.44}$$

Notice that the sum of the first two terms is positive by the diagonally strict concavity of the game, the concavity of the components of g_k implies that

$$(s_k^{(2)} - s_k^{(1)})^T \nabla_k g_{kj}(s_k^{(1)}) \geq g_{kj}(s_k^{(2)}) - g_{kj}(s_k^{(1)})$$

and

$$(s_k^{(1)} - s_k^{(2)})^T \nabla_k g_{kj}(s_k^{(2)}) \geq g_{kj}(s_k^{(1)}) - g_{kj}(s_k^{(2)}) .$$

Therefore from (7.44) we have

$$\begin{aligned} 0 &> \sum_{k=1}^N \sum_{j=1}^{m_k} r_k [u_{kj}^{(1)} (g_{kj}(s_k^{(2)}) - g_{kj}(s_k^{(1)})) + u_{kj}^{(2)} (g_{kj}(s_k^{(1)}) - g_{kj}(s_k^{(2)}))] \\ &= \sum_{k=1}^N \sum_{j=1}^{m_k} r_k [u_{kj}^{(1)} g_{kj}(s_k^{(2)}) + u_{kj}^{(2)} g_{kj}(s_k^{(1)})] \geq 0 , \end{aligned}$$

where we used the fact that for all k and l,

$$0 = \mathbf{u}_k^{(l)^T} \mathbf{g}_k(s_k^{(l)}) = \sum_{j=1}^{m_k} u_{kj}^{(l)} g_{kj}(s_k^{(l)}).$$

This is an obvious contradiction, which completes the proof. ■

Checking for Uniqueness of Equilibrium

In practical cases the following result is very useful in checking diagonally strict concavity of N-person games.

Theorem 7.16 *Assume S is convex, f_k is twice continuously differentiable for all k, and $\mathbf{J}(\mathbf{s},\mathbf{r}) + \mathbf{J}(\mathbf{s},\mathbf{r})^T$ is negative definite with some $\mathbf{r} \geq \mathbf{0}$, where $\mathbf{J}(\mathbf{s},\mathbf{r})$ is the Jacobian of $\mathbf{h}(\mathbf{s},\mathbf{r})$. Then the game is diagonally strictly concave.*

Proof. Let $\mathbf{s}^{(1)} \neq \mathbf{s}^{(2)}$, $\mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in S$. Then for all $\alpha \in [0,1]$, $\mathbf{s}(\alpha) = \alpha\mathbf{s}^{(1)} + (1-\alpha)\mathbf{s}^{(2)} \in S$ and

$$\frac{d}{d\alpha}\mathbf{h}(\mathbf{s}(\alpha),\mathbf{r}) = \mathbf{J}(\mathbf{s}(\alpha),\mathbf{r})(\mathbf{s}^{(1)} - \mathbf{s}^{(2)}) \ .$$

Integrate both side in [0, 1] to have

$$\mathbf{h}(\mathbf{s}^{(1)},\mathbf{r}) - \mathbf{h}(\mathbf{s}^{(2)},\mathbf{r}) = \int_0^1 \mathbf{J}(\mathbf{s}(\alpha),\mathbf{r})(\mathbf{s}^{(1)} - \mathbf{s}^{(2)})d\alpha \ ,$$

and by premultiplying both sides by $(\mathbf{s}^{(1)} - \mathbf{s}^{(2)})^T$ we see that

$$(\mathbf{s}^{(1)} - \mathbf{s}^{(2)})^T(\mathbf{h}(\mathbf{s}^{(1)},\mathbf{r}) - \mathbf{h}(\mathbf{s}^{(2)},\mathbf{r})) = \int_0^1 (\mathbf{s}^{(1)} - \mathbf{s}^{(2)})^T \mathbf{J}(\mathbf{s}(\alpha),\mathbf{r})(\mathbf{s}^{(1)} - \mathbf{s}^{(2)})d\alpha$$
$$= \frac{1}{2}\int_0^1 (\mathbf{s}^{(1)} - \mathbf{s}^{(2)})^T(\mathbf{J}(\mathbf{s}(\alpha),\mathbf{r}) + \mathbf{J}(\mathbf{s}(\alpha),\mathbf{r})^T)(\mathbf{s}^{(1)} - \mathbf{s}^{(2)})d\alpha < 0 \ ,$$

completing the proof. ■

Example 7.20 Consider a simple two-person game with strategy sets $S_1 = S_2 = [0,1]$, and payoff functions

$$f_1(s_1,s_2) = -s_1^2 + s_1 - s_1 s_2$$

and

$$f_2(s_1,s_2) = -s_2^2 + s_2 - s_1 s_2 \ .$$

Clearly all conditions, except diagonally strict concavity, are satisfied. We will use Theorem 7.16 to show this additional property. In this case

$$\nabla_1 f_1(s_1,s_2) = -2s_1 + 1 - s_2, \quad \nabla_2 f_2(s_1,s_2) = -2s_2 + 1 - s_1 \ ,$$

so

$$\mathbf{h}(\mathbf{s},\mathbf{r}) = \begin{pmatrix} r_1(-2s_1 + 1 - s_2) \\ r_2(-2s_2 + 1 - s_1 \end{pmatrix}$$

with Jacobian

$$\mathbf{J}(\mathbf{s},\mathbf{r}) = \begin{pmatrix} -2r_1 & -r_1 \\ -r_2 & -2r_2 \end{pmatrix}.$$

We will show that

$$\mathbf{J}(\mathbf{s},\mathbf{r}) + \mathbf{J}(\mathbf{s},\mathbf{r})^T = \begin{pmatrix} -4r_1 & -r_1 - r_2 \\ -r_1 - r_2 & -4r_2 \end{pmatrix}$$

is negative definite with some $\mathbf{r} \geq \mathbf{0}$. For example, select $r_1 = r_2 = 1$, then this matrix becomes

$$\begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}$$

with characteristic polynomial

$$\phi(\lambda) = \det\begin{pmatrix} -4-\lambda & -2 \\ -2 & -4-\lambda \end{pmatrix} = \lambda^2 + 8\lambda + 12 ,$$

having negative eigenvalues $\lambda_1 = -2$, $\lambda_2 = -6$.

Iterative Computation of Equilibrium

We have see earlier in Theorem 7.4 that $\mathbf{s}^\star \in S$ is an equilibrium if and only if

$$\mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s}^\star) \geq \mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s}) \tag{7.45}$$

for all $\mathbf{s} \in S$, where $\mathbf{H}_\mathbf{r}$ is the aggregation function (7.4). In the following analysis we assume that the N-person game satisfies all conditions presented at the beginning of Section 7.2.9 and (7.42) holds with some positive $\mathbf{r}$.

We first show the equivalence of (7.45) and a variational inequality.

Theorem 7.17 *A vector $\mathbf{s}^\star \in S$ satisfies (7.45) if and only if*

$$\mathbf{h}(\mathbf{s}^\star, \mathbf{r})^T(\mathbf{s} - \mathbf{s}^\star) \leq 0 \tag{7.46}$$

for all $\mathbf{s} \in S$, where $\mathbf{h}(\mathbf{s},\mathbf{r})$ is defined in (7.41).

Proof. Assume $\mathbf{s}^\star$ satisfies (7.45). Then $\mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s})$ as function of $\mathbf{s}$ obtains maximum at $\mathbf{s} = \mathbf{s}^\star$, therefore

$$\nabla_\mathbf{s}\mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s}^\star)(\mathbf{s} - \mathbf{s}^\star) \leq 0$$

for all $\mathbf{s} \in S$, and since $\nabla_\mathbf{s}\mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s}^\star)$ is $\mathbf{h}(\mathbf{s}^\star, \mathbf{r})$, we proved that $\mathbf{s}^\star$ satisfies (7.46).

Assume next that $\mathbf{s}^\star$ satisfies (7.46). By the concavity of $\mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s})$ in $\mathbf{s}$ and the diagonally strict concavity of the game we have

$$\mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s}^\star) - \mathbf{H}_\mathbf{r}(\mathbf{s}^\star, \mathbf{s}) \geq \mathbf{h}(\mathbf{s},\mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}) \geq \mathbf{h}(\mathbf{s},\mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}) + \mathbf{h}(\mathbf{s}^\star,\mathbf{r})^T(\mathbf{s} - \mathbf{s}^\star) > 0 ,$$

so $\mathbf{s}^\star$ satisfies (7.45). ■

Hence any method available for solving variational inequalities can be used to find equilibria.

Next we construct a special two-person, game the equilibrium problem of which is equivalent to the equilibrium problem of the original N-person game.

Theorem 7.18 *Vector $\mathbf{s}^\star \in S$ satisfies (7.45) if and only if $(\mathbf{s}^\star, \mathbf{s}^\star)$ is an equilibrium of the two-person game $D = \{2; S, S; f, -f\}$ where $f(\mathbf{s},\mathbf{z}) = \mathbf{h}(\mathbf{z},\mathbf{r})^T(\mathbf{s} - \mathbf{z})$.*

Proof.

- Assume first that $\mathbf{s}^\star \in S$ satisfies (7.45). Then it satisfies (7.46) as well, so

$$f(\mathbf{s}, \mathbf{s}^\star) \leq 0 = f(\mathbf{s}^\star, \mathbf{s}^\star).$$

We need in addition to show that

$$-f(\mathbf{s}^\star, \mathbf{s}) \leq 0 = -f(\mathbf{s}^\star, \mathbf{s}^\star).$$

In contrary assume that with some $\mathbf{s}$, $f(\mathbf{s}^\star, \mathbf{s}) < 0$. Then

$$\begin{aligned} 0 > f(\mathbf{s}^\star, \mathbf{s}) &= \mathbf{h}(\mathbf{s}, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}) > \mathbf{h}(\mathbf{s}, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}) + (\mathbf{s} - \mathbf{s}^\star)^T(\mathbf{h}(\mathbf{s}, \mathbf{r}) - \mathbf{h}(\mathbf{s}^\star, \mathbf{r})) \\ &= \mathbf{h}(\mathbf{s}^\star, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}) \geq 0, \end{aligned}$$

where we used (7.42) and (7.46). This is a clear contradiction.

- Assume next that $(\mathbf{s}^\star, \mathbf{s}^\star)$ is an equilibrium of game D. Then for any $\mathbf{s}, \mathbf{z} \in S$,

$$f(\mathbf{s}, \mathbf{s}^\star) \leq f(\mathbf{s}^\star, \mathbf{s}^\star) = 0 \leq f(\mathbf{s}^\star, \mathbf{z}).$$

The first part can be rewritten as

$$\mathbf{h}(\mathbf{s}^\star, \mathbf{r})^T(\mathbf{s} - \mathbf{s}^\star) \leq 0,$$

showing that (7.46) is satisfied, so is (7.45).

■

Consider the following iteration procedure.
Let $\mathbf{s}^{(1)} \in S$ be arbitrary, and solve problem

$$\begin{array}{ll} \text{maximize} & f(\mathbf{s}, \mathbf{s}^{(1)}) \\ \text{subject to} & \mathbf{s} \in S. \end{array} \tag{7.47}$$

Let $\mathbf{s}^{(2)}$ denote an optimal solution and define $\mu_1 = f(\mathbf{s}^{(2)}, \mathbf{s}^{(1)})$. If $\mu_1 = 0$, then for all $\mathbf{s} \in S$,

$$f(\mathbf{s}, \mathbf{s}^{(1)}) = \mathbf{h}(\mathbf{s}^{(1)}, \mathbf{r})^T(\mathbf{s} - \mathbf{s}^{(1)}) \leq 0,$$

so by Theorem 7.17, $\mathbf{s}^{(1)}$ is an equilibrium. Since $f(\mathbf{s}^{(1)}, \mathbf{s}^{(1)}) = 0$, we assume that $\mu_1 > 0$. In the general step $k \geq 2$ we have already k vectors $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \ldots, \mathbf{s}^{(k)}$, and $k - 1$ scalers $\mu_1, \mu_2, \ldots, \mu_{k-1} > 0$. Then the next vector $\mathbf{s}^{(k+1)}$ and next scaler μ_k are the solutions of the following problem:

$$\begin{array}{lll} \text{maximize} & \mu & \\ \text{subject to} & f(\mathbf{s}, \mathbf{s}^{(i)}) \geq \mu & (i = 1, 2, \ldots, k) \\ & \mathbf{s} \in S. & \end{array} \tag{7.48}$$

Notice that

$$f(\mathbf{s}^{(k)}, \mathbf{s}^{(i)}) \geq \mu_{k-1} \geq 0 \qquad (i = 1, 2, \ldots, k - 1)$$

and

$$f(\mathbf{s}^{(k)}, \mathbf{s}^{(k)}) = 0,$$

so we know that $\mu_k \geq 0$.

The formal algorithm is as follows:

$k \leftarrow 1$
solve problem (7.47), let $\mathbf{s}^{(2)}$ be optimal solution
if $f(\mathbf{s}^{(2)}, \mathbf{s}^{(1)}) = 0$
    then $\mathbf{s}^{(1)}$ is equilibrium and stop
$k \leftarrow k + 1$
solve problem (7.48), let $\mathbf{s}^{(k+1)}$ be optimal solution
if $\|\mathbf{s}^{(k+1)} - \mathbf{s}^{(k)}\| < \varepsilon$
    then $\mathbf{s}^{(k+1)}$ is equilibrium
else go to 4

Before stating the convergence theorem of the algorithm we notice that in the special case when the strategy sets are defined by linear inequalities (that is, all functions g_k are linear) then all constraints of problem (7.48) are linear, so at each iteration step we have to solve a linear programming problem.

In this linear case the simplex method has to be used in each iteration step with exponential computational cost, so the overall cost is also exponential (with prefixed number of steps).

Theorem 7.19 *There is a subsequence $\{\mathbf{s}^{(k_i)}\}$ of $\{\mathbf{s}^{(k)}\}$ generated by the method that converges to the unique equilibrium of the N-person game.*

Proof. The proof consists of several steps.

First we show that $\mu_k \to 0$ as $k \to \infty$. Since at each new iteration an additional constraint is added to (7.48), sequence $\{\mu_k\}$ is nonincreasing. Since it is also nonnegative, it must be convergent. Sequence $\{\mathbf{s}^{(k)}\}$ is bounded, since it is from the bounded set S, so it has a convergent subsequence $\{\mathbf{s}^{(k_i)}\}$. Notice that from (7.48) we have

$$0 \le \mu_{k_i-1} = \min_{1 \le k \le k_i-1} \mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r})^T(\mathbf{s}^{(k_i)} - \mathbf{s}^{(k)}) \le \mathbf{h}(\mathbf{s}^{(k_{i-1})}, \mathbf{r})^T(\mathbf{s}^{(k_i)} - \mathbf{s}^{(k_{i-1})}) ,$$

where the right hand side tends to zero. Thus $\mu_{k_i-1} \to 0$ and since the entire sequence $\{\mu_k\}$ is monotonic, the entire sequence converges to zero.

Let next $\mathbf{s}^\star$ be an equilibrium of the N-person game, and define

$$\delta(t) = \min\{(\mathbf{h}(\mathbf{s}, \mathbf{r}) - \mathbf{h}(\mathbf{z}, \mathbf{r}))^T(\mathbf{z} - \mathbf{s}) \mid \|\mathbf{s} - \mathbf{z}\| \ge t,\ \mathbf{z}, \mathbf{s} \in S\}. \tag{7.49}$$

By (7.42), $\delta(t) > 0$ for all $t > 0$. Define the indices k_i so that

$$\delta(\|\mathbf{s}^{(k_i)} - \mathbf{s}^\star\|) = \min_{1 \le k \le i} \delta(\|\mathbf{s}^{(k)} - \mathbf{s}^\star\|) \quad (i = 1, 2, \ldots),$$

then for all $k = 1, 2, \ldots, i$,

$$\begin{aligned} \delta(\|\mathbf{s}^{(k_i)} - \mathbf{s}^\star\|) &\le (\mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r}) - \mathbf{h}(\mathbf{s}^\star, \mathbf{r}))^T(\mathbf{s}^\star - \mathbf{s}^{(k)}) \\ &= \mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}^{(k)}) - \mathbf{h}(\mathbf{s}^\star, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}^{(k)}) \\ &\le \mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}^{(k)}), \end{aligned}$$

which implies that

$$\begin{aligned}
\delta(\|\mathbf{s}^{(k_i)} - \mathbf{s}^\star\|) &\le \min_{1\le k\le i} \mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r})^T(\mathbf{s}^\star - \mathbf{s}^{(k)}) \\
&\le \max_{\mathbf{s}\in S} \min_{1\le k\le i} \mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r})^T(\mathbf{s} - \mathbf{s}^{(k)}) \\
&= \min_{1\le k\le i} \mathbf{h}(\mathbf{s}^{(k)}, \mathbf{r})^T(\mathbf{s}^{(i+1)} - \mathbf{s}^{(k)}) \\
&= \mu_i
\end{aligned}$$

where we used again problem (7.48). From this relation we conclude that $\delta(\|\mathbf{s}^{(k_i)} - \mathbf{s}^\star\|) \to 0$ as $i \to \infty$. And finally, notice that function $\delta(t)$ satisfies the following properties:

1. $\delta(t)$ is continuous in t;
2. $\delta(t) > 0$ if $t > 0$ (as it was shown just below relation (7.49));
3. if for a convergent sequence $\{t^{(k)}\}$, $\delta(t^{(k)}) \to 0$, then necessarily $t^{(k)} \to 0$.

By applying property 3. with sequence $\{\|\mathbf{s}^{(k_i)} - \mathbf{s}^\star\|\}$ it is clear that $\|\mathbf{s}^{(k_i)} - \mathbf{s}^\star\| \to 0$ so $\mathbf{s}^{(k_i)} \to \mathbf{s}^\star$. Thus the proof is complete. ■

Exercises

7.2-1 Consider a 2-person game with strategy sets $S_1 = S_2 = [0, 1]$, and payoff functions $f_1(x_1, x_2) = x_1^2 + x_1x_2 + 2$ and $f_2(x_1, x_2) = x_1 + x_2$. Show the existence of a unique equilibrium point by computing it. Show that Theorem 7.3. cannot be applied to prove existence.

7.2-2 Consider the "price war" game in which two firms are price setting. Assume that p_1 and p_2 are the strategies of the players, $p_1, p_2 \in [0, p_{max}]$ and the payoff functions are:

$$f_1(p_1, p_2) = \begin{cases} p_1, & \text{if } p_1 \le p_2, \\ p_1 - c, & \text{if } p_1 > p_2, \end{cases}$$

$$f_2(p_1, p_2) = \begin{cases} p_2, & \text{if } p_2 \le p_1, \\ p_2 - c, & \text{if } p_2 > p_1, \end{cases}$$

by assuming that $c < p_{max}$. Is there an equilibrium? How many equilibria were found?

7.2-3 A portion of the sea is modeled by the unit square in which a submarine is hiding. The strategy of the submarine is the hiding place $\mathbf{x} \in [0, 1] \times [0, 1]$. An airplane drops a bomb in a location $\mathbf{y} = [0, 1] \times [0, 1]$,j which is its strategy. The payoff of the airplane is the damage $\alpha e^{-\beta\|\mathbf{x}-\mathbf{y}\|}$ occurred by the bomb, and the payoff of the submarine is its negative. Does this 2-person game have an equilibrium?

7.2-4 In the second-price auction they sell one unit of an item to N bidders. They value the item as $v_1 < v_2 < \cdots < v_N$. Each of them offers a price for the item simultaneously without knowing the offers of the others. The bidder with the highest offer will get the item, but he has to pay only the second highest price. So the strategy of bidder k is $[0, \infty]$, so $x_k \in [0, \infty]$, and the payoff function for this bidder is:

$$f_k(x_1, x_2, \ldots, x_N) = \begin{cases} v_k - \max_{j\ne k} x_j, & \text{if } x_k = \max_j x_j, \\ 0 & \text{otherwise.} \end{cases}$$

What is the best response function of bidder k? Does this game have equilibrium?

7.2-5 Formulate Fan's inequality for the Problem 7.2-1.

7.2-6 Formulate and solve Fan's inequality for Problem 7.2-2.

7.2-7 Formulate and solve Fan's inequality for Problem 7.2-4.
7.2-8 Consider a 2-person game with strategy sets $S_1 = S_2 = [0, 1]$, and payoff functions

$$f_1(x_1, x_2) = -(x_1 - x_2)^2 + 2x_1 - x_2 + 1$$

$$f_2(x_1, x_2) = -(x_1 - 2x_2)^2 - 2x_1 + x_2 - 1.$$

Formulate Fan's inequality.
7.2-9 Let $n = 2$, $S_1 = S_2 = [0, 10]$, $f_1(x_1, x_2) = f_2(x_1, x_2) = 2x_1 + 2x_2 - (x_1 + x_2)^2$. Formulate the Kuhn–Tucker conditions to find the equilibrium. Solve the resulted system of inequalities and equations.
7.2-10 Consider a 3-person game with $S_1 = S_2 = S_3 = [0, 1]$, $f_1(x_1, x_2, x_3) = (x_1-x_2)^2+x_3$, $f_2(x_1, x_2, x_3) = (x_2-x_3)^2+x_1$ and $f_3(x_1, x_2, x_3) = (x_3-x_1)^2+x_2$. Formulate the Kuhn–Tucker condition.
7.2-11 Formulate and solve system (7.9) for Problem 7.2-8.
7.2-12 Repeat the previous problem for the game given in Problem 7.2-1.
7.2-13 Rewrite the Kuhn–Tucker conditions for Problem 7.2-8. into the optimization problem (7.10) and solve it.
7.2-14 Formulate the mixed extension of the finite game given in Problem 7.1-1.
7.2-15 Formulate and solve optimization problem (7.10) for the game obtained in the previous problem.
7.2-16 Formulate the mixed extension of the game introduced in Problem 7.2-3.
Formulate and solve the corresponding linear optimization problems (7.22) with $\alpha = 5$, $\beta = 3$, $\gamma = 1$.
7.2-17 Use fictitious play method for solving the matrix game of Problem 7.2-16.
7.2-18 Generalize the fictitious play method for bimatrix games.
7.2-19 Generalize the fictitious play method for the mixed extensions of finite n-person games.
7.2-20 Solve the bimatrix game with matrics $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ with the method you have developed in Problem 7.2-18.
7.2-21 Solve the symmetric matrix game $\mathbf{A} = \begin{pmatrix} 0 & 1 & 5 \\ -1 & 0 & -3 \\ -5 & 3 & 0 \end{pmatrix}$ by linear programming.
7.2-22 Repeat problem 7.2-21. with the method of fictitious play.
7.2-23 Develop the Kuhn–Tucker conditions (7.9) for the game given in Problem 7.2-21. above.
7.2-24★ Repeat Problems 7.2-21., 7.2-22. and 7.2-23. for the matrix game $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$. (First find the equivalent symmetric matrix game!).
7.2-25 Formulate the linear programming problem to solve the matrix game with matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.
7.2-26 Formulate a linear programming solver based on the method of fictitious play and

solve the LP problem:

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1, x_2 \geq 0 \\ & 3x_1 + x_2 \leq 4 \\ & x_1 + 3x_2 \leq 4. \end{aligned}$$

7.2-27 Solve the LP problem given in Example 8.17 by the method of fictitious play.
7.2-28 Solve problem 7.2-21. by the method of von Neumann.
7.2-29 Solve Problem 7.2-24. by the method of von Neumann.
7.2-30 Solve Problem 7.2-17. by the method of von Neumann.
7.2-31⋆ Check the solution obtained in the previous problems by verifying that all constraints of (7.21) are satisfied with zero objective function. *Hint.* What α and β should be selected?
7.2-32 Solve problem 7.2-26. by the method of von Neumann.
7.2-33 Let $N = 2$, $S_1 = S_2 = [0, 10]$, $f_1(x_1, x_2) = f_2(x_1, x_2) = 2x_1 + 2x_2 - (x_1 + x_2)^2$. Show that both payoff functions are strictly concave in x_1 and x_2 respectively. Prove that there are infinitely many equilibria, that is , the strict concavity of the payoff functions does not imply the uniqueness of the equilibrium.
7.2-34 Can matrix games be strictly diagonally concave?
7.2-35 Consider a two-person game with strategy sets $S_1 = S_2 = [0, 1]$, and payoff functions $f_1(x_1, x_2) = -2x_1^2 + x_1(1 - x_2)$, $f_2(x_1, x_2) = -3x_2^2 + x_2(1 - x_1)$. Show that this game satisfies all conditions of Theorem 7.16.
7.2-36 Solve the problem of the previous exercise by algorithm (7.47)–(7.48).

7.3. The Oligopoly Problem

The previous sections presented general methodology, however special methods are available for almost all special classes of games. In the following parts of this chapter a special game, the ***oligopoly game*** will be examined. It describes a real-life economic situation when N-firms produce a homogeneous good to a market, or offers the same service. This model is known as the ***classical Cournot model***. The firms are the players. The strategy of each player is its production level x_k with strategy set $S_k = [0, L_k]$, where L_k is its capacity limit. It is assumed that the market price depends on the total production level $s = x_1 + x_2 + \cdots + x_N$ offered to the market: $p(s)$, and the cost of each player depends on its own production level: $c_k(x_k)$. The profit of each firm is given as

$$f_k(x_1, \ldots, x_N) = x_k p\left(\sum_{l=1}^{N} x_l\right) - c_k(x_k). \tag{7.50}$$

In this way an N-person game $G = \{N; S_1, \ldots, S_N; f_1, \ldots, f_N\}$ is defined.

It is usually assumed that functions p and c_k $(k = 1, 2, \ldots, N)$ are twice continuously differentiable, furthermore

1. $p'(s) < 0$;

2. $p'(s) + x_k p''(s) \le 0$;
3. $p'(s) - c_k''(x_k) < 0$

for all k, $x_k \in [0, L_k]$ and $s \in [0, \sum_{l=1}^{N} L_l]$. Under assumptions 1.–3. the game satisfies all conditions of Theorem 7.3, so there is at least one equilibrium.

Best Reply Mappings

Notice that with the notation $s_k = \sum_{l \ne k} x_l$, the payoff function of player $\mathcal{P}_k$ can be rewritten as

$$x_k p(x_k + s_k) - c_k(x_k) \ . \tag{7.51}$$

Since S_k is a compact set and this function is strictly concave in x_k, with fixed s_k there is a unique profit maximizing production level of player $\mathcal{P}_k$, which is its best reply and is denoted by $B_k(s_k)$.

It is easy to see that there are three cases: $B_k(s_k) = 0$ if $p(s_k) - c_k'(0) \le 0$, $B_k(s_k) = L_k$ if $p(s_k + L_k) + L_k p'(s_k + L_k) - c_k'(L_k) \ge 0$, and otherwise $B_k(s_k)$ is the unique solution of the monotonic equation

$$p(s_k + x_k) + x_k p'(s_k + x_k) - c_k'(x_k) = 0.$$

Assume that $x_k \in (0, L_k)$. Then implicit differentiation with respect to s_k shows that

$$p'(1 + B_k') + B_k' p' + x_k p''(1 + B_k') - c_k'' B_k' = 0$$

showing that

$$B_k'(s_k) = -\frac{p' + x_k p''}{2p' + x_k p'' - c_k''}.$$

Notice that from assumptions 2. and 3.,

$$-1 < B_k'(s_k) \le 0 \ , \tag{7.52}$$

which is also true for the other two cases except for the break points.

As in Section 7.2.1. we can introduce the best reply mapping:

$$\mathbf{B}(x_1, \ldots, x_N) = \left(B_1 \left(\sum_{l \ne 1} x_l \right), \ldots, B_N \left(\sum_{l \ne N} x_l \right) \right) \tag{7.53}$$

and look for its fixed points. Another alternative is to introduce dynamic process which converges to the equilibrium.

Similarly to the method of fictitious play a discrete system can be developed in which each firm selects its best reply against the actions of the competitors chosen at the previous time period:

$$x_k(t+1) = B_k(\sum_{l \ne k} x_l(t)) \qquad (k = 1, 2, \ldots, N) \ . \tag{7.54}$$

Based on relation (7.52) we see that for $N = 2$ the right hand side mapping $\mathbb{R}^2 \to \mathbb{R}^2$ is a contraction, so it converges, however if $N > 2$, then no convergence can be established. Consider next a slight modification of this system: with some $K_k > 0$:

$$x_k(t+1) = x_k(t) + K_k(B_k(\sum_{l \ne k} x_l(t)) - x_k(t)) \tag{7.55}$$

for $k = 1, 2, \ldots, N$. Clearly the steady-states of this system are the equilibria, and it can be proved that if K_k is sufficiently small, then sequences $x_k(0), x_k(1), x_k(2), \ldots$ are all convergent to the equilibrium strategies.

Consider next the continuous counterpart of model (7.55), when (similarly to the method of von Neumann) continuous time scales are assumed:

$$\dot{x}_k(t) = K_k(B_k(\sum_{l \neq k} x_l(t)) - x_k(t)) \quad (k = 1, 2, \ldots, N) \ . \tag{7.56}$$

The following result shows the convergence of this process.

Theorem 7.20 *Under assumptions 1–3, system (7.56) is asymptotically stable, that is, if the initial $x_k(0)$ values are selected close enough to the equilibrium, then as $t \to \infty$, $x_k(t)$ converges to the equilibrium strategy for all k.*

Proof. It is sufficient to show that the eigenvalues of the Jacobian of the system have negative real parts. Clearly the Jacobian is as follows:

$$\mathbf{J} = \begin{pmatrix} -K_1 & K_1 b_1 & \cdots & K_1 b_1 \\ K_2 b_2 & -K_2 & \cdots & K_2 b_2 \\ \vdots & \vdots & & \vdots \\ K_N b_N & K_N b_N & \cdots & -K_N \end{pmatrix} , \tag{7.57}$$

where $b_k = B_k'(\sum_{l \neq k} x_l)$ at the equilibrium. From (7.52) we know that $-1 < b_k \leq 0$ for all k. In order to compute the eigenvalues of $\mathbf{J}$ we will need a simple but very useful fact. Assume that $\mathbf{a}$ and $\mathbf{b}$ are N-element real vectors. Then

$$\det(\mathbf{I} + \mathbf{a}\mathbf{b}^T) = 1 + \mathbf{b}^T \mathbf{a} \ , \tag{7.58}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix. This relation can be easily proved by using finite induction with respect to N. By using (7.58), the characteristic polynomial of $\mathbf{J}$ can be written as

$$\begin{aligned} \phi(\lambda) &= \det(\mathbf{J} - \lambda\mathbf{I}) = \det(\mathbf{D} + \mathbf{a}\mathbf{b}^T - \lambda\mathbf{I}) \\ &= \det(\mathbf{D} - \lambda\mathbf{I})\det(\mathbf{I} + (\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{a}\mathbf{b}^T) \\ &= \det(\mathbf{D} - \lambda\mathbf{I})[1 + \mathbf{b}^T(\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{a}] \\ &= \Pi_{k=1}^{N}(-K_k(1 + b_k) - \lambda)[1 + \sum_{k=1}^{N} \frac{K_k b_k}{-K_k(1 + b_k) - \lambda}], \end{aligned}$$

where we used the notation

$$\mathbf{a} = \begin{pmatrix} K_1 b_1 \\ K_2 b_2 \\ \vdots \\ K_N b_N \end{pmatrix}, \quad \mathbf{b}^T = (1, 1, \ldots, 1), \quad \mathbf{D} = \begin{pmatrix} -K_1(1 + b_1) & & \\ & \ddots & \\ & & -K_N(1 + b_N) \end{pmatrix}.$$

The roots of the first factor are all negative: $\lambda = -K_k(1 + b_k)$, and the other eigenvalues are the roots of equation

$$1 + \sum_{k=1}^{N} \frac{K_k b_k}{-K_k(1+b_k) - \lambda} = 0.$$

Notice that by adding the terms with identical denominators this equation becomes

$$1 + \sum_{l=1}^{m} \frac{\alpha_k}{\beta_k + \lambda} = 0 \tag{7.59}$$

with $\alpha_k, \beta_k > 0$, and the $\beta_k s$ are different. If $g(\lambda)$ denotes the left hand side then clearly the values $\lambda = -\beta_k$ are the poles,

$$\lim_{\lambda \to \pm\infty} g(\lambda) = 1, \quad \lim_{\lambda \to -\beta_k \pm 0} g(\lambda) = \pm\infty,$$

$$g'(\lambda) = \sum_{l=1}^{m} \frac{-\alpha_l}{(\beta_l + \lambda)^2} < 0,$$

so $g(\lambda)$ strictly decreases locally. The graph of the function is shown in Figure 7.3. Notice first that (7.59) is equivalent to a polynomial equation of degree m, so there are m real or complex roots. The properties of function $g(\lambda)$ indicate that there is one root below $-\beta_1$, and one root between each $-\beta_k$ and $-\beta_{k+1}$ $(k = 1, 2, \ldots, m-1)$. Therefore all roots are negative, which completes the proof. ∎

The general discrete model (7.55) can be examined in the same way. If $K_k = 1$ for all k, then model (7.55) reduces to the simple dynamic process (7.54).

Example 7.21 Consider now a 3-person oligopoly with price function

$$p(s) = \begin{cases} 2 - 2s - s^2, & \text{if} \quad 0 \le s \le \sqrt{3} - 1 \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

strategy sets $S_1 = S_2 = S_3 = [0, 1]$, and cost functions

$$c_k(x_k) = k x_k^3 + x_k \qquad (k = 1, 2, 3) \,.$$

The profit of firm k is therefore the following:

$$x_k(2 - 2s - s^2) - (k x_k^3 + x_k) = x_k(2 - 2x_k - 2s_k - x_k^2 - 2x_k s_k - s_k^2) - k x_k^3 - x_k \,.$$

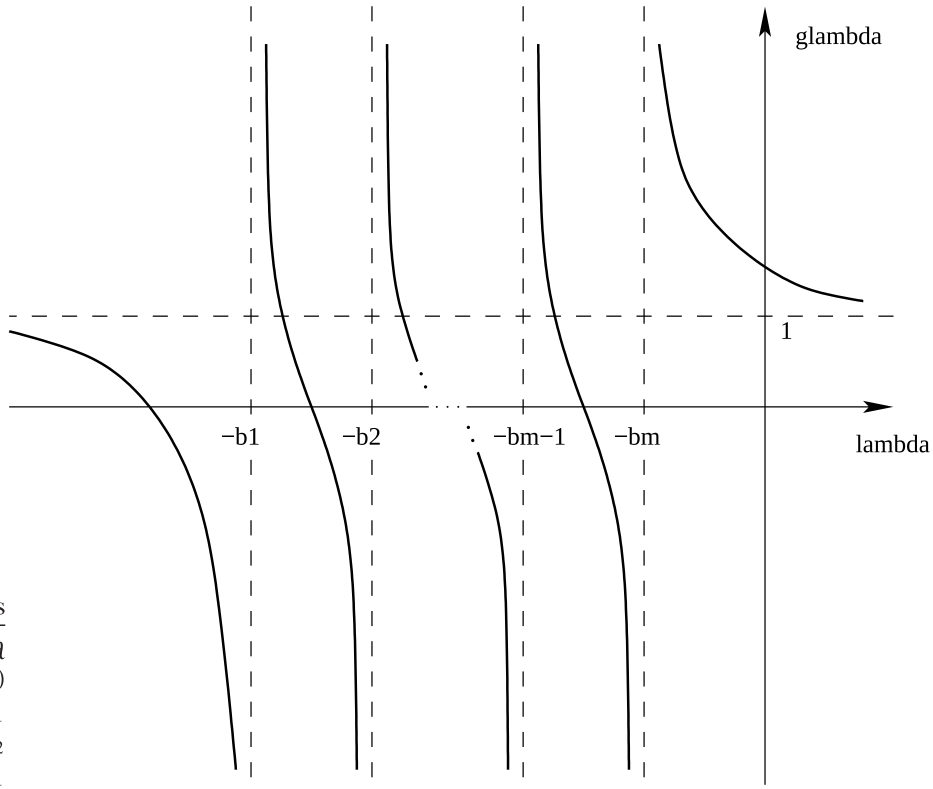
The best reply of play k can be obtained as follows. Following the method outlined at the beginning of Section 7.3. we have the following three cases. If $1 - 2s_k - s_k^2 \le 0$, then $x_k = 0$ is the best choice. If $(-6 - 3k) - 6s_k - s_k^2 \ge 0$, then $x_k = 1$ is the optimal decision. Otherwise x_k is the solution of equation

$$\begin{aligned} &\frac{\partial}{\partial x_k}[x_k(2 - 2x_k - 2s_k - s_k^2 - 2s_k x_k - x_k^2) - k x_k^3 - x_k] \\ &= 2 - 4x_k - 2s_k - s_k^2 - 4s_k x_k - 3x_k^2 - 3k x_k^2 - 1 = 0 \,, \end{aligned}$$

where the only positive solution is

$$x_k = \frac{-(4 + 4s_k) + \sqrt{(4 + 4s_k)^2 - 12(1 + k)(s_k^2 + 2s_k - 1)}}{6(1 + k)} \,.$$

After the best replies are found, we can easily construct any of the methods presented before.



PSfrag replacements

λ
$g(\lambda)$
$-\beta_1$
$-\beta_2$
$-\beta_{m-1}$
$-\beta_m$

Figure 7.5. The graph of function $g(\lambda)$

Reduction to Single-Dimensional Fixed Point Problems

Consider an N-firm oligopoly with price function p and cost functions c_k $(k = 1, 2, \ldots, N)$. Introduce the following function

$$\Psi_k(s, x_k, t_k) = t_k p(s - x_k + t_k) - c_k(t_k) \ , \tag{7.60}$$

and define

$$X_k(s) = \{x_k | x_k \in S_k, \ \ \Psi_k(s, x_k, x_k) = \max_{t_k \in S_k} \Psi_k(s, x_k, t_k)\} \tag{7.61}$$

for $k = 1, 2, \ldots, N$ and let

$$X(s) = \{u | u = \sum_{k=1}^{N} x_k, \ \ x_k \in X_k(s), \ \ k = 1, 2, \ldots, N\}. \tag{7.62}$$

Notice that if $s \in [0, \sum_{k=1}^{N} L_k]$, then all elements of $X(s)$ are also in this interval, therefore X is a single-dimensional point-to-set mapping. Clearly $(x_1^\star, \ldots, x_N^\star)$ is an equilibrium of the N-firm oligopoly game if and only if $s^\star = \sum_{k=1}^{N} x_k^\star$ is a fixed point of mapping X and for all

k, $x_k^\star \in X_k(s^\star)$. Hence the equilibrium problem has been reduced to find fixed points of only one-dimensional mappings. This is a significant reduction in the difficulty of the problem, since best replies are N-dimensional mappings.

If conditions 1–3 are satisfied, then $X_k(s)$ has exactly one element for all s and k:

$$X(s) = \begin{cases} 0, & \text{if } p(s) - c_k'(0) \le 0 \ , \\ L_k, & \text{if } p(s) + L_k p_k'(s) - c_k'(L_k) \ge 0, \\ z^\star & \text{otherwise,} \end{cases} \tag{7.63}$$

where $z^\star$ is the unique solution of the monotonic equation

$$p(s) + zp'(s) - c_k'(z) = 0 \tag{7.64}$$

in the interval $(0, L_k)$. In the third case, the left hand side is positive at $z = 0$, negative at $z = L_k$, and by conditions 2–3, it is strictly decreasing, so there is a unique solution.

In the entire interval $[0, \sum_{k=1}^N L_k]$, $X_k(s)$ is nonincreasing. In the first two cases it is constant and in the third case strictly decreasing. Consider finally the single-dimensional equation

$$\sum_{k=1}^N X_k(s) - s = 0. \tag{7.65}$$

At $s = 0$ the left hand side is nonnegative, at $s = \sum_{k=1}^N L_k$ it is nonpositive, and is strictly decreasing. Therefore there is a unique solution (that is, fixed point of mapping X), which can be obtained by any method known to solve single-dimensional equations.

Let $[0, S_{max}]$ be the initial interval for the solution of equation (7.65). After K bisection steps the accuracy becomes $S_{max}/2^K$, which will be smaller than an error tolerance $\epsilon > 0$ if $K > \log_2(S_{max}/\epsilon)$.

```
solve equation (7.65) for s
for k ← 1 to n
    do solve equation (7.64), and let x_k ← z
(x_1, ..., x_N) is equilibrium
```

Example 7.22 Consider the 3-person oligopoly examined in the previous example. From (7.63) we have

$$X(s) = \begin{cases} 0, & \text{if} \quad 1 - 2s - s^2 \le 0, \\ 1, & \text{if} \quad -(1+3k) - 4s - s^2 \ge 0, \\ z^\star & \text{otherwise,} \end{cases}$$

where $z^\star$ is the unique solution of equation

$$3kz^2 + z(2s+2) + (-1 + 2s + s^2) = 0.$$

The first case occurs for $s \ge \sqrt{2} - 1$, the second case never occurs, and in the third case there is a unique positive solution:

$$z^\star = \frac{-(2s+2) + \sqrt{(2s+2)^2 - 12k(-1 + 2s + s^2)}}{6k} \ . \tag{7.66}$$

And finally equation (7.65) has the special form

$$\sum_{k=1}^{3} \frac{-(s+1) + \sqrt{(s+1)^2 - 3k(-1+2s+s^2)}}{3k} - s = 0\ .$$

A single program based on the bisection method gives the solution $s^\star \approx 0.2982$ and then equation (7.66) gives the equilibrium strategies $x_1^\star \approx 0.1077$, $x_2^\star \approx 0.0986$, $x_3^\star \approx 0.0919$.

Methods Based on Kuhn–Tucker Conditions

Notice first that in the case of N-player oligopolies $S_k = \{x_k | x_k \geq 0,\ L_k - x_k \geq 0\}$, so we select

$$\mathbf{g}_k(x_k) = \begin{pmatrix} x_k \\ L_k - x_k \end{pmatrix}, \tag{7.67}$$

and since the payoff functions are

$$f_k(x_1, \ldots, x_N) = x_k p(x_k + s_k) - c_k(x_k)\ , \tag{7.68}$$

the Kuhn–Tucker conditions (7.9) have the following form. The components of the 2-dimensional vectors $\mathbf{u}_k$ will be denoted by $u_k^{(1)}$ and $u_k^{(2)}$. So we have for $k = 1, 2, \ldots, N$,

$$\begin{array}{rcl} u_k^{(1)}, u_k^{(2)} & \geq & 0 \\ x_k & \geq & 0 \\ L_k - x_k & \geq & 0 \\ p(\sum_{l=1}^N x_l) + x_k p'(\sum_{l=1}^N x_l) - c_k'(x_k) + (u_k^{(1)}, u_k^{(2)})\binom{1}{-1} & = & 0 \\ u_k^{(1)} x_k + u_k^{(2)}(L_k - x_k) & = & 0. \end{array} \tag{7.69}$$

One might either look for feasible solutions of these relations or rewrite them as the optimization problem (7.10), which has the following special form in this case:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^N (u_k^{(1)} x_k + u_k^{(2)}(L_k - x_k)) \\ \text{subject to} & u_k^{(1)}, u_k^{(2)} \geq 0 \\ & x_k \geq 0 \\ & L_k - x_k \geq 0 \\ & p(\sum_{l=1}^N x_l) + x_k p'(\sum_{l=1}^N x_l) - c_k'(x_k) + u_k^{(1)} - u_k^{(2)} = 0 \\ & \qquad\qquad\qquad (k = 1, 2, \ldots, N). \end{array} \tag{7.70}$$

Computational cost in solving (7.69) or (7.70) depends on the type of functions p and c_k. No general characterization can be given.

Example 7.23 In the case of the three-person oligopoly introduced in Example 7.21. we have

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{3} (u_k^{(1)} x_k + u_k^{(2)}(1 - x_k)) \\ \text{subject to} & u_k^{(1)}, u_k^{(2)} \geq 0 \\ & x_k \geq 0 \\ & 1 - x_k \geq 0 \\ & 1 - 2s - s^2 - 2x_k - 2x_k s - 3kx_k^2 + u_k^{(1)} - u_k^{(2)} = 0 \\ & x_1 + x_2 + x_3 = s. \end{array}$$

A professional optimization software was used to obtain the optimal solutions:

$$x_1^\star \approx 0.1077, \quad x_2^\star \approx 0.0986, \quad x_3^\star \approx 0.0919\ ,$$

and all $u_k^{(1)} = u_k^{(2)} = 0$.

Reduction to Complementarity Problems

If $(x_1^\star, \dots, x_N^\star)$ is an equilibrium of an N-person oligopoly, then with fixed $x_1^\star, \dots, x_{k-1}^\star, x_{k+1}^\star, \dots, x_N^\star$, $x_k = x_k^\star$ maximizes the payoff f_k of player $\mathcal{P}_k$. Assuming that condition 1–3 are satisfied, f_k is concave in x_k, so $x_k^\star$ maximizes f_k if and only if at the equilibrium

$$\frac{\partial f_k}{\partial x_k}(\mathbf{x}^\star) = \begin{cases} \le 0, & \text{if} \quad x_k^\star = 0, \\ = 0, & \text{if} \quad 0 < x_k^\star < L_k, \\ \ge 0, & \text{if} \quad x_k^\star = L_k. \end{cases}$$

So introduce the slack variables

$$z_k = \begin{cases} = 0, & \text{if} \quad x_k > 0, \\ \ge 0, & \text{if} \quad x_k = 0 \end{cases}$$

$$v_k = \begin{cases} = 0, & \text{if} \quad x_k < L_k, \\ \ge 0, & \text{if} \quad x_k = L_k \end{cases}$$

and

$$w_k = L_k - x_k. \tag{7.71}$$

Then clearly at the equilibrium

$$\frac{\partial f_k}{\partial x_k}(\mathbf{x}) - v_k + z_k = 0 \tag{7.72}$$

and by the definition of the slack variables

$$z_k x_k = 0 \tag{7.73}$$

$$v_k w_k = 0, \tag{7.74}$$

and if we add the nonnegativity conditions

$$x_k, z_k, v_k, w_k \ge 0\ , \tag{7.75}$$

then we obtain a system of nonlinear relations (7.71)–(7.75) which are equivalent to the equilibrium problem.

We can next show that relations (7.71)–(7.75) can be rewritten as a nonlinear complementarity problem, for the solution of which standard methods are available. For this purpose introduce the notation

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{pmatrix}, \quad \mathbf{h}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f_N}{\partial x_N}(\mathbf{x}) \end{pmatrix},$$

$$\mathbf{t} = \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix}, \quad \text{and} \quad \mathbf{g}(\mathbf{t}) = \begin{pmatrix} -\mathbf{h}(\mathbf{x}) + \mathbf{v} \\ \mathbf{L} - \mathbf{x} \end{pmatrix},$$

then system (7.72)–(7.75) can be rewritten as

$$\begin{aligned} \mathbf{t} &\geq \mathbf{0} \\ \mathbf{g}(\mathbf{t}) &\geq \mathbf{0} \\ \mathbf{t}^T\mathbf{g}(\mathbf{t}) &= 0. \end{aligned} \tag{7.76}$$

This problem is the usual formulation of ***nonlinear complementarity*** problems. Notice that the last condition requires that in each component either **t** or **g**(**t**) or both must be zero.

The computational cost in solving problem (7.76) depends on the type of the involved functions and the choice of method.

Example 7.24 In the case of the 3-person oligopoly introduced and examined in the previous examples we have:

$$\mathbf{t} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{and} \quad \mathbf{g}(\mathbf{t}) = \begin{pmatrix} -1 + 2\sum_{l=1}^3 x_l + (\sum_{l=1}^3 x_l)^2 + 2x_1 + 2x_1\sum_{l=1}^3 x_l + 3x_1^2 + v_1 \\ -1 + 2\sum_{l=1}^3 x_l + (\sum_{l=1}^3 x_l)^2 + 2x_2 + 2x_2\sum_{l=1}^3 x_l + 6x_2^2 + v_2 \\ -1 + 2\sum_{l=1}^3 x_l + (\sum_{l=1}^3 x_l)^2 + 2x_3 + 2x_3\sum_{l=1}^3 x_l + 9x_3^2 + v_3 \\ 1 - x_1 \\ 1 - x_2 \\ 1 - x_3 \end{pmatrix}.$$

Linear Oligopolies and Quadratic Programming

In this section N-player oligopolies will be examined under the special condition that the price and all cost functions are linear :

$$p(s) = As + B, \quad c_k(x_k) = b_k x_k + c_k \quad (k = 1, 2, \ldots, N)\ ,$$

where B, b_k, and c_k are positive, but $A < 0$. Assume again that the strategy set of player $\mathcal{P}_k$ is the interval $[0, L_k]$. In this special case

$$f_k(x_1, \ldots, x_N) = x_k(Ax_1 + \cdots + Ax_N + B) - (b_k x_k + c_k) \tag{7.77}$$

for all k, therefore

$$\frac{\partial f_k}{\partial x_k}(\mathbf{x}) = 2Ax_k + A\sum_{l \neq k} x_l + B - b_k, \tag{7.78}$$

and relations (7.71)–(7.75) become more special:

$$\begin{aligned} 2Ax_k + A\sum_{l \neq k} x_l + B - b_k - v_k + z_k &= 0 \\ z_k x_k = v_k w_k &= 0 \\ x_k + w_k &= L_k \\ x_k, v_k, z_k, w_k &\geq 0, \end{aligned}$$

where we changed the order of them. Introduce the following vectors and matrixes:

$$\mathbf{Q} = \begin{pmatrix} 2A & A & \cdots & A \\ A & 2A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & 2A \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B \\ B \\ \vdots \\ B \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix},$$

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}, \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{pmatrix}.$$

Then the above relations can be summarized as:

$$\begin{aligned} \mathbf{Qx} + \mathbf{B} - \mathbf{b} - \mathbf{v} + \mathbf{z} &= \mathbf{0} \\ \mathbf{x} + \mathbf{w} &= \mathbf{L} \\ \mathbf{x}^T\mathbf{z} = \mathbf{v}^T\mathbf{w} &= 0 \\ \mathbf{x}, \mathbf{v}, \mathbf{z}, \mathbf{w} &\geq \mathbf{0}\ . \end{aligned} \tag{7.79}$$

Next we prove that matrix $\mathbf{Q}$ is negative definite. With any nonzero vector $\mathbf{a} = (a_i)$,

$$\mathbf{a}^T\mathbf{Qa} = 2A\sum_i a_i^2 + A\sum_i\sum_{j\neq i} a_i a_j = A(\sum_i a_i^2 + (\sum_i a_i)^2) < 0\ ,$$

which proves the assertion.

Observe that relations (7.79) are the Kuhn–Tucker conditions of the strictly concave quadratic programming problem:

$$\begin{aligned} &\text{maximize} && \tfrac{1}{2}\mathbf{x}^T\mathbf{Qx} + (\mathbf{B} - \mathbf{b})\mathbf{x} \\ &\text{subject to} && \mathbf{0} \leq \mathbf{x} \leq \mathbf{L}, \end{aligned} \tag{7.80}$$

and since the feasible set is a bounded linear polyhedron and the objective function is strictly concave, the Kuhn–Tucker conditions are sufficient and necessary. Consequently a vector $\mathbf{x}^\star$ is an equilibrium if and only if it is the unique optimal solution of problem (7.80). There are standard methods to solve problem (7.80) known from the literature.

Since (7.79) is a convex quadratic programming problem, several algorithms are available. Their costs are different, so computation cost depends on the particular method being selected.

Example 7.25 Consider now a duopoly (two-person oligopoly) where the price function is $p(s) = 10 - s$ and the cost functions are $c_1(x_1) = 4x_1 + 1$ and $c_2(x_2) = x_2 + 1$ with capacity limits $L_1 = L_2 = 5$. That is,

$$B = 10,\ \ A = -1,\ \ b_1 = 4,\ \ b_2 = 1,\ \ c_1 = c_2 = 1\ .$$

Therefore,

$$\mathbf{Q} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 5 \\ 5 \end{pmatrix},$$

so the quadratic programming problem can be written as:

$$\begin{aligned} &\text{maximize} && \frac{1}{2}(-2x_1^2 - 2x_1x_2 - 2x_2^2) + 6x_1 + 9x_2 \\ &\text{subject to} && 0 \leq x_1 \leq 5 \\ & && 0 \leq x_2 \leq 5. \end{aligned}$$

It is easy to see by simple differentiation that the global optimum at the objective function without the constraints is reached at $x_1^\star = 1$ and $x_2^\star = 4$. They however satisfy the constraints, so they are the optimal solutions. Hence they provide the unique equilibrium of the duopoly.

Exercises

7.3-1 Consider a duopoly with $S_1 = S_2 = [0, 1]$, $p(s) = 2 - s$ and costs $c_1(x) = c_2(x) = x^2 + 1$. Examine the convergence of the iteration scheme (7.55).

7.3-2 Select $n = 2$, $S_1 = S_2 = [0, 1.5]$, $c_k(x_k) = 0.5x_k$ $(k = 1, 2)$ and

$$p(s) = \begin{cases} 1.75 - 0.5s, & \text{if} \quad 0 \le s \le 1.5, \\ 2.5 - s, & \text{if} \quad 1.5 \le s \le 2.5, \\ 0, & \text{if} \quad 2.5 \le s. \end{cases}$$

Show that there are infinitely many equilibria:

$$\{(x_1^\star, x_2^\star) | 0.5 \le x_1 \le 1, \;\; 0.5 \le x_2 \le 1, \;\; x_1 + x_2 = 1.5\}.$$

7.3-3 Consider the duopoly of Problem 7.3-1. above. Find the best reply mappings of the players and determine the equilibrium.

7.3-4 Consider again the duopoly of the previous problem.
(a) Construct the one-dimensional fixed point problem of mapping (7.62) and solve it to obtain the equilibrium.
(b) Formulate the Kuhn–Tucker equations and inequalities (7.69).
(c) Formulate the complementarity problem (7.76) in this case.

Chapter notes

(Economic) Nobel Prize was given only once, in 1994 in the field of game theory. One of the winner was John Nash , who received this honor for his equilibrium concept, which was introduced in 1951 [11].

Backward induction is a more restrictive equilibrium concept. It was developed by Kuhn and can be found in [7]. Since it is more restrictive equilibrium, it is also a Nash equilibrium.

The existence and computation of equilibria can be reduced to those of fixed points. the different variants of fixed point theorems-such as that of Brouwer [2], Kakutani[5], Tarski [21] are successfully used to prove existence in many game classes. The article [13] uses the fixed point theorem of Kakutani. The books [20] and [3] discuss computer methods for computing fixed points. The most popular existence result is the well known theorem of Nikaido and Isoda [13].

The Fan inequality is discussed in the book of Aubin [1]. The Kuhn–Tucker conditions are presented in the book of Martos [9]. By introducing slack and surplus variables the Kuhn–Tucker conditions can be rewritten as a system of equations. For their computer solutions well known methods are available ([20] and [9]).

The reduction of bimatrix games to mixed optimization problems is presented in the papers of Mills [10] and Shapiro [18]. The reduction to quadratic programming problem is given in ([8]).

The method of fictitious play is discussed in the paper of Robinson [16]. In order to use the Neumann method we have to solve a system of nonlinear ordinary differential equations. The Runge–Kutta method is the most popular procedure for doing it. It can be found in [20].

The paper of Rosen [17] introduces diagonally strictly concave games. The computer method to find the equilibria of N-person concave games is introduced in Zuhovitsky et al. [22].

The different extensions and generalizations of the classical Cournot model can be found in the books of Okuguchi and Szidarovszky [14, 15]. The proof of Theorem 7.20 is given in [19]. For the proof of lemma (7.58) see the monograph [15]. The bisection method is described in [20]. The paper [6] contains methods which are applicable to solve nonlinear complementarity problems. The solution of problem (7.80) is discussed in the book of Hadley [4].

The book of von Neumann and Morgenstern [12] is considered the classical textbook of game theory. There is a large variety of game theory textbooks (see for example [3]).

Bibliography

[1] J-P. Aubin. *Mathematical Methods of Game and Economic Theory*. North-Holland, 1979. 346

[2] L. E. J. Brouwer. Über Abbildung von Manningfaltigkeiten. *Mathematische Annalen*, pp. 97–115. 346

[3] F. Forgó, J. Szép, F. Szidarovszky. *Introduction to the Theory of Games: Concepts, Methods and Applications*. Kluwer Academic Publishers, 1999. 346, 347

[4] G. Hadley. *Nonlinear and Dynamic Programming*. Addison-Wesley, 1964. 347

[5] S. Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8:457–459, 1941. 346

[6] S. Karamardian. The nonlinear complementarity problems with applications. I, II. *Journal of Optimization Theory and Applications*, 4:87–98 and 167–181, 1969. 347

[7] H. W. Kuhn, A. Tucker (szerkesztők). *Contributions to the Theory of Games. II*. Princeton University Press, 1953. 346

[8] O. Mangasarian, H. Stone. Two-person zero-sum games and quadratic programming. *Journal of Mathematical Analysis and its Applications*, 9:348–355, 1964. 346

[9] B. Martos. *Nonlinear Programming Theory and Methods*. Akadémiai Kiadó, 1975. 346

[10] H. Mills. Equilibrium points of finite games. *SIAM Journal of Applied Mathematics*, 8:397–402, 1976. 346

[11] J. Nash. Noncooperative games. *Annals of Mathematics*, 54:286–295, 1951. 346

[12] J. Neumann, O. Morgenstern. *Theory of Games and Economical Behaviour*. Princeton University Press, 1947 (2. edition). 347

[13] H. Nikaido, K. Isoda. Note on noncooperative games. *Pacific Journal of Mathematics*, 5:807–815, 1955. 346

[14] K. Okuguchi. *Expectation and Stability of Oligopoly Models*. Springer, 1976. 347

[15] K. Okuguchi, F. Szidarovszky. *The Theory of Oligopoly with Multi-Product Firms*. Springer, 1999 (2. kiadás). 347

[16] J. Robinson. An iterative method of solving a game. *Annals of Mathematics*, 154:296–301, 1951. 347

[17] J. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33:520–534, 1965. 347

[18] H. N. Shapiro. Note on a computation method in the theory of games. *Communications on Pure and Applied Mathematics*, 11:587–593, 1958. 346

[19] F. Szidarovszky, C. Chiarella. Dynamic oligopolies, stability and bifurcation. *Cubo Mathemática Educational*, 3(2):267–284, 2001. 347

[20] F. Szidarovszky, S. Yakowitz. *Principles and Procedures of Numerical Analysis*. Plenum Press, 1998. 346, 347

[21] A. Tarski. A lattice-theoretical fixpoint theorem and its application. *Pacific Journal of Mathematics*, 5:285–308, 1955. 346

[22] S. I. Zuhovitsky, R. A. Polyak, M. E. Primak. Concave n-person games (numerical methods). *Ékonomika i Matematicheskie Methody*, 7:888–900, 1971 (in Russian). 347

Name index

Subject Index

Contents