Sets of complex numbers generated from a polynomial functional equation

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To the memory of I. Környei and B. Kovács

Abstract

Non-linear mappings of complex numbers has gained widespread attention in recent years playing a very important role in the present-day mathematics and physics. The purpose of the present paper is to study those possible sets of periodic points generated by a special kind of quadratic iteration, which have symmetric features. To be more precise, these sets Γ must have the following properties:

$$\Gamma \subset \mathbb{C} \setminus \mathbb{R}, \quad \Gamma = -\Gamma, \quad \Gamma = \overline{\Gamma}, \quad \Gamma = R(\Gamma), \quad \gamma_i \neq \gamma_j, \quad \forall i \neq j, \quad \gamma_i, \gamma_j \in \Gamma$$

where $R(x) = Ax^2 + Ax + E, \quad A, E \in \mathbb{R}, \quad A \neq 0$.

Periodic points of quadratic maps was investigated from a general algebraic and number theoretic point of view for periodic points of order 3 by Morton ([2]). In this note a functional equation approach was chosen to investigate the elements of Γ, i.e. they can be generated by the roots of a polynomial functional equation. It appears to be very difficult to determine in algebraic way all the appropriate Γ's, therefore the author presents an algorithmic method to verify his conjecture in case of $\mathrm{card}(\Gamma) \leq 100$.

Let $\mathbb{K}$ be a field. Denote $\mathbb{K}[x]$ the ring of polynomials over $\mathbb{K}$. Let $\mathbb{Z}$ denote the ring of integers, $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively.

The original question was to determine all non-trivial $Q \in \mathbb{K}[x]$ and all $S \in \mathbb{K}[x]$ with $\deg(S) = 2$ for which

$$Q(S(x)) = c\, Q(x)Q(x+\gamma) \tag{1}$$

holds with a suitable $\gamma \in \mathbb{K}$ and $c \in \mathbb{K}$. Kátai and Daróczy proved (see [1]), that the general problem can be reduced to the case of $\gamma = 1$, $S(x) = Ax^2 + B$ which is equivalent to the following assertion: the polynomial $Q(x)=\prod_{j=1}^{N}(x-\beta_j)$ is a solution of (1) if and only if the relations

$$\begin{aligned}\mathcal{A}:=\{\beta_1,\beta_2,\dots,\beta_N\}&=\{1-\beta_1,1-\beta_2,\dots,1-\beta_N\}=\\ &=\ \{S(\beta_1),S(\beta_2),\dots,S(\beta_N)\}\end{aligned}\tag{2}$$

hold.

The above mentioned authors could determine the complete solution of (1) if $\mathbb{K}=\mathbb{R}$ and Q has a real root, i.e. it has been given all the primitive solution sets $\mathcal{A}$, which contain at least one real element. We shall say that the set $\mathcal{A}$ is primitive if there exists no proper subset $\mathcal{B}\subset\mathcal{A}$ for which (2) holds.

Further questions and problems arise naturally.

1. Let $\mathbb{K}=\mathbb{R}$. Give all the possible primitive sets Γ in the complex plane, i.e., if Q has no real root.
2. What are the possible solutions in case of $\mathbb{K}=\mathbb{C}$?
3. What can we assert in the most general case, when $\mathbb{K}$ is an arbitrary field?

In this paper we shall restrict our attention to the problem Nr 1. Substituting $\beta_j=\gamma_j+1/2$ in (2) and taking into account that

$$\Gamma\subset\mathbb{C}\setminus\mathbb{R}\tag{3}$$

we get the following relations:

$$\Gamma=\{\gamma_1,\gamma_2,\dots,\gamma_N\}=\{-\gamma_1,-\gamma_2,\dots,-\gamma_N\}=-\Gamma\tag{4}$$

$$\Gamma=\{S(\beta_1),S(\beta_2),\dots,S(\beta_N)\}=\{R(\gamma_1),R(\gamma_2),\dots,R(\gamma_N)\}=R(\Gamma)\tag{5}$$

$$\Gamma=\overline{\Gamma}\tag{6}$$

where $R(\gamma)=A\gamma^2+A\gamma+E,\quad A\neq 0$.

The aim of this note is to examine the possible primitive sets Γ according to (3) - (6). First, we make some simple remarks.

1. We may assume that the leading coefficient of the polynomial $Q(x)$ in (1) is one, i.e. Q is monic.
2. The case $E=0$ was examined in [1, p.307], therefore throughout this article we assume that $E\neq 0$.
3. It follows from (4) and (6) that the points of Γ have a symmetry to the real and the imaginary axis in the complex plain, so because of (3) Γ contains even number of points.
4. Let $\gamma\in\Gamma$ an arbitrary element. Suppose that γ and $\overline{\gamma}$ fall into the same orbit with order k. Then k is even and it follows from the equation $R(\overline{\gamma})=\overline{R(\gamma)}$ that this orbit has a symmetry, namely for the point γ it is $\gamma,R(\gamma),R^2(\gamma),\cdots,R^{k/2}(\gamma),\overline{\gamma},\overline{R(\gamma)},\overline{R^2(\gamma)},\cdots,\overline{R^k(\gamma)}=\gamma,\cdots$.

Lemma 1 *If Γ is a proper primitive set satisfying (3) - (6) then $|A\ | \leq 1$.*

Proof: Let K denote the following:

$$K := \max\{\ |Im\ \gamma_j|\ ,\gamma_j \in \Gamma\ \} = |Im\ \gamma^*|.$$

Then

$$|\ Im\ \gamma_i - Im\ \gamma_j\ |\ \leq\ 2K$$

for all γ_i, γ_j. On the other hand

$$2K \geq\ |\ Im\ R(\gamma^*)\ -\ Im\ R(-\gamma^*)\ |\ = |\ 2A\ Im\ (\gamma^*)\ |\ =\ 2K\ |A|.$$

It means that $|A| \leq 1$. $\square$

A primitive set Γ contains no points from the real axis but it may have points from the imaginary one. It happens when the primitive set has only two elements. What can we assert in this case about the quadratic iteration R?

Lemma 2 *If Γ is a primitive set with two elements satisfying (3) - (6) then $A = \pm 1$ and $\Gamma = \{\ bi, -bi\ \} = \{\ \sqrt{E}, -\sqrt{E}\ \},\ E < 0,\ b \neq 0$.*

Proof: It is obvious, that for this Γ (3), (4) and (6) are valid. Solving the equalities $R(\gamma) = \gamma,\ \ R(\gamma) = -\gamma$ our statement immediately follows. $\square$

Example A sample for a primitive set with 2 elements is the case $b = \pm 1$, i.e. $\Gamma = \{\ i,\ -i\}$.

Theorem *If $A = \pm 1$ and (3) - (6) holds then $\Gamma = \{\ \sqrt{E}, -\sqrt{E}\ \}, E < 0$.*

Proof: Let $A = 1$. Let $Q(x) = \prod_{j=1}^{N}(x - \gamma_j)$ be a solution of the equation

$$Q(R(x)) = c\ Q(x)Q(x+1),$$

where $R(x) = x^2 + x + E$ and the γ_j-s satisfy (3) - (6). For simplicity we replace E by $-F$. It is obvious, that

$$\{\gamma^2 + \gamma \mid \gamma \in \Gamma\} = \{\gamma + F \mid \gamma \in \Gamma\}.$$

Consequently

$$\sum_{\gamma\in\Gamma}(\gamma^2 + \gamma)^k = \sum_{\gamma\in\Gamma}(\gamma + F)^k \qquad (k = 0, 1, 2, \ldots). \tag{7}$$

Further on let σ_l denote the following power sum:

$$\sigma_l = \sum_{j=1}^{N} \gamma_j^l \quad (l = 0, 1, 2 \ldots). \tag{8}$$

It is easy to see that

$$\sigma_0 = N \tag{9}$$

and by (4)

$$\sigma_{2k+1} = 0 \quad (k = 0, 1, 2 \ldots). \tag{10}$$

It follows from (7) - (8) that

$$\sum_{j=0}^{k} \binom{k}{j} \sigma_{k+j} = \sum_{j=0}^{k} \binom{k}{j} F^{k-j} \ \sigma_j \quad (k = 0, 1, 2 \ldots). \tag{11}$$

Next we shall prove the following statement by induction:

$$\sigma_{2k} = F^k \ N \quad (k = 0, 1, 2, \ldots). \tag{12}$$

We have seen that for $k = 0$ the statement (12) is true. For $k = 1$ from (11) immediately follows that $\sigma_2 = F \ \sigma_0$. From (9) we get that $\sigma_2 = F \ N$.
Suppose that (12) is true for $t = 1, 2, \ldots, k-1$.
Case 1. k is even, $k = 2 \ K$.
Then

$$\sum_{2t=0}^{2K} \binom{2K}{2t} \sigma_{2K+2t} = \sum_{2t=0}^{2K} \binom{2K}{2t} F^{2K-2t} \ \sigma_{2t}.$$

Using the induction hypothesis we get

$$\sigma_{4K} + \sum_{t=0}^{K-1} \binom{2K}{2t} F^{K+t} N = F^{2K} N + \sum_{t=1}^{K} \binom{2K}{2t} F^{2K-2t} F^t N.$$

Observe, that

$$\binom{2K}{2t} F^{K+t} N = \binom{2K}{2t^*} F^{2K-t^*} N$$

when $t + t^* = K$. Consequently

$$\sigma_{2k} = \sigma_{4K} = F^{2K} N = F^k N$$

which was to be proved.
Case 2. k is odd, $k = 2K + 1$.
Then

$$\sum_{t=0}^{K} \binom{2K+1}{2t+1} \sigma_{2K+2t+2} = \sum_{t=0}^{K} \binom{2K+1}{2t} F^{2K-2t+1} \ \sigma_{2t}.$$

Using the induction hypothesis we get

$$\sigma_{4K+2} + \sum_{t=0}^{K} \binom{2K+1}{2t+1} F^{K+t+1} N = F^{2K+1} N + \sum_{t=0}^{K} \binom{2K+1}{2t+1} F^{2K-t+1} N.$$

By similar arguments as in Case 1. we have

$$\sigma_{2k} = \sigma_{4K+2} = F^{2K+1}N = F^k N.$$

This completes the proof of our statement (12).
Let φ denote the logarithmic derivative of the polynomial $Q(x)$, i.e.

$$\varphi(x) = \frac{Q'(x)}{Q(x)} = \sum_{\gamma\in\Gamma} \frac{1}{(x-\gamma)}.$$

It means that

$$\varphi(x) = \frac{1}{x} \sum_{\gamma\in\Gamma} \frac{1}{(1-\gamma/x)} = \frac{1}{x} \sum_{\gamma\in\Gamma}\sum_{n=0}^{\infty} \frac{\gamma^n}{x^n} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{x^n}\ \sigma_n. \tag{13}$$

By using (10) and (12) we obtain

$$\varphi(x) = \frac{1}{x} \sum_{m=0}^{\infty} \frac{F^m N}{x^{2m}} = \frac{N}{x} \sum_{m=0}^{\infty} \frac{F^m}{x^{2m}} = \frac{N}{x} \quad \frac{1}{(1-F/x^2)} = \frac{Nx}{(x^2-F)}.$$

Finally, taking the poles of the function $\varphi(x)$ one can see that the only primitive set Γ satisfying (3) - (6) is

$$\Gamma = \{\ \sqrt{F}, -\sqrt{F}\ \},\ F<0.$$

Thus the theorem is proved in the case $A = 1$.
The proof in the case $A = -1$ is carried out analogously. Then we have the equations

$$\sum_{j=0}^{k} (-1)^k \begin{pmatrix} k \\ j \end{pmatrix} \sigma_{k+j} = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} F^{k-j}\ \sigma_j \quad (k = 0, 1, 2 \ldots)$$

and

$$\sigma_{2k} = (-1)^k\ F^k\ N \quad (k = 0, 1, 2, \ldots)$$

whose proof we leave to the reader. In this case

$$\varphi(x) = \frac{N}{x} \sum_{m=0}^{\infty} \frac{(-1)^m\ F^m}{x^{2m}} = \frac{N}{x} \quad \frac{1}{(1+F/x^2)} = \frac{Nx}{(x^2+F)}\ ,$$

which means that the only appropriate primitive set satisfying (3) - (6) is

$$\Gamma = \{\ \sqrt{-F}, -\sqrt{-F}\ \},\ F>0.$$

The proof of the theorem is complete. □

Conjecture *The only primitive set satisfying (3) - (6) is*

$$\Gamma = \{\ \sqrt{E}, -\sqrt{E}\ \}, E < 0.$$

Lemma 2 states the truth of the conjecture in case of card(Γ) = 2. For the other cases it would be enough to show, that if the conditions (3) - (6) hold then $A = \pm 1$. It is clear, that if the conjecture is true, the solutions of the polynomial functional equation (1) can be exactly given, namely depending on the value of A they are:

$$Q(x) = \sum_{j=0}^{L} \left(\begin{array}{c} L \\ j \end{array} \right) E^{L-j}\ x^{2j} \quad or \quad \sum_{j=0}^{L} \left(\begin{array}{c} L \\ j \end{array} \right) (-E)^{L-j}\ x^{2j}$$

where $\deg(Q) = 2L$. ($L = 1, 2, 3 \ldots$). The roots of the polynomial $Q(x)$ satisfy the conditions (3) - (6) but they form obviously only primitive set when $L = 1$.

The general algebraic solution of the problem seems to be hard. However, using an idea coming from the theory of power sums expressed by elementary symmetric polynomials (Newton–Girard formulas) we shall construct an algorithm to verify the conjecture for lower degrees of the polynomial Q(x), namely when $\deg(Q) \leq 100$ (or equivalently card(Γ) ≤ 100).

Let

$$Q(z) = \prod_{j=1}^{N} (z - \gamma_j) = p_0 + p_1 z + \ldots + p_N z^N \tag{14}$$

be a solution of the equation $Q(R(z)) = c\ Q(z)Q(z+1)$, where $R(z) = Az^2 + Az + E$ and γ_j-s satisfy (3) - (6). For simplicity we shall use the following notations:

$$F := -E,\ \xi := \frac{F}{A},\ \eta := \frac{1}{F},\ \kappa := \xi\eta = \frac{1}{A}.$$

It is clear, that

$$A\{\gamma^2 + \gamma \mid \gamma \in \Gamma\} = \{\gamma + F \mid \gamma \in \Gamma\}.$$

Consequently, using the notations already adopted we have

$$\sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \sigma_{k+j} = \xi^k \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \eta^j\ \sigma_j \quad (k = 0, 1, 2 \ldots)\ ,$$

from which we get the recurrence relation

$$\sigma_{2k} = \kappa^k \sigma_k + \sum_{j=0}^{k-1} \left(\begin{array}{c} k \\ j \end{array} \right) (\xi^{k-j} \kappa^j \sigma_j - \sigma_{k+j}). \tag{15}$$

Hence, because of (4) and (8) we have

$$\sigma_{2k+1} = 0 \quad (k = 0, 1, \ldots), \quad \sigma_0 = N. \tag{16}$$

Obviously, $\sigma \in \mathbb{Z}[\xi, \kappa]$ for a fixed N. Let us observe, that the powers of κ in σ_{2k} are divisible by 2. Taking the derivative of the polynomial $Q(z)$ in (14) and multiplying by z we get

$$z\ Q'(z) = zp_1 + 2z^2 p_2 + \ldots + Np_N z^N. \tag{17}$$

On the other hand we have seen in (13) that

$$\frac{Q'(z)}{Q(z)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{\sigma_k}{z^k}.$$

Rearranging this equation we have

$$z\ Q'(z) = Q(z) \sum_{k=0}^{\infty} \frac{\sigma_k}{z^k}. \tag{18}$$

Let $1 \le k \le N$ and compare the coefficients of z^k in (17) and (18):

$$k\ p_k = \sigma_0 p_k + \sigma_1 p_{k+1} + \ldots + \sigma_{N-k} p_N$$

i.e.,

$$(k - N)\ p_k = \sigma_1 p_{k+1} + \ldots + \sigma_{N-k} p_N. \tag{19}$$

It means, that for a fixed N using the recurrence relations (15) - (16) we should first compute the polynomials σ_{2k} and then substituting $k = N-1, \ldots, 0$ to the (19) formula — keeping in mind that $p_N = 1$ — it allows a ready computation of the coefficients $p_{N-1}, \ldots, p_0$. Observe, that in (19) $p_{2k+1} = 0 \ \ (k = 0, 1, \ldots)$. In the right hand side of (18) the coefficients of the factors $1/z^k (k = 0, 1, \ldots)$ must be equal to 0, so

$$p_0 \sigma_k + p_1 \sigma_{k+1} + \ldots + p_N \sigma_{k+N} = 0 \ \ (k = 0, 1, 2, \ldots).$$

Clearly,

$$\sigma_{k+N} = -(p_0 \sigma_k + p_1 \sigma_{k+1} + \ldots + p_{N-1} \sigma_{N+k-1}) \ \ (k = 0, 1, 2, \ldots).$$

From this recurrence formula we can explicitly determine the power sums σ_N, σ_{N+1}, σ_{N+2}, ... Next, we shall introduce the polynomial $\omega \in \mathbb{Z}[\xi, \kappa]$ by

$$\omega_k = \begin{cases} \sigma_k & \text{if } k \le N \\ -(p_0 \sigma_{k-N} + p_1 \sigma_{k-N+1} + \ldots + p_{N-1} \sigma_{k-1}) & \text{otherwise} \end{cases} \tag{20}$$

It is plausible, that ω_{2k+1} must be zero $(k = 0, 1, 2, \ldots)$. Let us define the polynomial

$$\Psi_k(\xi, \kappa) \ := \sigma_k(\xi, \kappa) \ - \ \omega_k(\xi, \kappa) \ \ (k = 0, 1, \ldots). \tag{21}$$

It is clear, that for a fixed N only those pairs (ξ, κ) can satisfy the conditions (3) - (6), for which the polynomials

$$\Psi_k(\xi, \kappa) = 0 \quad (k = 0, 1, \ldots). \tag{22}$$

It follows from our construction, that if k is odd or $k \leq N$ then (22) is an identity, but else it is not. Therefore for a fixed N we should determine the common solutions of the equations (22) by $k = N+2, N+4, \ldots$. If the only solutions are $\kappa = \xi\eta = 1/A = \pm 1$ then the conditions of the theorem are satisfied, i.e. the conjecture is true for the case of $\deg(Q) = N$.

Algorithm verifying the conjecture for polynomials $Q(x)$, $n \leq \deg(Q) \leq m$.
Input n, m: positive integers divisible by 2
for i **from** n **to** m **by** 2 **do**
    $N := i$;
    compute the coefficients $p_{N-1}, p_{N-2}, \ldots p_0$ *using the (19) formula*;
    poly := **gcd**(Ψ_{i+2}, Ψ_{i+4});
    if (*solving the equation* $poly = 0$ *for* κ *is equal to* ± 1)
    then *the conjecture is true*
    else *counter-example found*
    fi
od

The algorithm clearly terminates. The polynomials $\sigma_k, \omega_k, \Psi_k$ can be determined in the outlined manner using the formulas (15),(16),(20) and (21) respectively. In the algorithm the function **gcd** finds the greatest common divisor of polynomials.

Implementing this algorithm the experiments with a simple Maple[1] program (MapleV Release 3) running on a Sun[2] workstation with the UltraSPARC-I[3] processor (clock frequency 143 Mhz, 64 MByte RAM) show, that the conjecture is true when $\deg(Q) \leq 100$. In order to solve the polynomial equation $poly = 0$ for κ we used the method of resultants. In the case of $N = 100$ the running time was 19,66 hours, total sum of 31290 Kbyte memory was used.

[1]Maple is a registered trademark of Waterloo Maple Software
[2]Sun is a registered trademark os Sun Microsystem Inc.
[3]SPARC is a registered trademark of SPARC International.

References

[1] I. Kátai, Z. Daróczy *On a functional equation with polynomials* Acta Math. Hung. 52. (3-4), 1988, 305-320.

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