On completely additive functions satisfying a congruence

Attila Kovács and Bui Minh Phong *

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Abstract

A fascinating connection was conjectured on sums of completely additive functions satisfying a congruence by Kátai. He could prove it for the case of $k = 3$. The general solution of the problem is far from beeing simple. The object of the present paper is to prove this conjecture for $k = 4$. Another purpose of the authors was to discuss the problem from group theoretical aspects and to describe an algorithm verifying the conjecture for finite number of functions under a certain bound.

1 Introduction

An arithmetical function $f(n)$ is said to be *completely additive* if the relation $f(nm) = f(n)+f(m)$ holds for every positive integer n and m. Let $\mathcal{A}^*$ denote the class of all real-valued completely additive functions. Throughout this paper we apply the usual notations, i.e. $\mathbb{P}$ denotes the set of primes, $\mathbb{N}$ the set of positive integers, $\mathbb{Q}$ and $\mathbb{R}$ the fields of rational and real numbers, respectively.

Let $P(z) = 1 + A_1 z + A_2 z^2 + \ldots + A_k z^k$ $(k \geq 1)$ be a polynomial with real coefficients. Let E denote the operator $Ez_n := z_{n+1}$ in the linear space of infinite sequences. For the polynomial $P(z)$ we have

$$P(E)f(n) = f(n) + A_1 f(n+1) + \ldots + A_k f(n+k).$$

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Conjecture I. (Kátai) *If $f \in \mathcal{A}^*$, $P(z) \notin \mathbb{Q}[z]$ and $P(E)f(n) \equiv 0 \pmod 1$ for all $n \in \mathbb{N}$ then $f(n)$ is identically zero.*

This conjecture was proved for the case of $k = 2$ (see [1]). Moreover, Kátai raised a more general question:

Conjecture II. (Kátai) *Let $f_j \in \mathcal{A}^*$ $(j = 0, 1, 2, \ldots, k)$. Assume that*

$$\sum_{j=0}^{k} f_j(n+j) \equiv 0 \pmod 1$$

for all $n \in \mathbb{N}$. Then $f_j(n) \equiv 0 \pmod 1$ for every $n \in \mathbb{N}$ and for every j.

In [2] Kátai proved it for the case of $k = 3$. The idea can be extended to the Gaussian integers, as was done in [5, 7]. It was examined the *analogy* of the conjecture for three *completely additive* [4], as well as for two [3] and three [6] *additive* functions.

2 Group theoretical approach

We extend the domain of an arbitrary real valued completely additive function $f(n)$ for the set of positive rational numbers $\mathbb{Q}_*$ by $f(a/b) := f(a) - f(b)$. Let us do it for $f_0, f_1, f_2, \ldots, f_k$ $(k \in \mathbb{N})$ and let us define the set

$$\Omega_k := \{\ (a_0, a_1, a_2, \ldots, a_k) \in \mathbb{Q}_*^{k+1} \quad : \quad \sum_{i=0}^{k} f_i(a_i) \equiv 0 \pmod 1\ \}.$$

Performing the multiplications component-wise it is easy to see that $(\mathbb{Q}_*^{k+1}; \cdot)$ is an abelian group. Hence, it follows from the additivity that if $(a_0, a_1, \ldots, a_k) \in \Omega_k$ and $(b_0, b_1, \ldots, b_k) \in \Omega_k$ then $(a_0 {b_0}^{-1}, a_1 {b_1}^{-1}, \ldots, a_k {b_k}^{-1}) \in \Omega_k$. Thus, the following assertion is obvious.

Assertion. *$(\Omega_k; \cdot)$ is an abelian group with respect to the above defined multiplication.*

In fact, we think that the next conjecture is true:

Conjecture III *If $(A, \cdot)$ is a subgroup of the group $(\mathbb{Q}_*^h; \cdot)$, where $h \in \mathbb{N}$ and $(n, n+1, \ldots, n+h-1) \in A$ for all $n \in \mathbb{N}$, then $A = \mathbb{Q}_*^h$.*

It is clear that the first conjecture follows from the second one and the second from the third one.

3 The main theorem

Theorem. *Let* $P(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4$ *be a polynomial with real coefficients,* $P(z) \notin \mathbb{Q}[z]$*. If* $f \in \mathcal{A}^*$ *satisfies the congruence relation*

$$L_n := \quad P(E)f(n) \equiv 0 \pmod 1 \tag{1}$$

for every $n \in \mathbb{N}$*, then* $f(n)$ *is identically zero.*

In order to prove the theorem we shall use an induction-like method.

3.1 Induction step

Lemma 1. *If* (1) *holds for every* $n \in \mathbb{N}$ *and* $f(m) = 0$ *is satisfied for every positive integer* $m \leq 23$ *then* $f(n) = 0$ *for every* $n \in \mathbb{N}$*.*

Proof: We prove the lemma indirectly. Let

$$\mathcal{K}_f := \{n \in \mathbb{N} \ : \ f(n) = 0\}.$$

Assume that $\mathcal{K}_f \neq \mathbb{N}$, i.e. there exist a smallest positive integer S such that

$$f(S) \neq 0. \tag{2}$$

By the assumption of the lemma it is pretty obvious that

$$S \in \mathbb{P} \tag{3}$$

and $S \geq 29$. First we shall prove that for each V satisfying the relations $V \in \{S+2, S+6, S+8, S+12\}$ and $V \equiv 1 \pmod 6$ we have

$$f(V) = 0 \text{ or } f(V) \notin \mathbb{Q}. \tag{4}$$

Suppose that (4) does not hold. Then for some U for which $U \in \{S+2, S+6, S+8, S+12\}$ with $U \equiv 1 \pmod 6$, we obtain

$$f(U) \neq 0 \text{ and } f(U) \in \mathbb{Q}. \tag{5}$$

By the fact $U \equiv 1 \pmod 6$ it is easily seen that

$$\{U \pm 1 - 6k, U + 2 - 6k, U + 3 - 6k, 2U \pm 2 \ : \ k = 0, 1\} \subseteq \mathcal{K}_f. \tag{6}$$

By using (1) and (6) we get that $L_{U-1} = A_1 f(U) \equiv 0 \pmod 1$, which with (5) implies that

$$A_1 \in \mathbb{Q}. \tag{7}$$

Let us observe that

$$f(U-2) \notin \mathbb{Q}, \tag{8}$$

since in the opposite case, by using (1) and (6) we get

$$\begin{aligned} L_{U-4} &= A_2 f(U-2) + A_4 f(U) \equiv 0 \pmod 1, \\ L_{U-3} &= A_1 f(U-2) + A_3 f(U) \equiv 0 \pmod 1, \\ L_{U-2} &= \quad f(U-2) + A_2 f(U) \equiv 0 \pmod 1, \end{aligned}$$

which with (5) and (7) would imply that $P(z) \in \mathbb{Q}[z]$. In virtue of (5), (8) and of the equation L_{U-2} it is obvious that

$$A_2 \notin \mathbb{Q}. \tag{9}$$

Since $3 \mid (2U + 1)$ and $(2U + 1)/3 < S$, we have $f(2U + 1) = 0$, hence by using (5), (6), (7), (9) and the equation

$$L_{2U-2} = A_1 f(2U-1) + A_2 f(U) \equiv 0 \pmod 1$$

we obtain

$$A_1 \neq 0, \quad f(2U-1) \notin \mathbb{Q}. \tag{10}$$

Consider the equation

$$L_{2U-1} = f(2U-1) + A_1 f(U) + A_4 f(2U+3) \equiv 0 \pmod 1,$$

which with (5), (7) and (10) implies that

$$A_4 \neq 0, \quad A_4 f(2U+3) \notin \mathbb{Q}. \tag{11}$$

By using relations (5), (8), (10) and (11) it is easy to see that U, $U-2$, $2U-1$, $2U+3 \in \mathbb{P}$, consequently $U \equiv 4 \pmod 5$. Since $U \equiv 1 \pmod 6$, $U \equiv 4 \pmod 5$, $U \leq S + 12$ and $S \geq 29$ the following statements hold:

$$3 \mid 2U - 5, \quad (2U-5)/3 < S, \quad 5 \mid 2U - 3, \quad (2U-3)/5 < S.$$

It means that $f(2U-5) = f(2U-3) = 0$. Hence, using (5), (6) and the equations

$$L_{2U-6} = A_2 f(U-2) \equiv 0 \pmod 1,$$

$$L_{U-4} = A_2 f(U-2) + A_4 f(U) \equiv 0 \pmod 1$$

we have

$$A_4 \in \mathbb{Q}. \tag{12}$$

On the other hand, using (5), (6), (7) and (10) the equation $L_{U-7} = A_1 f(U-6) \equiv 0 \pmod 1$ gives that $f(U-6) \in \mathbb{Q}$, which with (11), (12) and by $L_{U-6} = f(U-6) + A_4 f(U-2) \equiv 0 \pmod 1$ implies $f(U-2) \in \mathbb{Q}$. This contradicts to (8). Thus, we have proved (4).

Now we are able to prove Lemma 1. We distinguish two cases:

(I) $S \equiv 1 \pmod 6$
(II) $S \equiv 5 \pmod 6$.

Case (I). In this case we have

$$\{S+1+6k, S+2+6k, S+3+6l, S+5+6l \ : \ k = 0,1,2,3 \ \ l = 0,1,2\} \subseteq \mathcal{K}_f, \tag{13}$$

and by virtue of the equations $L_{S-1}, L_{S-2}, L_{S-3}, L_{S-4}$ we get

$$A_1 f(S) \equiv A_2 f(S) \equiv A_3 f(S) \equiv A_4 f(S) \equiv 0 \pmod 1.$$

If $f(S) \in \mathbb{Q}$ then $A_j \in \mathbb{Q}$ $(j = 1,2,3,4)$ but this is a contradiction, since $P(z) \notin \mathbb{Q}[z]$. Then we have

$$f(S) \notin \mathbb{Q}. \tag{14}$$

Observe that if $A_j \in \mathbb{Q}$ for some $1 \le j \le 4$ then $A_j = 0$. It means that

$$A_j \notin \mathbb{Q} \ \text{ or } \ A_j = 0 \quad (j = 1,2,3,4). \tag{15}$$

Let $i = 0$. It follows from (14), (15) and from the equation $L_{S+6i} = f(S+6i) + A_4 f(S+4+6i) \equiv 0 \pmod 1$ that

$$A_4 \notin \mathbb{Q}, \quad f(S+4+6i) \neq 0. \tag{16}$$

The reader can readily verify that $f(S+6+6i) \neq 0$ since in the opposite case using the equation $L_{S+4+6i} = f(S+4+6i) \equiv 0 \pmod 1$ we get

$f(S+4+6i)\in\mathbb{Q}$. This with (13), (15), (16) and by $L_{S+1+6i}, L_{S+2+6i}, L_{S+3+6i}$ implies $A_1 = A_2 = A_3 = 0$. Hence the equation $L_{2S+4+12i}$ would infer $A_4 f(S+4+6i)\equiv 0 \pmod 1$, i.e., $A_4\in\mathbb{Q}$ which is a contradiction. So we have $f(S+6+6i)\neq 0$. By similar arguments for $i=1,2$ and by using (4) we obtain $f(S+4)\neq 0$ and $f(S+6i)\neq 0 \quad (i=0,1,2,3)$. It means that $S, S+4, S+6, S+12, S+18$ are primes which is impossible since $S\geq 29$ and one of these numbers is a multiple of 5. So we proved that Case (I) cannot occur.

Case (II). In this case it is no hard to see that

$$\{S+1+6k, S+3+6k, S+4+6k, S+5+6l \ : \ k=0,1,2 \ \ l=0,1\}\subseteq\mathcal{K}_f.$$

Let $i=0$. Because of $P(z)\notin\mathbb{Q}[z]$ it is easily seen that $f(S+2+6i)\neq 0$ and by (4) we have

$$f(S+2+6i)\notin\mathbb{Q}. \tag{17}$$

Hence by using the equation

$$L_{S+2+6i} = f(S+2+6i) + A_4 f(S+6+6i)\equiv 0 \pmod 1$$

it follows $f(S+6+6i)\neq 0$. Suppose that $f(S+8+6i)=0$. Then from L_{S+6+6i} we get $f(S+6+6i)\in\mathbb{Q}$, therefore by L_{S+3+6i}, L_{S+4+6i}, L_{S+5+6i} we infer $A_1, A_2, A_3\in\mathbb{Q}$. Moreover, by the equations L_{S-1+6i}, L_{S+1+6i} and by (17) immediately follows that $A_1 = A_3 = 0$. Since $2S+8+12i\in\mathcal{K}_f$, $2S+10+12i\in\mathcal{K}_f$, the equation

$$L_{2S+8+12i} = A_4 f(2S+12+12i)\equiv A_4 f(S+6+6i)\equiv 0 \pmod 1$$

gives $A_4\in\mathbb{Q}$ which is contradiction. So $f(S+8+6i)\neq 0$. Similarly, for $i=1$ using (4) we obtain $S, S+2, S+6, S+8, S+14$ are primes, but this is impossible since $S\geq 29$ and one of these numbers is a multiple of 5. The proof of Lemma 1 is finished. □

The proof of the theorem will be completed by proving the following

Lemma 2. *If (1) holds for every $n\in\mathbb{N}$ then $f(m)=0$ for every positive integer $m\leq 23$.*

To verify this lemma we discuss the next conjecture which is a step by step approach of the second one.

3.2 Algorithm for the bounded case

Conjecture IV *Let* $f_0, f_1, \ldots, f_k \in \mathcal{A}^*$. *Given any* $e \in \mathbb{N}$ *there exist a* $d \in \mathbb{N}$ *such that if* $L_n := \sum_{j=0}^{k} f_j(n+j) \equiv 0 \pmod 1$ *holds for every* $n \le d$ $(n \in \mathbb{N})$ *then* $f_j(n) \equiv 0 \pmod 1$ *for every* $n \le e$ $(j = 0, 1, \ldots, k)$.

We introduce some notations.

Let us denote the n-tuple $\mathcal{L}_n := (n, n+1, \ldots, n+k) \in \Omega_k$. Let furthermore

$$\Gamma_k := \{(a_0, a_1, \ldots, a_k) \ : \ a_j \text{ are squarefree integers}, a_j > 1 \ (0 \le j \le k)\}.$$

For an arbitrary $\gamma = (a_0, a_1, \ldots, a_k) \in \Gamma_k$ we define the vector $\underline{v} = \underline{v}(\gamma)$ as follows. Let the prime decomposition of a_j be $a_j = p_1^{(j)} p_2^{(j)} \ldots p_{l_j}^{(j)}$, where the factors are written in ascending order. Then

$$\underline{v}(\gamma) := [f_0(p_1^{(0)}), \ldots, f_0(p_{l_0}^{(0)}), f_1(p_1^{(1)}), \ldots, f_1(p_{l_1}^{(1)}), \ldots, f_k(p_1^{(k)}), \ldots, f_k(p_{l_k}^{(k)})] =$$

$$= [v_1, v_2, \ldots, v_\mu], \quad \mu = l_0 + l_1 + \ldots + l_k.$$

Example Let $k = 4, \gamma = (6, 30, 10, 15, 3)$.
Then $\underline{v} = [f_0(2), f_0(3), f_1(2), f_1(3), f_1(5), f_2(2), f_2(5), f_3(3), f_3(5), f_4(3)]$.

Let $\Delta_{\underline{v}}^{\mu} := \{[b_1, b_2, \ldots, b_\mu] \ : \ \sum_{j=1}^{\mu} b_j v_j \equiv 0 \pmod 1, \ b_i \in \mathbb{Z}\}$. Let $\underline{b} = [b_1, \ldots, b_\mu] \in \Delta_{\underline{v}}^{\mu}$. Then

$$b_1 f_0(p_1^{(0)}) + \ldots + b_{l_0}(p_{l_0}^{(0)}) + \ldots + b_\mu f_k(p_{l_k}^{(k)}) \equiv 0 \pmod 1,$$

which can be rewritten as

$$f_0(\beta_0) + f_1(\beta_1) + \ldots + f_k(\beta_k) \equiv 0 \pmod 1,$$

where $\beta_t = \prod_{k=1}^{l_t} (p_k^{(t)})^{b_{s+k}}$, $s = l_0 + \ldots + l_{t-1}$. Let $\underline{\beta}(= \underline{\beta}(\underline{b})) = (\beta_0, \beta_1, \ldots, \beta_k)$. Clearly, $\underline{\beta} \in \mathbb{Q}_*^k$. Let $\Delta_{\underline{v}}^k = \{\underline{\beta}(\underline{b}) \ : \ \underline{b} \in \Delta_{\underline{v}}^{\mu}\}$.

Remarks
1. It follows from our construction that there is a one-to-one correspondence between $\Delta_{\underline{v}}^{\mu}$ and $\Delta_{\underline{v}}^k$.
2. $\Delta_{\underline{v}}^k \subset \Omega_k$.

Example Let $k, \gamma, \underline{v}$ as before. Suppose that $\underline{b}$ = (1,2,3,1,-2,-3,1,-2,3,1) $\in \Delta_{\underline{v}}^{\mu}$. Then $\underline{\beta} = (18, \frac{24}{25}, \frac{5}{8}, \frac{125}{9}, 3) \in \Delta_{\underline{v}}^k$.

Algorithm *to verify Conjecture IV.*
Input k, $e \in \mathbb{N}$

1. Let γ be an arbitrary element from Γ_k and let $\underline{v}$ be the vector generated by γ as before.
2. By using the equations $L_1, L_2, \ldots, L_i$ and the fact that $f_0, f_1, \ldots, f_k \in \mathcal{A}^*$ express all the possible $F_1, F_2, \ldots, F_j \in \Delta_{\underline{v}}^k$. The corresponding vectors $G_1, G_2, \ldots, G_j \in \Delta_{\underline{v}}^\mu$ can be considered as rows of a matrix $M \in \mathbb{R}^{\mu \times \mu}$. Examine so many equations that the rank of the matrix M will be equal to μ.
3. Using Gaussian elimination over the integers it can be solved the linear equation $M\underline{v} \equiv 0 \pmod 1$. If the only solution is $\underline{v} \equiv 0 \pmod 1$ then go to the next step, otherwise go to the step 2, increase i and find a new matrix M or go to the step 1 and choose a new γ.
4. Investigate so many equations $L_1, L_2, \ldots, L_i, \ldots$ while all $f_j(p)$ can be expressed in the form $f_j(p) \equiv \sum_{l=1}^{\mu} b_l v_l \pmod 1$ $(p \le e, p \in \mathbb{P}, b_l \in \mathbb{Z}, 0 \le j \le k)$. Then the conjecture is true for k, e and d is equal to i, which is the number of the examined equations.

If the conjecture is true, the algorithm terminates.

Remark For a given k, e the d is not unique, it depends on the selection of γ.

4 Examples

Implementing this algorithm the experiments with a simple Maple[1] program show the following results:

Example Let $k = 4, \gamma = (210, 210, 210, 210, 2310)$.
Then we have

$$F_1 = \frac{\mathcal{L}_9^5 \mathcal{L}_{11}^2 L_{12}^4 \mathcal{L}_{14}^5 \mathcal{L}_{17} \mathcal{L}_{21} \mathcal{L}_{23}^2 \mathcal{L}_{30}^2 \mathcal{L}_{32}^2 \mathcal{L}_{33} \mathcal{L}_{37} \mathcal{L}_{45}^2 \mathcal{L}_{54}^3 \mathcal{L}_{58} \mathcal{L}_{62}^4 \mathcal{L}_{91} \mathcal{L}_{141} \mathcal{L}_{143}}{\mathcal{L}_8 \mathcal{L}_{10}^6 \mathcal{L}_{13} \mathcal{L}_{15} \mathcal{L}_{16} \mathcal{L}_{18} \mathcal{L}_{22}^3 \mathcal{L}_{24}^4 \mathcal{L}_{25}^4 \mathcal{L}_{27} \mathcal{L}_{29} \mathcal{L}_{31}^4 \mathcal{L}_{44} \mathcal{L}_{47} \mathcal{L}_{48} \mathcal{L}_{70} \mathcal{L}_{74} \mathcal{L}_{92}^2 \mathcal{L}_{117} \mathcal{L}_{119}} =$$

$$= \left(\frac{3^{19} 7^5}{2^8 5^{12}}, \frac{3^{13}}{2^{12}}, \frac{2^{33} 5^2 7^5}{3^{27}}, \frac{2^{10} 3^3}{7^2}, \frac{3^{20} 5}{2^5 7^7}\right),$$

$$F_2 = \frac{\mathcal{L}_8^3 \mathcal{L}_{10}^3 \mathcal{L}_{11}^3 \mathcal{L}_{15} \mathcal{L}_{16}^2 \mathcal{L}_{18}^2 \mathcal{L}_{23} \mathcal{L}_{25}^2 \mathcal{L}_{27} \mathcal{L}_{29} \mathcal{L}_{30} \mathcal{L}_{31}^4 \mathcal{L}_{38} \mathcal{L}_{44} \mathcal{L}_{45} \mathcal{L}_{47} \mathcal{L}_{48}^3 \mathcal{L}_{57} \mathcal{L}_{59} \mathcal{L}_{65}^2 \mathcal{L}_{70}}{\mathcal{L}_9^3 \mathcal{L}_{12}^2 \mathcal{L}_{14}^8 \mathcal{L}_{19}^2 \mathcal{L}_{21} \mathcal{L}_{24}^2 \mathcal{L}_{28} \mathcal{L}_{32} \mathcal{L}_{33} \mathcal{L}_{54} \mathcal{L}_{62}^4 \mathcal{L}_{91} \mathcal{L}_{92} \mathcal{L}_{116} \mathcal{L}_{118} \mathcal{L}_{120} \mathcal{L}_{132}^2 \mathcal{L}_{141} \mathcal{L}_{143}} =$$

[1] Maple is a registered trademark of Waterloo Maple Software

$$= (\frac{5^{12}}{2^3 3^4 7^{10}}, \frac{2^{32}}{3^4 5^{15}}, \frac{3^{14} 5^{10}}{2^{43} 7^4}, \frac{2^{12} 7^5}{3^{13} 5^6}, \frac{5^4 7^7}{2^{21} 3^8 11}),$$

$$F_3 = \frac{\mathcal{L}_9^2 \mathcal{L}_{13}^2 \mathcal{L}_{14}^4 \mathcal{L}_{19} \mathcal{L}_{21} \mathcal{L}_{22} \mathcal{L}_{23}^2 \mathcal{L}_{24}^2 \mathcal{L}_{32}^3 \mathcal{L}_{34} \mathcal{L}_{37} \mathcal{L}_{55} \mathcal{L}_{56} \mathcal{L}_{58} \mathcal{L}_{62} \mathcal{L}_{91} \mathcal{L}_{92} \mathcal{L}_{132}}{\mathcal{L}_{10}^3 \mathcal{L}_{11}^3 \mathcal{L}_{12} \mathcal{L}_{15}^2 \mathcal{L}_{18}^3 \mathcal{L}_{26}^2 \mathcal{L}_{27} \mathcal{L}_{29} \mathcal{L}_{30} \mathcal{L}_{31} \mathcal{L}_{44} \mathcal{L}_{45} \mathcal{L}_{46} \mathcal{L}_{47} \mathcal{L}_{48}^2 \mathcal{L}_{54} \mathcal{L}_{65}}$$

$$\frac{\mathcal{L}_{140} \mathcal{L}_{141} \mathcal{L}_{142} \mathcal{L}_{143} \mathcal{L}_{144}}{\mathcal{L}_{69} \mathcal{L}_{70} \mathcal{L}_{71} \mathcal{L}_{74} \mathcal{L}_{114} \mathcal{L}_{115} \mathcal{L}_{117} \mathcal{L}_{119}} = (\frac{2^{17} 7^6}{3^{13} 5^8}, \frac{5^5}{2^{14} 3^2}, \frac{2^4 3^3 5}{7^6}, \frac{5^5}{3^4 7^4}, \frac{2^9 3^{14}}{5^5 7^3 11^5}),$$

$$F_4 = \frac{\mathcal{L}_9^2 \mathcal{L}_{11}^2 \mathcal{L}_{12}^6 \mathcal{L}_{14}^3 \mathcal{L}_{18}^2 \mathcal{L}_{23}^2 \mathcal{L}_{30}^3 \mathcal{L}_{33} \mathcal{L}_{45}^3 \mathcal{L}_{54}^5 \mathcal{L}_{62}^4 \mathcal{L}_{65} \mathcal{L}_{115} \mathcal{L}_{141} \mathcal{L}_{143} \mathcal{L}_{152}}{\mathcal{L}_8^2 \mathcal{L}_{10}^6 \mathcal{L}_{13}^2 \mathcal{L}_{16}^2 \mathcal{L}_{19} \mathcal{L}_{22}^3 \mathcal{L}_{24}^4 \mathcal{L}_{25}^6 \mathcal{L}_{28} \mathcal{L}_{31}^4 \mathcal{L}_{47} \mathcal{L}_{56} \mathcal{L}_{70} \mathcal{L}_{92}^3 \mathcal{L}_{132}} =$$

$$= (\frac{3^{35}}{2^{17} 5^{11}}, \frac{3^{11}\ 5\ 7}{2^{10}}, \frac{2^{39} 5^3 7^{10}}{3^{39}}, \frac{2^4 3^{10}}{5\ 7^2}, \frac{3^{14} 11^6}{2^8 5^4 7^5}),$$

$$F_5 = \frac{\mathcal{L}_9 \mathcal{L}_{14} \mathcal{L}_{18} \mathcal{L}_{21} \mathcal{L}_{30} \mathcal{L}_{33} \mathcal{L}_{45} \mathcal{L}_{48} \mathcal{L}_{54} \mathcal{L}_{62}^2 \mathcal{L}_{91} \mathcal{L}_{115} \mathcal{L}_{121} \mathcal{L}_{141} \mathcal{L}_{143} \mathcal{L}_{168}}{\mathcal{L}_8 \mathcal{L}_{10}^2 \mathcal{L}_{11} \mathcal{L}_{13} \mathcal{L}_{15} \mathcal{L}_{16} \mathcal{L}_{22}^2 \mathcal{L}_{24} \mathcal{L}_{25}^2 \mathcal{L}_{28} \mathcal{L}_{31}^2 \mathcal{L}_{39} \mathcal{L}_{44} \mathcal{L}_{47} \mathcal{L}_{56} \mathcal{L}_{60} \mathcal{L}_{70} \mathcal{L}_{92}} =$$

$$= (\frac{3^{11} 7}{2^{13} 5^6}, \frac{7^3}{2^{12}\ 3\ 5}, \frac{2^7 5^3}{3^{14}}, \frac{2^6 3^5}{5^5 7^4}, \frac{5^2 11^3}{2^{23} 3}),$$

$$F_6 = \frac{\mathcal{L}_{11}^4 \mathcal{L}_{12}^2 \mathcal{L}_{18}^2 \mathcal{L}_{23}^2 \mathcal{L}_{26} \mathcal{L}_{27} \mathcal{L}_{30}^2 \mathcal{L}_{31}^2 \mathcal{L}_{42} \mathcal{L}_{45}^2 \mathcal{L}_{48}^8 \mathcal{L}_{54}^2 \mathcal{L}_{74} \mathcal{L}_{114} \mathcal{L}_{117} \mathcal{L}_{119} \mathcal{L}_{174}}{\mathcal{L}_8 \mathcal{L}_9^6 \mathcal{L}_{10} \mathcal{L}_{13}^2 \mathcal{L}_{14}^{10} \mathcal{L}_{19} \mathcal{L}_{22}^3 \mathcal{L}_{24}^3 \mathcal{L}_{25}^2 \mathcal{L}_{32} \mathcal{L}_{37} \mathcal{L}_{55} \mathcal{L}_{56} \mathcal{L}_{58} \mathcal{L}_{62}^2 \mathcal{L}_{85} \mathcal{L}_{92}^2} =$$

$$= (\frac{2^6 3^{19}}{5^3 7^9}, \frac{2^{15} 7^{13}}{3^9 5^{15}}, \frac{5^{18} 7^5}{2^{31} 3^{15}}, \frac{3^{10} 7^4}{2^{11} 5^7}, \frac{5^6 7^2 11^2}{2^4 3^{15}}),$$

$$F_7 = \frac{\mathcal{L}_{11}^8 \mathcal{L}_{12} \mathcal{L}_{15} \mathcal{L}_{16} \mathcal{L}_{18}^2 \mathcal{L}_{23}^3 \mathcal{L}_{26} \mathcal{L}_{29} \mathcal{L}_{30}^3 \mathcal{L}_{31}^4 \mathcal{L}_{45}^3 \mathcal{L}_{47} \mathcal{L}_{48}^8 \mathcal{L}_{54}^2 \mathcal{L}_{65} \mathcal{L}_{70} \mathcal{L}_{74}}{\mathcal{L}_8^3 \mathcal{L}_9^5 \mathcal{L}_{13}^3 \mathcal{L}_{14}^{13} \mathcal{L}_{19} \mathcal{L}_{21} \mathcal{L}_{22}^4 \mathcal{L}_{24}^6 \mathcal{L}_{25} \mathcal{L}_{32}^2 \mathcal{L}_{33} \mathcal{L}_{34} \mathcal{L}_{37} \mathcal{L}_{55} \mathcal{L}_{58} \mathcal{L}_{60} \mathcal{L}_{62}^4 \mathcal{L}_{92}^3}$$

$$\frac{\mathcal{L}_{114} \mathcal{L}_{117} \mathcal{L}_{119} \mathcal{L}_{182}}{\mathcal{L}_{132} \mathcal{L}_{141} \mathcal{L}_{143}} = (\frac{3^{11} 5^5}{2^{20} 7^{11}}, \frac{2^{40} 7^6}{3^{14} 5^{25}}, \frac{5^{15} 7^5}{2^{61} 3^8}, \frac{2\ 7^7}{3^9 5^{14}}, \frac{5^{11} 7^2 11}{2^{35} 3^{17}}),$$

$$F_8 = \frac{\mathcal{L}_9^2 \mathcal{L}_{12}^2 \mathcal{L}_{14}^3 \mathcal{L}_{18} \mathcal{L}_{33} \mathcal{L}_{54} \mathcal{L}_{62}^3 \mathcal{L}_{92} \mathcal{L}_{115} \mathcal{L}_{141} \mathcal{L}_{143} \mathcal{L}_{183}}{\mathcal{L}_{10} \mathcal{L}_{11}^2 \mathcal{L}_{13} \mathcal{L}_{16} \mathcal{L}_{23}^2 \mathcal{L}_{25}^2 \mathcal{L}_{28} \mathcal{L}_{31}^3 \mathcal{L}_{35} \mathcal{L}_{45} \mathcal{L}_{47} \mathcal{L}_{56} \mathcal{L}_{61} \mathcal{L}_{70}} =$$

$$= (\frac{2^3 3^{12}}{5^7 7}, \frac{3^4 5^6 7^2}{2^{23}}, \frac{2^{30} 7}{3^{16} 5^2}, \frac{3^{10} 5^4}{2^{14} 7^3}, \frac{2^{10} 11^5}{5^7 7^3}),$$

$$F_9 = \frac{\mathcal{L}_8 \mathcal{L}_{10}^5 \mathcal{L}_{13}^3 \mathcal{L}_{14}^5 \mathcal{L}_{15} \mathcal{L}_{19} \mathcal{L}_{22}^5 \mathcal{L}_{24}^9 \mathcal{L}_{25}^3 \mathcal{L}_{28} \mathcal{L}_{44} \mathcal{L}_{55} \mathcal{L}_{56} \mathcal{L}_{92}^3 \mathcal{L}_{132} \mathcal{L}_{184}}{\mathcal{L}_{11}^8 \mathcal{L}_{12}^2 \mathcal{L}_{18}^2 \mathcal{L}_{23}^3 \mathcal{L}_{26} \mathcal{L}_{30}^3 \mathcal{L}_{32} \mathcal{L}_{36} \mathcal{L}_{45}^3 \mathcal{L}_{48}^5 \mathcal{L}_{54}^3 \mathcal{L}_{65} \mathcal{L}_{90} \mathcal{L}_{91} \mathcal{L}_{114} \mathcal{L}_{115}} =$$

$$= (\frac{2^{21} 5^4 7^6}{3^{23}}, \frac{5^{22}}{2^{18} 3^3 7^6}, \frac{2^{20} 3^{23}}{5^{13} 7^5}, \frac{3^{10} 5^{11}}{2^6 7^8}, \frac{2^{32} 7^7}{3^5 5^{10} 11^2}),$$

$$F_{10} = \frac{\mathcal{L}_8 \mathcal{L}_{12}^2 \mathcal{L}_{13} \mathcal{L}_{14} \mathcal{L}_{19} \mathcal{L}_{21} \mathcal{L}_{22} \mathcal{L}_{24}^3 \mathcal{L}_{33} \mathcal{L}_{48} \mathcal{L}_{54} \mathcal{L}_{55} \mathcal{L}_{62} \mathcal{L}_{132} \mathcal{L}_{141} \mathcal{L}_{143} \mathcal{L}_{185}}{\mathcal{L}_9 \mathcal{L}_{11}^4 \mathcal{L}_{16} \mathcal{L}_{18} \mathcal{L}_{25}^2 \mathcal{L}_{26} \mathcal{L}_{30} \mathcal{L}_{31} \mathcal{L}_{32} \mathcal{L}_{44} \mathcal{L}_{47} \mathcal{L}_{65} \mathcal{L}_{70} \mathcal{L}_{74} \mathcal{L}_{114}} =$$

$$= (\frac{2^9 3^7 7}{5^5}, \frac{5^4 7^6}{2^7 3^5}, \frac{2^5 5^5 7^2}{3^7}, \frac{2^4 3^{16} 5^3}{7^9}, \frac{2^{11} 7^5}{3^3 5^4}),$$

$$F_{11} = \frac{\mathcal{L}_9^4 \mathcal{L}_{12} \mathcal{L}_{14}^9 \mathcal{L}_{22} \mathcal{L}_{24} \mathcal{L}_{32} \mathcal{L}_{54} \mathcal{L}_{62}^3 \mathcal{L}_{186}}{\mathcal{L}_{10}^3 \mathcal{L}_{11} \mathcal{L}_{15} \mathcal{L}_{16} \mathcal{L}_{25} \mathcal{L}_{31}^4 \mathcal{L}_{45} \mathcal{L}_{48}^5} = (\frac{3^6 7^9}{2^2 5^7}, \frac{3^{15} 5^{16}}{2^{24} 7^7}, \frac{2^{53} 7^2}{3^{11} 5^{10}}, \frac{3^4 5^7}{2^4}, \frac{2^8 3^{22} 11^3}{5^5 7^8}),$$

$$F_{12} = \frac{\mathcal{L}_8^2 \mathcal{L}_9^5 \mathcal{L}_{13}^2 \mathcal{L}_{14}^{14} \mathcal{L}_{19}^2 \mathcal{L}_{21}^2 \mathcal{L}_{22} \mathcal{L}_{24} \mathcal{L}_{25} \mathcal{L}_{32}^3 \mathcal{L}_{33} \mathcal{L}_{37} \mathcal{L}_{55} \mathcal{L}_{56} \mathcal{L}_{58} \mathcal{L}_{61} \mathcal{L}_{62}^4 \mathcal{L}_{91}^2 \mathcal{L}_{92}^2}{\mathcal{L}_{10} \mathcal{L}_{11}^2 \mathcal{L}_{15}^2 \mathcal{L}_{18}^4 \mathcal{L}_{23} \mathcal{L}_{26}^2 \mathcal{L}_{27} \mathcal{L}_{29} \mathcal{L}_{30}^3 \mathcal{L}_{31}^4 \mathcal{L}_{38} \mathcal{L}_{44}^2 \mathcal{L}_{45}^2 \mathcal{L}_{47} \mathcal{L}_{48}^7 \mathcal{L}_{54}^2 \mathcal{L}_{65}^2 \mathcal{L}_{66} \mathcal{L}_{70} \mathcal{L}_{74}}$$

$$\frac{\mathcal{L}_{132}^2 \mathcal{L}_{141} \mathcal{L}_{143} \mathcal{L}_{200} \mathcal{L}_{202} \mathcal{L}_{204}}{\mathcal{L}_{99} \mathcal{L}_{101} \mathcal{L}_{114} \mathcal{L}_{115} \mathcal{L}_{117} \mathcal{L}_{119} \mathcal{L}_{122}} = (\frac{2^{11} 7^{17}}{3^{21} 5^7}, \frac{3^{13} 5^{13}}{2^{26} 7^5}, \frac{2^{42} 3^7}{5^{14} 7^3}, \frac{2^{10} 5^8}{3^6 7^2}, \frac{2^{17} 3^{26}}{5\ 7^{11} 11^3}),$$

$$F_{13} = \frac{\mathcal{L}_8^3 \mathcal{L}_{10}^6 \mathcal{L}_{11}^3 \mathcal{L}_{13} \mathcal{L}_{16} \mathcal{L}_{18}^2 \mathcal{L}_{22}^2 \mathcal{L}_{24} \mathcal{L}_{25}^5 \mathcal{L}_{27} \mathcal{L}_{29} \mathcal{L}_{31}^7 \mathcal{L}_{44} \mathcal{L}_{47}^2 \mathcal{L}_{48}^4 \mathcal{L}_{49} \mathcal{L}_{65} \mathcal{L}_{70}^2 \mathcal{L}_{74} \mathcal{L}_{84}}{\mathcal{L}_9^7 \mathcal{L}_{12}^5 \mathcal{L}_{14}^{12} \mathcal{L}_{19} \mathcal{L}_{21}^2 \mathcal{L}_{23}^2 \mathcal{L}_{30}^2 \mathcal{L}_{32}^2 \mathcal{L}_{33}^2 \mathcal{L}_{37} \mathcal{L}_{41} \mathcal{L}_{45}^2 \mathcal{L}_{54}^3 \mathcal{L}_{55} \mathcal{L}_{58} \mathcal{L}_{62}^7 \mathcal{L}_{91}}$$

$$\frac{\mathcal{L}_{92}^2 \mathcal{L}_{114} \mathcal{L}_{117} \mathcal{L}_{119} \mathcal{L}_{205}}{\mathcal{L}_{102} \mathcal{L}_{132} \mathcal{L}_{141}^2 \mathcal{L}_{143}^2} = (\frac{2^6 5^{15}}{3^{26} 7^9}, \frac{2^{27}}{3^{16} 5^{12}}, \frac{3^{35} 5^5}{2^{77} 7^6}, \frac{7^8}{2^8 3^{19} 5^5}, \frac{5^3 7^6}{2^4 3^{30}}),$$

$$F_{14} = \frac{\mathcal{L}_{12}^7 \mathcal{L}_{18} \mathcal{L}_{20} \mathcal{L}_{23}^3 \mathcal{L}_{26}^2 \mathcal{L}_{30}^4 \mathcal{L}_{31}^2 \mathcal{L}_{33}^2 \mathcal{L}_{44} \mathcal{L}_{45}^4 \mathcal{L}_{48}^9 \mathcal{L}_{53} \mathcal{L}_{54}^5 \mathcal{L}_{90} \mathcal{L}_{115}}{\mathcal{L}_8^4 \mathcal{L}_9^6 \mathcal{L}_{10} \mathcal{L}_{11}^2 \mathcal{L}_{13}^3 \mathcal{L}_{14}^{11} \mathcal{L}_{16} \mathcal{L}_{21}^2 \mathcal{L}_{22}^3 \mathcal{L}_{24} \mathcal{L}_{25}^7 \mathcal{L}_{27} \mathcal{L}_{28} \mathcal{L}_{43} \mathcal{L}_{47}^2 \mathcal{L}_{56} \mathcal{L}_{62}^2 \mathcal{L}_{70}^2 \mathcal{L}_{84}}$$

$$\frac{\mathcal{L}_{141}^2 \mathcal{L}_{143}^2 \mathcal{L}_{213} \mathcal{L}_{215}}{\mathcal{L}_{91} \mathcal{L}_{92}^4 \mathcal{L}_{106} \mathcal{L}_{142}} = (\frac{2^{12} 3^{33}}{5^5 7^{19}}, \frac{2^6 7^{14}}{3^{10} 5^{14}}, \frac{5^{22} 7^{13}}{2^{31} 3^{39}}, \frac{3^{29}}{2^8 5^{11} 7^7}, \frac{2^3, 7^{14}}{3^{20} 5^4 11^3}),$$

$$F_{15} = \frac{\mathcal{L}_9^3 \mathcal{L}_{12}^7 \mathcal{L}_{14} \mathcal{L}_{15}^2 \mathcal{L}_{18}^2 \mathcal{L}_{23}^2 \mathcal{L}_{30}^3 \mathcal{L}_{33}^3 \mathcal{L}_{44} \mathcal{L}_{45}^3 \mathcal{L}_{52} \mathcal{L}_{54}^6 \mathcal{L}_{62}^2 \mathcal{L}_{65} \mathcal{L}_{115} \mathcal{L}_{122}}{\mathcal{L}_8^3 \mathcal{L}_{10}^4 \mathcal{L}_{11}^2 \mathcal{L}_{13}^2 \mathcal{L}_{16}^3 \mathcal{L}_{21} \mathcal{L}_{22}^2 \mathcal{L}_{25}^7 \mathcal{L}_{28} \mathcal{L}_{31}^2 \mathcal{L}_{32}^2 \mathcal{L}_{40} \mathcal{L}_{42} \mathcal{L}_{47} \mathcal{L}_{53} \mathcal{L}_{56} \mathcal{L}_{61} \mathcal{L}_{70}^2}$$

$$\frac{\mathcal{L}_{141} \mathcal{L}_{143} \mathcal{L}_{212} \mathcal{L}_{214} \mathcal{L}_{216}}{\mathcal{L}_{91} \mathcal{L}_{92}^3 \mathcal{L}_{105} \mathcal{L}_{107} \mathcal{L}_{132}} = (\frac{3^{49}}{2^{17} 5^{12} 7^7}, \frac{5^{12}}{2^4 3^7}, \frac{2^{32} 5^3 7^{12}}{3^{38}}, \frac{3^{28}}{2^7 7^9}, \frac{2^4 3^2 7^5 11^4}{5^{10}}),$$

$$F_{16} = \frac{\mathcal{L}_9^{12} \mathcal{L}_{12}^7 \mathcal{L}_{13} \mathcal{L}_{14}^{24} \mathcal{L}_{19}^2 \mathcal{L}_{21}^2 \mathcal{L}_{23}^2 \mathcal{L}_{24}^4 \mathcal{L}_{32}^6 \mathcal{L}_{33}^4 \mathcal{L}_{34}^2 \mathcal{L}_{37}^2 \mathcal{L}_{54}^4 \mathcal{L}_{55}^2 \mathcal{L}_{58}^2 \mathcal{L}_{62}^{10} \mathcal{L}_{91}^2 \mathcal{L}_{132}^2}{\mathcal{L}_8^2 \mathcal{L}_{10}^{11} \mathcal{L}_{11}^{10} \mathcal{L}_{15}^4 \mathcal{L}_{16}^4 \mathcal{L}_{18}^4 \mathcal{L}_{25}^8 \mathcal{L}_{26}^3 \mathcal{L}_{27}^2 \mathcal{L}_{29}^2 \mathcal{L}_{31}^{11} \mathcal{L}_{44}^2 \mathcal{L}_{46} \mathcal{L}_{47}^4 \mathcal{L}_{48}^{11} \mathcal{L}_{65}^2 \mathcal{L}_{69} \mathcal{L}_{70}^4 \mathcal{L}_{71}}$$

$$\frac{\mathcal{L}_{140} \mathcal{L}_{141}^4 \mathcal{L}_{142} \mathcal{L}_{143}^4 \mathcal{L}_{217}}{\mathcal{L}_{74}^2 \mathcal{L}_{108} \mathcal{L}_{114}^2 \mathcal{L}_{117}^2 \mathcal{L}_{119}^2} = (\frac{2^6 3^{20} 7^{24}}{5^{34}}, \frac{3^{20} 5^{39}}{2^{73} 7^8}, \frac{2^{120} 7^4}{3^{45} 5^{14}}, \frac{2^5 3^{25} 5^{17}}{7^{20}}, \frac{2^{28} 3^{52}}{5^{22} 7^{11}}),$$

$$F_{17} = \frac{\mathcal{L}_8^2 \mathcal{L}_{10}^7 \mathcal{L}_{13} \mathcal{L}_{15} \mathcal{L}_{19} \mathcal{L}_{22}^4 \mathcal{L}_{24}^7 \mathcal{L}_{25}^3 \mathcal{L}_{29} \mathcal{L}_{31}^5 \mathcal{L}_{44} \mathcal{L}_{48} \mathcal{L}_{92}^3 \mathcal{L}_{132} \mathcal{L}_{234}}{\mathcal{L}_9^5 \mathcal{L}_{11}^6 \mathcal{L}_{12}^2 \mathcal{L}_{14}^4 \mathcal{L}_{18} \mathcal{L}_{21} \mathcal{L}_{23}^2 \mathcal{L}_{30}^3 \mathcal{L}_{32} \mathcal{L}_{45}^3 \mathcal{L}_{46} \mathcal{L}_{54}^3 \mathcal{L}_{62}^5 \mathcal{L}_{65} \mathcal{L}_{76} \mathcal{L}_{91} \mathcal{L}_{116}} =$$

$$= (\frac{2^{23} 5^7}{3^{21} 7^6}, \frac{2^6 5^6}{3^{16} 7^2}, \frac{3^{24}}{2^{38} 7^4}, \frac{3^{10} 5^4}{2^{14} 7^8}, \frac{2^{20} 7^{14}}{3^{22} 5^8 11^5}),$$

$$F_{18} = \frac{\mathcal{L}_{10} \mathcal{L}_{11}^7 \mathcal{L}_{16}^2 \mathcal{L}_{18}^3 \mathcal{L}_{23} \mathcal{L}_{25}^3 \mathcal{L}_{26} \mathcal{L}_{27} \mathcal{L}_{30} \mathcal{L}_{31}^3 \mathcal{L}_{38} \mathcal{L}_{45} \mathcal{L}_{47}^2 \mathcal{L}_{48}^2 \mathcal{L}_{65}^2 \mathcal{L}_{70}^2 \mathcal{L}_{74} \mathcal{L}_{114} \mathcal{L}_{242}}{\mathcal{L}_9 \mathcal{L}_{12}^4 \mathcal{L}_{13} \mathcal{L}_{14}^7 \mathcal{L}_{19}^2 \mathcal{L}_{22}^2 \mathcal{L}_{24}^4 \mathcal{L}_{33}^2 \mathcal{L}_{37} \mathcal{L}_{54} \mathcal{L}_{55} \mathcal{L}_{62}^3 \mathcal{L}_{92} \mathcal{L}_{120} \mathcal{L}_{132}^2 \mathcal{L}_{141}^2 \mathcal{L}_{143}^2} =$$

$$= (\frac{5^{11}}{2^{14} 3^5 7^5}, \frac{2^{26} 3^4}{5^{16} 7^2}, \frac{3^{14} 5^5}{2^{33} 7^4}, \frac{2^2 7^{16}}{3^{23} 5^9}, \frac{5^{11}}{2^{30} 3^2 7}),$$

$$F_{19} = \frac{\mathcal{L}_9^2\mathcal{L}_{14}^6\mathcal{L}_{19}\mathcal{L}_{21}\mathcal{L}_{25}\mathcal{L}_{32}^2\mathcal{L}_{62}^2\mathcal{L}_{91}\mathcal{L}_{92}\mathcal{L}_{132}\mathcal{L}_{243}}{\mathcal{L}_{10}\mathcal{L}_{15}^2\mathcal{L}_{18}\mathcal{L}_{23}\mathcal{L}_{30}\mathcal{L}_{31}^2\mathcal{L}_{38}\mathcal{L}_{44}\mathcal{L}_{45}\mathcal{L}_{48}^3\mathcal{L}_{54}\mathcal{L}_{60}\mathcal{L}_{65}} =$$

$$= (\frac{2\ 7^8}{3^3 5^5}, \frac{3^8 5^7}{2^{13} 7^3}, \frac{2^{20} 3^2 7^2}{5^9}, \frac{2\ 5^6 7}{3^6}, \frac{3^{13} 5\ 11}{2\ 7^6}),$$

$$F_{20} = \frac{\mathcal{L}_{10}^4\mathcal{L}_{13}\mathcal{L}_{15}\mathcal{L}_{22}^2\mathcal{L}_{24}^2\mathcal{L}_{25}^2\mathcal{L}_{28}\mathcal{L}_{31}^5\mathcal{L}_{44}\mathcal{L}_{48}^4\mathcal{L}_{56}\mathcal{L}_{60}\mathcal{L}_{92}\mathcal{L}_{244}}{\mathcal{L}_9^5\mathcal{L}_{11}\mathcal{L}_{12}^2\mathcal{L}_{14}^9\mathcal{L}_{18}\mathcal{L}_{21}\mathcal{L}_{27}\mathcal{L}_{32}\mathcal{L}_{45}\mathcal{L}_{54}^2\mathcal{L}_{61}\mathcal{L}_{62}^5\mathcal{L}_{91}\mathcal{L}_{115}\mathcal{L}_{121}} =$$

$$= (\frac{2^{15} 5^8}{3^{18} 7^9}, \frac{2^{15} 7^5}{3^{17} 5^{10}}, \frac{3^{14} 5^9}{2^{51} 7^5}, \frac{3}{2^{15} 5^4}, \frac{2^{16} 7^8}{3^{23} 5^2 11^6}),$$

$$F_{21} = \frac{\mathcal{L}_9^5\mathcal{L}_{11}^{11}\mathcal{L}_{12}^4\mathcal{L}_{14}\mathcal{L}_{18}^3\mathcal{L}_{21}\mathcal{L}_{23}^4\mathcal{L}_{26}\mathcal{L}_{30}^4\mathcal{L}_{32}\mathcal{L}_{38}\mathcal{L}_{45}^5\mathcal{L}_{48}\mathcal{L}_{54}^5\mathcal{L}_{61}\mathcal{L}_{62}^4\mathcal{L}_{65}^2\mathcal{L}_{91}}{\mathcal{L}_8\mathcal{L}_{10}^9\mathcal{L}_{13}^3\mathcal{L}_{15}\mathcal{L}_{19}^2\mathcal{L}_{20}\mathcal{L}_{22}^7\mathcal{L}_{24}^{13}\mathcal{L}_{25}^5\mathcal{L}_{28}\mathcal{L}_{31}^4\mathcal{L}_{44}\mathcal{L}_{52}\mathcal{L}_{55}\mathcal{L}_{56}}$$

$$\frac{\mathcal{L}_{114}\mathcal{L}_{115}\mathcal{L}_{158}\mathcal{L}_{245}}{\mathcal{L}_{79}\mathcal{L}_{92}^5\mathcal{L}_{122}\mathcal{L}_{132}^2} = (\frac{3^{36} 7^3}{2^{46} 5^9}, \frac{2^{10} 3^{21}}{5^{18}\ 7}, \frac{2^{20} 5^{10} 7^9}{3^{44}}, \frac{2^{32} 7^{11}}{3^{17} 5^{16}}, \frac{3^{29} 5^{11} 11^7}{2^{52} 7^{16}}).$$

From these we can be give a non-singular matrix M corresponding to the algorithm. Going on to the step 4 the next chart shows the appropriate d values belonging to the different e-s.

e	1-97	98-137	138-179	180-191	192-239	240-269	270-419
d	245	299	411	721	788	954	1076

e	420-431	432-439	440-599	600-659
d	1674	1725	1765	2401

The attentive reader can observe that by this example Lemma 2 (even more) is proved and so the proof of the theorem is completed. □

Without giving the exact vectors $F_1, F_2, \ldots, F_j \in \Delta_{\underline{v}}^{\mu}$ we insert some computer tests verifying Conjecture IV for higher degree. For brevity let us denote $P_n := \prod a_j, a_j \in \mathbb{P}, a_j \leq n$.

Let $k = 5, \gamma = (P_{23}, P_{23}, P_{23}, P_{23}, P_{23}, P_{23})$. Then

e	1-109	110-179	180-191	192-229	230-307	308-313	314-397
d	624	815	853	960	1145	1240	1252

e	398-419	420-431	432-439	440-457	458-599	600-643
d	1587	1674	1726	1768	1833	2401

Let $k = 6, \gamma = (P_{37}, P_{41}, P_{41}, P_{43}, P_{43}, P_{43}, P_{43})$. Then

e	1-227	228-419	420-431	432-541	542-599
d	1336	1589	1674	2154	2164

Our method is clearly not appropriate for large number of completely additive functions, for large e or μ. Even if we could prove Conjecture IV for a given k to prove the induction step seems to be very hard.

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Attila Kovács and Bui Minh Phong
Eötvös Loránd University
Deptartment of Computer Algebra
H–1117 Budapest, Pázmány P. sétány I/D, Hungary
e-mail: `attila@compalg.inf.elte.hu` `bui@compalg.inf.elte.hu`