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ON EXPANSIONS OF GAUSSIAN INTEGERS WITH NON-NEGATIVE DIGITS

Attila **Kovács**

Department of Computer Algebra, Eötvös L. University, H–1117 Budapest, Pázmány Péter sétány I/D Inf. ép., Hungary

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Abstract: It is well-known ([11]) that for a given Gaussian integer θ of norm $N \geq 2$ and canonical digit set $\mathcal{A} = \{0, 1, \dots, N-1\}$ every Gaussian integer γ has a unique expansion in the form $\gamma = \sum_{i=0}^{m} a_i \theta^i$, $a_i \in \mathcal{A}$ if and only if $\theta = -n \pm i$ for some positive integer n. In this paper we shall generalize this result in the meaning that we shall completely describe the location, number and structural properties of the attractors generated by the discrete dynamic of an expansion determined by $(\theta, \mathcal{A})$.

1. Expansions of the integers

Let θ be any rational integer greater than one. It is well-known that every non-negative integer n has a unique representation of the form $n = a_0 + a_1\theta + \ldots + a_k\theta^k$, where the integers a_j are selected from the set $\{0, 1, \dots, \theta - 1\}$. The decimal ($\theta = 10$) and binary ($\theta = 2$) systems are the most familiar. Both positive and negative integers can be uniquely represented without a sign prefix in any negative base $\theta < -1$ using the digits from $\{0, 1, \dots, |\theta| - 1\}$. The concept of radix representation can

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be extended to the integers of the algebraic number fields, or in a more general settings to the lattices of the k-dimensional Euclidean space.

The first non-real base system was introduced by Knuth [12], who suggested that $\theta = 2i$ can be used as base for a complex number system with the digit set $\mathcal{A} = \{0, 1, 2, 3\}$. However, in order to represent all the Gaussian integers, it is necessary to use one negative radix place; for example $1 + 5i = 3(2i)^1 + 1(2i)^0 + 2(2i)^{-1}$. Penney [12] noticed, that every complex number can be represented in binary form using the base $-1 + i$, moreover, all the Gaussian integers can be written in the form $\sum_{j=0}^{n} a_j(-1 + i)^j$, where $a_j = 0, 1$. The systematic research of positional number systems in algebraic extensions was initiated by Kátai and Szabó.

Theorem (Kátai and Szabó [11])**.** *Let θ be a Gaussian integer of norm $N \geq 2$ and let $\mathcal{A} = \{0, 1, \ldots, N-1\}$. Then every Gaussian integer γ can be uniquely represented as $\gamma = a_0 + a_1\theta + \ldots + a_m\theta^m$, $a_j \in \mathcal{A}$, $a_m \neq 0$, $j = 0, 1, \ldots, m$ if and only if $\theta = -n \pm i$ for some positive integer n.*

Let θ be an algebraic integer for which θ and all its conjugates have moduli greater than one. Let $N = |\operatorname{Norm}(\theta)| \geq 2$ and let $\mathcal{A}$ be a complete residue system modulo θ. The pair $(\theta, \mathcal{A})$, $\mathcal{A} = \{0, a_1, \ldots, a_{N-1}\}$ is called a *number system in* $\mathbb{Z}[\theta]$ if for each $\gamma \in \mathbb{Z}[\theta]$ there exist an $m \in \mathbb{N}_0$ and $a_j \in \mathcal{A}$ such that

$$\gamma = a_0 + a_1\theta + \ldots + a_m\theta^m,\ a_j \in \mathcal{A},\ a_m \neq 0,\ j = 0, 1, \ldots, m. \tag{1}$$

The uniqueness of the expansion follows from the assumption that any two elements of $\mathcal{A}$ are incongruent modulo θ. The reason that the norm N yields the correct number of digits is due to the fact that the quotient ring $\mathbb{Z}[\theta]/(\theta)$ is isomorphic to $\mathbb{Z}_N$ by the map which takes a polynomial in θ to its constant term modulo N. Hence the possible digits for a_0 in (1) must form a complete set of representatives for $\mathbb{Z}_N$.

The exact characterization of those algebraic integers for which $(\theta, \mathcal{A})$ is a number system with a suitable digit set $\mathcal{A}$ seems to be hard. Some necessary conditions are $|\theta^{(j)}| > 1$ $(j = 1, \ldots, n)$ and $1 - \theta^{(j)}$ is not unit, where $\theta^{(j)}$ denotes the j-th conjugate of $\theta \in \mathbb{Q}(\Gamma)$ and Γ is an algebraic number of degree n. It was shown in [19] that for $\mathbb{Q}(i)$ the above conditions are also sufficient, i.e. most Gaussian integers can be used as a base of a number system if the digit sets are chosen appropriately. For imaginary quadratic extension fields Kátai [7] proved the sufficiency of the above conditions. Using his construction Farkas [1, 2] has the same result for real quadratic extension fields. In general, Kátai [5] proved for arbitrary algebraic number fields that if $|\theta^{(j)}| > 2$ $(j = 1, \ldots, n)$ then the set $\mathcal{A}$ can be constructed in such a way that $(\theta, \mathcal{A})$ is number system.

If the digit set $\mathcal{A}$ is restricted to be a set of non-negative numbers, we get a straightforward generalization of the traditional number systems in $\mathbb{Z}$. The set $\mathcal{A}=\{0,1,\ldots,N-1\}$ is called *canonical digit set*. In this case all those integers θ in quadratic number fields can be given, for which $(\theta,\mathcal{A})$ are number systems (see [4, 8, 9]). Using canonical digit sets Körmendi [17] determined all the integers $\theta\in\mathbb{Q}(\sqrt[3]{2})$ for which $(\theta,\mathcal{A})$ is a number system. B. Kovács [15] gave a necessary and sufficient condition for the existence of canonical number systems in $\mathbb{Z}[\theta]$. Kovács and Pethő [16] characterized all those integral domains that have number systems. The concept of number systems can also be generalized to the lattices of the k-dimensinal Euclidean space. In [13] an effective algorithm was presented to decide whether the system $(M,\mathcal{A})$ is a number system or not for a given invertible expanding linear operator M and given digit set $\mathcal{A}$.

2. Expansions and discrete dynamics

Let θ be an algebraic integer for which θ and all its conjugates have moduli greater than one. Let $N=|\operatorname{Norm}(\theta)|\geq 2$ and let $\mathcal{A}$ be a complete residue system modulo θ. The system $(\theta,\mathcal{A})$ can be used to represent all the integers even if it is not a number system. Clearly, for each $\gamma\in\mathbb{Z}[\theta]$ there exist a unique $a_j\in\mathcal{A}$ such that $\theta|\gamma-a_j$. Let $\gamma_1=\frac{\gamma-a_j}{\theta}$ and let us define the function $\Phi:\mathbb{Z}[\theta]\to\mathbb{Z}[\theta]$ by $\Phi(\gamma)=\gamma_1$. Let Φ^l denote the l-fold iterate of Φ, $\Phi^0(\gamma)=\gamma_0$. The sequence of integer vectors $\Phi^j(\gamma_0)=\gamma_j$ $(j=0,1,2,\ldots)$ is called the *path of the dynamical system* generated by Φ. It is also called the *orbit of* γ_0 generated by Φ.

The function Φ was introduced by Matula [18] for $\mathbb{Z}$ in order to construct number systems. Somewhat later, independently, Kátai [10, 3] used it for constructing number systems in algebraic extensions. From dynamical point of view the research was initiated by Kátai [6, 13].

In this paper we restrict our attention to the ring of Gaussian integers $\mathbb{Z}[i]$. Let $\theta\in\mathbb{Z}[i]$ be given. The number of elements in a complete residue system modulo $\theta=A+Bi$ is $N=\operatorname{Norm}(\theta)=A^2+B^2$. Gauss showed that if A and B are relative prime then the natural numbers $0,1,\ldots,N-1$ form a complete residue system modulo θ. Moreover, if A and B have a common factor then any complete residue system modulo θ must contain some numbers with nonzero imaginary parts. In the following we shall consider the canonical digit set $\mathcal{A}=\{0,1,\ldots,N-1\}$, where $N\geq 2$; so we always assume that $(A,B)=1$. Let furthermore $L:=\frac{N-1}{|\theta|-1}$. Then for every $\gamma\in\mathbb{Z}[i]$ the next lemma is obviously true.

Lemma 1. *If* $|\gamma| \leq L$ *then* $|\Phi(\gamma)| \leq (L+N-1)/(|\theta|) = L$. *If* $|\gamma| > L$ *then* $|\Phi(\gamma)| \leq (|\gamma|+N-1)/(|\theta|) < |\gamma|$.

Since the inequality $|\delta| \leq L$ holds only for finitely many Gaussian integers δ, therefore the path $\gamma, \Phi(\gamma), \Phi^2(\gamma), \ldots$ is ultimately periodic for all $\gamma \in \mathbb{Z}[i]$. An element $\pi \in \mathbb{Z}[i]$ is called *periodic* if there exist a $j \in \mathbb{N}$ such that $\Phi^j(\pi) = \pi$. The smallest such j is the *length of the period of* π generated by Φ. Let $\mathcal{P}$ denote the set of all periodic elements. Let $\pi \in \mathcal{P}$ be of period length l. The set of the periodic elements $\{\Phi(\pi), \ldots, \Phi^l(\pi)\}$ will be denoted by $\mathcal{C}(\pi)$. Suppose that $\pi \in \mathcal{P}$. Then the *basin of attraction of* π consists of all $\gamma \in \mathbb{Z}[i]$ for which there exists a $j \in \mathbb{N}_0$ such that $\Phi^j(\gamma) = \pi$ and is denoted by $\mathcal{B}(\pi)$. Let $X \subseteq \mathcal{P}$. In a similar way, $\mathcal{B}(X)$ denotes all the elements $\gamma \in \mathbb{Z}[i]$ for which there exists a $j \in \mathbb{N}_0$ and $\pi' \in X$ such that $\Phi^j(\gamma) = \pi'$.

The basin of attraction of an element $\pi \in \mathcal{P}$ can be obtained in a simple way. Let $\Psi^0(\pi) = \{\pi\}$, $\Psi^{l+1}(\pi) = \bigcup_{b \in \mathcal{A}} \{\theta \Psi^l(\pi) + b\}$, $l = 0, 1, \ldots$. Observe that Ψ — which acts as a left shift operator — is an inverse of Φ, which acts as a right shift operator. Obviously, $\mathcal{B}(\pi) = \bigcup_{l=0}^{\infty} \Psi^l(\pi)$. The following assertions are clearly true. 1) $\mathcal{P}$ is finite. 2) If $\pi \in \mathcal{P}$ then $\Phi(\pi) \in \mathcal{P}$. 3) If $\pi \in \mathcal{P}$ then $|\pi| \leq L$. 4) $\pi \in \mathcal{P}$ iff there is an $l > 0$

$$\pi = a_0 + a_1\theta + \ldots + a_{l-1}\theta^{l-1} + \pi\theta^l, \quad a_j \in \mathcal{A}. \tag{2}$$

5) If $\pi_1, \pi_2 \in \mathcal{P}$, $\pi_1 \neq \pi_2$ and $\mathcal{C}(\pi_1) = \mathcal{C}(\pi_2)$ then their length of period are equal. 6) $\mathcal{B}(\mathcal{P}) = \mathbb{Z}[i]$. 7) If $\pi_1, \pi_2 \in \mathcal{P}$ then $\mathcal{B}(\pi_1) = \mathcal{B}(\pi_2)$ iff $\mathcal{C}(\pi_1) = \mathcal{C}(\pi_2)$.

Let $\mathcal{G}(P)$ be the directed graph defined on the set $\mathcal{P}$ by drawing an edge from $\pi \in \mathcal{P}$ to $\Phi(\pi)$. Then $\mathcal{G}(P)$ is a disjoint union of directed cycles, where loops are allowed. $\mathcal{G}(P)$ is also called the *attractor of* $\mathbb{Z}[i]$ generated by Φ.

Observe that $(\theta, \mathcal{A})$ is a number system iff for each $\gamma \in \mathbb{Z}[i]$ there is an $m \in \mathbb{N}_0$ such that $\Phi^m(\gamma) = 0$. Due to Kátai and Szabó we know that this is the case iff $\theta = -n \pm i$ for some positive integer n. The following questions arise naturally (see also [13]). (a) What can be stated about the attractors of an arbitrary canonical system $(\theta, \mathcal{A})$ generated by Φ, where $\theta \in \mathbb{Z}[i]$ and $\mathrm{Norm}(\theta) \geq 2$? (b) How the structure of the periodic elements looks like? (c) Is there a good upper estimation for the number of the different sets $\mathcal{C}(\pi)$? (d) It is known that if $\pi \in \mathcal{P}$ then the maximum of the period length of π can be estimated with the number of Gaussian integers covered by the disk with radius L centered at the origin. Is there a better estimation? The purpose of this note is to answer these questions.

3. Periodic elements of period length one

Let $\theta = A+Bi, \mathcal{A} = \{0,1,\dots,N-1\}, N = A^2+B^2$. First we shall examine the periodic elements of period length one (also called *loops*).

Theorem 1. *The loops in the system* $(\theta,\mathcal{A})$ *are* $\pi_j = \frac{1-A+Bi}{(1-A,B)}j, \quad j = 0,1,\dots,k,$ *where* $k = \left\lfloor (1-A,B)\left(1+\frac{2(A-1)}{N-2A+1}\right)\right\rfloor$.

Proof. It follows from (2) that $\pi \in \mathcal{P}$ is a loop if and only if $\pi = b+\theta\pi$ for some $b \in \mathcal{A}$. It means that $(1-\theta)\pi = b \in \mathcal{A}$, hence $\pi = \frac{b}{1-\theta} = \frac{b}{1-A-Bi} = \frac{(1-A)b}{(1-A)^2+B^2}+\frac{Bb}{(1-A)^2+B^2}i$. Since $\pi \in \mathbb{Z}[i]$ therefore $(1-A)^2+B^2|(1-A,B)b$. On the other hand $0 \le b \le N-1$, which means that $\pi_j = \frac{1-A+Bi}{(1-A,B)}\,j, \quad j = 0,1,\dots,k$, where $k = \left\lfloor \frac{(N-1)(1-A,B)}{(1-A)^2+B^2}\right\rfloor$. The proof is complete.$\Diamond$

Before we continue the examinations for searching the periodic elements of longer periods it is worth to analyze some special cases.

4. Some special cases

Let us consider the system $(A\pm i, \{0,1,\dots,A^2\})$, where $A > 0$. Recall that Kátai and Szabó determined the attractors for the cases $A < 0$, namely $\mathcal{G}(P) = \{0 \to 0\}$.

4.1. The case $\theta = A \pm i, \ A \ge 1$. It is well-known that the attractors of the system $(1+i,\{0,1\})$ are $\mathcal{G}(P) = \{0 \to 0, i \to i\}$. It was also shown (e.g. in [13]) that the attractors of the system $(2+i,\{0,1,2,3,4\})$ are $\mathcal{G}(P) = \{0 \to 0, \ -1+i \to -1+i, \ -2+2i \to -2+2i\}$.

Lemma 2. *Let* $\theta = A+i, \ A \ge 3$. *Then the attractors are* $\mathcal{G}(P) = \{0 \to \ 0, \ 1-A+i \to 1-A+i\}$.

Proof. It is easy to see that $\Phi(0) = 0$ and $\Phi(1-\overline{\theta}) = 1-\overline{\theta}$. We have to prove that there are no more periodic elements. It is not hard to check (e.g. using the algorithm presented in [13]) that the lemma is true for $A = 3$. Let $A \ge 4$. Suppose that $U+Vi = \pi \in \ \mathcal{P}, \Phi(\pi) = \pi_1 = U_1+V_1i \in \mathcal{P}$. Then $U+Vi = b+(A+i)(U_1+iV_1)$ for some $b \in \mathcal{A}$. Hence

$$U = b+AU_1-V_1, \qquad V = U_1+AV_1. \tag{3}$$

We have some useful observations. 1. For every periodic elements π

$$|U|,|V| \le A+1. \tag{4}$$

Indeed, it is clear that if $\pi \in \mathcal{P}$ then $|\pi| \le L = \frac{A^2}{|\theta|-1} = |\theta|+1$. Therefore $|\pi|^2 = U^2+V^2 \le |\theta|^2+2|\theta|+1$. On the other hand $2|\theta| = 2\sqrt{A^2+1} <$

$< 2(A+1/2)$, hence $|\pi|^2 < A^2+2A+3$. But $|\pi|^2 \in \mathbb{Z}[i]$, so $|\pi|^2 \leq$ $\leq (A+1)^2+1$, which means that the inequality (4) holds.

2. Nonzero rational integer can not be periodic. For the proof suppose indirectly, that $\pi = U \in \mathcal{P}$ $(U \neq 0, V = 0)$. From (3) we get that $U_1 = -AV_1$. If $V_1 = 0$ then $U_1 = 0$, therefore $\pi_1 = 0$ which contradicts the periodicity of π. Hence $V_1 \neq 0$. By using (4) it is clear that $|U_1| < 2A$, therefore $V_1 = \pm 1$. If $V_1 = 1$ then keeping in mind that $\theta + \overline{\theta} = 2A$ we have $\pi_1 = -A + i = -\overline{\theta} = \theta - 2A = (N-2A) - \theta\overline{\theta} + \theta$. Hence $\Phi(\pi_1) = 1 - \overline{\theta}$, and the fact $\Phi(1-\overline{\theta}) = 1-\overline{\theta}$ contradicts to $\pi \in \mathcal{P}$. If $V_1 = -1$ then $\pi_1 = A - i = \overline{\theta} = 2A - \theta$, which means that $\Phi(\pi_1) = -1$. Clearly, $-1 = N - 1 - \theta\overline{\theta}$, so $\Phi(-1) = -\overline{\theta}$. Since $\Phi(-\overline{\theta}) = 1 - \overline{\theta}$ and $\Phi(1-\overline{\theta}) = 1 - \overline{\theta}$ therefore $\pi \notin \mathcal{P}$, which is contradiction.

3. Pure imaginary Gaussian integer can not be periodic. Again, suppose indirectly that $\pi_1 = iV_1 \in \mathcal{P}$ $(U_1 = 0, V_1 \neq 0)$. From (3) we get that $V = AV_1$. By the inequality (4) it follows that $|V| < 2A$, therefore $V_1 = \pm 1$, so $\pi_1 = \pm i$. If $\pi_1 = i$ then $\Phi(\pi_1) = 1 - \overline{\theta}$, since $i = -A + \theta = (N-A) + \theta - \theta\overline{\theta}$. If $\pi_1 = -i$ then $\Phi(\pi_1) = \Phi(A - \theta) = -1$. In both cases we can use the same argumentation as before, so π_1 can not be periodic, which is a contradiction.

4. If $\pi_1 \in \mathcal{P}$ then $|V_1| \leq 2$. Indeed from (3) and (4) it follows that $|AV_1| = |V - U_1| \leq 2A + 2 < 3A$.

5. If $\pi_1 \in \mathcal{P}$ then $V_1 > 0$. In virtue of (3) and (4) we get that $U = b + A(V - AV_1) - V_1 = b - A^2V_1 + (AV - V_1)$. If $V_1 < 0$ then $U \geq A^2 - 2A$, so $A + 2 > A^2 - 2A$ which contradicts the fact that $A^2 \geq 3A + 2$.

According to the observations we have only to check 4 cases. Before we analyze these cases we remark the fact that if $A \geq 4$ then $A^2 - 2A + 3$, $A^2 - 4A + 2$ and $A^2 - 4A + 3 \in \mathcal{A}$.

(a) If $V = V_1 = 1$ then $U_1 = 1 - A$, hence $\pi_1 = 1 - A + i$. π_1 is listed in Lemma 2.

(b) If $V = V_1 = 2$ then $U_1 = 2 - 2A$, so $\pi_1 = 2 - 2A + 2i$. The expansion of π_1 is $2 - 2A + 2i = A^2 - 4A + 3 + \theta(2 - \overline{\theta})$. The expansion of $2 - \overline{\theta}$ is $2 + (\theta - 2A) = A^2 + 3 - 2A + \theta(1 - \overline{\theta})$, therefore $\pi_1 \in \mathcal{B}(1 - \overline{\theta})$.

(c) If $V = 1, V_1 = 2$ then $U_1 = 1 - 2A$, so $\pi_1 = 1 - 2A + 2i$. But $1 - 2A + 2i = 1 - 4A + 2\theta = A^2 - 4A + 2 + \theta(2 - \overline{\theta})$, therefore using the expansion of $2 - \overline{\theta}$ we get that $1 - 2A + 2i \in \mathcal{B}(1 - \overline{\theta})$.

(d) If $V = 2, V_1 = 1$ then $U_1 = 2 - A$ and $\pi_1 = 2 - A + i$. Its expansion is $2 - A + i = 2 - 2A + \theta = A^2 + 3 - 2A + \theta(1 - \overline{\theta})$, so $\pi_1 \in \mathcal{B}(1 - \overline{\theta})$. The proof of Lemma 2 is complete.$\Diamond$

Let us notice that if $\pi = a_0 + a_1\theta + \ldots + a_{l-1}\theta^{l-1} + \pi\theta^l$ then

$$\overline{\pi} = a_0 + a_1\overline{\theta} + \ldots + a_{l-1}\overline{\theta}^{l-1} + \overline{\pi}\overline{\theta}^l, \quad a_j \in \mathcal{A}. \tag{5}$$

Together with this observation we proved the following

Theorem 2. *Let $\theta = A + i$, $A \geq 1$. Then the attractors of the system $(\theta, \mathcal{A})$ are $\mathcal{G}(P) = \{0 \to 0,\ 1-A+i \to 1-A+i\}$ for all A except $A = 2$, in which case $\mathcal{G}(P) = \{0 \to 0,\ -1+i \to -1+i,\ -2+2i \to -2+2i\}$. If $\theta = A - i$ then using (5) the attractor set $\mathcal{G}(P)$ can be obtained from the previous ones.*

4.2. The case $\theta = 1 \pm Bi$. Suppose that $B \geq 1$. Applying Th. 1 it is easy to see that the periodic elements of period length one in the system $(1 + Bi, \mathcal{A} = \{0, \ldots, B^2\})$ are $\pi_j = ji$ $(j = 0, 1, \ldots, B)$. We state that there are no other periodic elements.

Lemma 3. *Let $\theta = 1 + Bi$, $B \geq 1$. The attractors in the system $(1 + Bi, \mathcal{A})$ are $\mathcal{G}(P) = \{ji \to ji,\ j = 0, 1, \ldots B\}$.*

Proof. Th. 2 shows that the lemma is true for $B = 1$. It is not hard to see that Lemma 3 also holds for $B = 2$. In the following, let $B \geq 3$. Let $U + Vi = \pi \in \mathcal{P}$, $\Phi(\pi) = \pi_1 = U_1 + V_1 i \in \mathcal{P}$. Then $U + Vi = b + (1 + Bi)(U_1 + iV_1)$ for some $b \in \mathcal{A}$. Hence

$$U = b + U_1 - BV_1, \qquad V = BU_1 + V_1. \tag{6}$$

Again, we have some useful observations:

$$-Bi,\ 2 - Bi,\ -2 + Bi,\ -j \in \mathcal{B}(Bi),\ j = 1, 2, \ldots, B + 1, \tag{7}$$

$$-1 + ji \in \mathcal{B}(ji),\ j = 1, 2, \ldots, B, \tag{8}$$

$$+1 + ji \in \mathcal{B}(ji),\ j = 0, 1, \ldots, B - 1, \tag{9}$$

$$+1 + Bi,\ 2 - (B - 1)i \in \mathcal{B}(0), \tag{10}$$

$$(B + 1)i \in \mathcal{B}(i), \tag{11}$$

$$-ji \in \mathcal{B}((B - j)i), j = 1, \ldots, B - 1, \tag{12}$$

$$-Bi - i, -1 - i, -2 + (B - 1)i \in \mathcal{B}((B - 1)i). \tag{13}$$

The proofs are easy, we left them to the reader. The inequalities

$$|U|, |V| \leq B + 1 \tag{14}$$

can be proven in the same way as we did it in the previous subsection. Statement (7) shows that a nonzero rational integer can not be periodic. In the same way, it follows from (7), (11), (12) and (13) that pure imaginary Gaussian integer periodic elements are exactly the given ones. By equation (6) we have that $|U_1| \leq 2$.

If $V = B+1$ then from (6) and (14) we get the following cases: (a) $U_1 = 0, V_1 = B+1$. Clearly, $\pi_1 \in \mathcal{B}(0)$. (b) $U_1 = 1, V_1 = 1$. Observation (9) shows that the path of π_1 does not generate a new periodic element. (c) $U_1 = 2, V_1 = -B+1$. See (10).

If $V = -B-1$ then we have: (a) $U_1 = 0, V_1 = -B-1$. See (13). (b) $U_1 = -1, V_1 = -1$. See (13). (c) $U_1 = -2, V_1 = B-1$. See (13).

If $V = B$ then the cases: (a) $U_1 = 0, V_1 = B$. Clearly, $\pi_1 \in \mathcal{B}(Bi)$. (b) $U_1 = 1, V_1 = 0$. Obviously, $\pi_1 \in \mathcal{B}(0)$. (c) $U_1 = 2, V_1 = -B$. See (7).

If $V = -B$ then we have: (a) $U_1 = 0, V_1 = -B$. See (7). (b) $U_1 = -1, V_1 = 0$. See (7). (c) $U_1 = -2, V_1 = B$. See (7).

We can conclude that the remaining cases are $|V|, |V_1| \leq B-1$. Then, from the equation (6) we have that $|U_1| \leq 1$. If $U = U_1 = \pm 1$ then by (6) we get that $V_1 = j,\ j = 0, 1, \ldots, B$. The statements (8), (9) and (10) show that the orbits of the different π_1-s do not generate a new attractor. If $U = -U_1 = 1$ then $V_1 = j,\ j = 0, 1, \ldots, B-1$. See (8). If $U = -U_1 = -1$ then $V_1 = j,\ j = 1, \ldots, B$. In this case see (9) and (10). The proof of Lemma 3 is finished.$\Diamond$

Taking (5) into consideration we proved the following theorem.

Theorem 3. *Let $\theta = 1 + Bi$. The attractors in the system $(\theta, \mathcal{A})$ are $\mathcal{G}(P) = \{ji \to ji,\ j = 0, 1, \ldots B\}$. If $\theta = 1 - Bi$ then $\mathcal{G}(P) = \{-ji \to -ji,$ $j = 0, 1, \ldots B\}$.*

Fig. 1 shows the basin of attractions of the system $(1+2i, \{0, \ldots, 4\})$ in a region of the complex plane. $\mathcal{B}(0)$ is gray, $\mathcal{B}(i)$ is white and $\mathcal{B}(2i)$ is black.

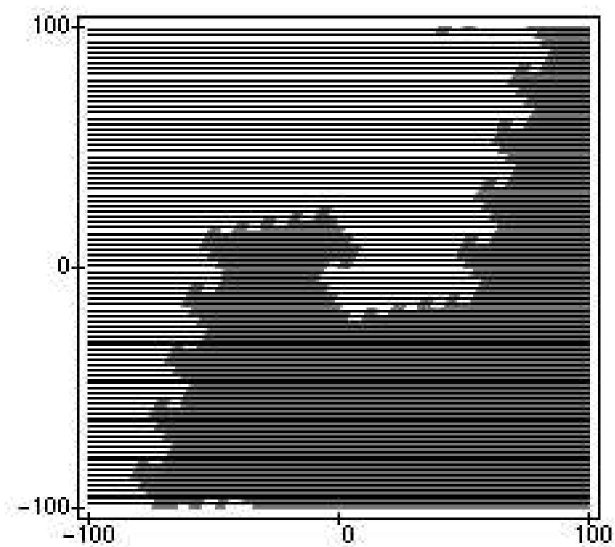

Fig. 1

5. Structural properties of the attractors

It is enough to consider the case $\theta = A + Bi$, $(A, B) = 1$, $B \geq 2$. Clearly, if $\pi \in \mathcal{P}$ then

$$(15) \quad |\pi| \leq L = \frac{A^2 + B^2 - 1}{|\theta| - 1} = |\theta| + 1 = \sqrt{A^2 + B^2} + 1 < |A| + |B|.$$

In this section we shall examine the structure of the orbit of π generated by Φ. Suppose that $U + Vi = \pi \in \mathcal{P}$, $\Phi(\pi) = \pi_1 = U_1 + V_1 i \in \mathcal{P}$. Then $U + Vi = b + (A + Bi)(U_1 + iV_1)$ for some $b \in \mathcal{A}$. Therefore,

$$(16) \qquad U = b + AU_1 - BV_1, \qquad V = BU_1 + AV_1.$$

From equation (16) we can get a congruence equation $V \equiv AV_1 \pmod{B}$. Applying the Euler-Fermat theorem we have that

$$(17) \qquad V_1 \equiv A^{\varphi(B)-1} V \pmod{B}.$$

Here φ denotes the Euler totient function. Let furthermore $\mathcal{L}_\delta = \{C + +Di \in \mathbb{Z}[i],\ (B, D) = \delta\}$. Clearly, $\mathbb{Z}[i] = \bigcup_{\delta|B} \mathcal{L}_\delta$. In virtue of (16) we can also notice that if $(V, B) = \delta$ then $(V_1, B) = \delta$. Hence the function Φ maps $\mathcal{L}_\delta$ to $\mathcal{L}_\delta$ for each $\delta|B$. We shall distinguish two cases.

5.1. The structure of $\mathcal{L}_\delta$, where $\delta < B$. Let $X = V/\delta$, $X_1 = V_1/\delta$, $B_\delta = B/\delta$. Then from (16) we have that

$$(18) \qquad X = B_\delta U_1 + AX_1.$$

Let us denote by $\mathbb{Z}^*_{B_\delta}$ the set of reduced residue classes modulo B_δ, i.e., $\mathbb{Z}^*_{B_\delta} = \{m \pmod{B_\delta},\ (m, B_\delta) = 1\}$. Since $(A, B) = 1$ therefore the residue classes $A, A^2, A^3, \dots \pmod{B_\delta}$ are all in $\mathbb{Z}^*_{B_\delta}$ constituting a subgroup. Let us denote by t_δ the smallest positive integer k for which $A^k \equiv 1 \pmod{B_\delta}$. Then $T_\delta = \{A, A^2, \dots, A^{t_\delta}\}$ is a subgroup of $\mathbb{Z}^*_{B_\delta}$. Obviously, $t_\delta | \varphi(B_\delta)$. Let $\varphi(B_\delta) = l_\delta t_\delta$. Hence the number of elements of the factor group $\mathbb{Z}^*_{B_\delta}/T_\delta$ is l_δ and we get the decomposition $\mathbb{Z}^*_{B_\delta} = H_0 \cup H_1 \cup \ldots \cup H_{l_\delta - 1}$, where $H_0 = T_\delta$. Let $\mathcal{L}^{(j)}_\delta = \{\alpha = = C + Di,\ \alpha \in \mathcal{L}_\delta,\ D/\delta \pmod{B_\delta} \in H_j\}$. Hence we have a decomposition of $\mathcal{L}_\delta$ by $\mathcal{L}_\delta = \mathcal{L}^{(0)}_\delta \cup \mathcal{L}^{(1)}_\delta \cup \ldots \cup \mathcal{L}^{(l_\delta - 1)}_\delta$. The next lemma follows from the above deduced construction.

Lemma 4. *If $\alpha \in \mathcal{L}^{(j)}_\delta$ then $\Phi(\alpha) \in \mathcal{L}^{(j)}_\delta$.*

5.2. The structure of $\mathcal{L}_B$. Theorem 4. *Let $\theta = A + Bi$, $A \geq 2$, $(A, B) = 1$ and $X + YBi = = \pi \in \mathcal{P}$. Then either $\pi = 0$ or $\pi = 1 - \overline{\theta}$.*

Proof. It is enough to prove it for the case $B \geq 2$. The following statements are clearly true. 1) $|X| \leq A + B$. 2) $|Y| \leq \left\lfloor \frac{A}{B} + 1 \right\rfloor$. 3) $0 \to 0 \in \mathcal{G}(P),\ 1 - \overline{\theta} \to 1 - \overline{\theta} \in \mathcal{G}(P)$.

The expansion of π is $X + YBi = X + Y(\theta - A) = X - AY + \theta Y$. If $X - AY \geq 0$ and $Y \geq 0$ then $\pi \in \mathcal{B}(0)$. (The case $X - AY \geq \geq A^2 + B^2$ can not occur.) Let $X - AY \geq 0$ and $Y < 0$. In this case the expansion of Y is $Y = Y + A^2 + B^2 - \theta\overline{\theta}$. The expansion of $-\overline{\theta}$ is $\theta - 2A = A^2 + B^2 - 2A + \theta(1 - \overline{\theta})$, so $\pi \in \mathcal{B}(1 - \overline{\theta})$. If $X - AY < 0$ then $\pi = A^2 + B^2 + X - AY + \theta(Y - \overline{\theta})$. The expansion of $Y - \overline{\theta}$ is $Y - \overline{\theta} = Y + \theta - 2A = A^2 + B^2 + Y - 2A + \theta(1 - \overline{\theta})$. Hence, the proof is complete.$\Diamond$

Recall that the elements of the ring $\mathbb{Z}[\theta]$ have the form $\{m + n\theta \ : \ : \ m, n \in \mathbb{Z}\}$.

Lemma 5. $\mathbb{Z}[\theta] = \{u + vBi \ : \ u, v \in \mathbb{Z}\} = \{\beta \in \mathbb{Z}[i] \ : \ B|\ \mathrm{Im}(\beta)\}$.

Proof. Obviously, $m + n\theta = (m + nA) + nBi$. Let $v := n, u := m + nA$.$\Diamond$

In the next theorem we shall prove somewhat more than what we really need. The proof follows the argument of Kátai and Szabó ([11]).

Theorem 5. *Let* $\theta = A + Bi,\ A = -E,\ E > 0,\ (E, B) = 1$. *Then* $\mathbb{Z}[\theta] = \mathcal{B}(0)$.

Proof. Let β be an arbitrary Gaussian integer such that $\beta \in \mathcal{B}(0)$. It means that $\beta = b_0 + b_1\theta + \ldots + b_k\theta^k$ for some $k \in \mathbb{N}$ and $b_j \in \mathcal{A}$. Clearly, $B|\ \mathrm{Im}(\theta^j)$. Since $\mathcal{A} \subset \mathbb{N}_0$ therefore $B|\ \mathrm{Im}(\beta)$. By Lemma 5 we have that $\beta \in \mathbb{Z}[\theta]$, hence $\mathcal{B}(0) \subseteq \mathbb{Z}[\theta]$. Suppose now that $\beta \in \mathbb{Z}[\theta]$. Then

$$\beta = m + n\theta. \tag{19}$$

On the other hand the expansion of -1 is

$$-1 = N - 1 + \theta\overline{\theta} = N - 1 + 2E\theta + \theta^2. \tag{20}$$

Since $1, N - 1, 2E \in \mathcal{A}$ therefore -1 has the finite expansion (20) in the system $(\theta, \mathcal{A})$. Equations (19) and (20) mean that β has also an expansion in the form

$$\beta = u_0 + u_1\theta + u_2\theta^2 + u_3\theta^3, \tag{21}$$

where $u_j \geq 0 \ \ (j = 0, 1, 2, 3)$.

Lemma 6. (Clearing Lemma.) *Suppose that* β *has the following expansion:*

$$\beta = u_0 + u_1\theta + \ldots + u_m\theta^m, \tag{22}$$

where $u_j \geq 0 \ \ (j = 0, \ldots, m)$. *Let* $T = \sum_{j=0}^{m} u_j$. *Then for every* $k \geq 0$ *there exists an expansion* $\beta = v_0 + v_1\theta + \ldots + v_k\theta^k + \ldots + v_l\theta^l$ *such that* $\sum_{j=0}^{l} v_j = T$ *and* $0 \leq v_j < N \ \ (j = 0, 1, \ldots, k)$.

Proof of Lemma 6. First we shall examine the expansion of $N = A^2 + B^2$.

$$\begin{aligned} N &= (-2E-\theta)\theta = \big((N-2E) - \theta\overline{\theta} - \theta\big)\theta = \\ &= (N-2E)\theta + (-\overline{\theta}-1)\theta^2 = (N-2E)\theta + (2E-1)\theta^2 + \theta^3. \end{aligned} \tag{23}$$

Observe that $N-2E, 2E-1, 1 \in \mathcal{A}$ and the sum of the digits of the expansion in (23) is N. Clearly, if $u_0 < N$ in (22) then the lemma holds for $k = 0$. In the opposite case, if $N \le u_0$, then let $u_0 = pN + q,\ p \ge 1,$ $0 \le q < N$. Let us take $u_0 = q + p\big(0 + (N-2E)\theta + (2E-1)\theta^2 + \theta^3\big)$ into the equation (22). Then we have that $\beta = u_0' + u_1'\theta + \ldots + u_m'\theta^{m'}$, where $u_0' = q,\ u_1' = u_1 + p(N-2E),\ u_2' = u_2 + p(2E-1),\ u_3' = u_3 + p,\ u_j' = u_j$ $(j \ge 4)$. Observe that $\sum u_j' = T$, so the sum of the digits does not change in the new expansion. Hence, the case $k = 0$ is satisfied. We can continue the process for $j = 1, 2, \ldots, k$.$\Diamond$

Now, we can apply Clearing Lemma for the expansion of β in (21). Let $T_0 = T = u_0 + u_1 + u_2 + u_3$. If $u_0 \ge N$ then by the lemma we have that $\beta = v_0 + \theta\beta_1,\ 0 \le v_0 < N,\ \beta_1 = y_1 + y_2\theta + \ldots + y_m\theta^m,\ y_j \ge 0.$ Let $T_1 = T_0 - v_0$. Clearly, $T_1 = y_1 + \ldots + y_m$. Applying the Clearing Lemma again and again we have a monotonically decreasing sequence $T_0, T_1, T_2, \ldots$, and expansions $\beta, \beta_1, \beta_2, \ldots$. If there exists an $h \in \mathbb{N}$ such that $T_h = 0$ then the expansion of β is finite having the digits from the set $\mathcal{A}$, so Th. 5 is proved. If such an h does not exist then there is a suitable large h_0 such that $T_{h_0} = T_{h_0+1} = \ldots = r > 0$. But in this case $\beta_{h_0} = \theta\beta_{h_0+1} = \theta^2\beta_{h_0+2} = \ldots = \theta^j\beta_{h_0+j} = \ldots$, therefore $\gamma := \beta - (v_0 + v_1\theta + \ldots + v_{h_0}\theta^{h_0}) = \theta^{h_0+t}\beta_{h_0+t}$ for every $t \in \mathbb{N}$. Observe that $\theta^{h_0+t}|\gamma\ \ (t = 1, 2, \ldots)$ and this holds only if $\gamma = 0$. This means that β has a finite expansion with digits from the set $\mathcal{A}$. The proof of Th. 5 is complete.$\Diamond$

6. Location and number of periodic elements

Let $\pi, \rho \in \mathcal{P}, \pi \to \pi_1, \rho \to \rho_1$ and $\mathcal{D} = \mathcal{A} - \mathcal{A} = \{-(N-1), \ldots, \ldots, N-1\}$. Clearly, $\pi - \rho = d + \theta(\pi_1 - \rho_1)$, where $d \in \mathcal{D}$. Let $M = \max_{\pi,\rho\in\mathcal{P}} |\pi - \rho| = |\pi_1 - \rho_1|$. Then $|\theta|M \le M + (N-1)$. This shows that $M \le \frac{N-1}{|\theta|-1} = L$. It means that the periodic elements are 'close' together. But even more is true.

Theorem 6. *Let* $\theta = A + Bi,\ A, B \ge 2,\ (A,B) = 1$ *and let* $S_1^+ = \{x + yi \in \mathbb{Z}[i],\ -A \le x \le 0,\ 0 < y \le B-1\}$, $S_2^+ = \{0, 1-\overline{\theta}\}$. *If* $A \ne B+1$ *then* $\mathcal{P} \subseteq S_1^+ \cup S_2^+$ *else* $\mathcal{P} \subseteq S_1^+ \cup S_2^+ \cup \{-A + Ai\}$, *where* $0, 1-\overline{\theta}, -A + Ai \in \mathcal{P}$ *with period length one.*

Proof. First we observe that the only real periodic element is 0. Indeed, if $\pi \in \mathbb{Z}$ then $\pi \in \mathcal{L}_B$, and by Lemma 4 and Th. 4 we obtain that $\pi = 0$. Let $\pi = P + Qi,\ \rho = U + Vi$ and $\kappa = X + Yi$ be three arbitrary periodic elements, such that $\Phi(\pi) = \rho,\ \Phi(\rho) = \kappa$. Then with suitable $b_1, b_2 \in \mathcal{A}$ we have

$$P = b_1 + AU - BV,\ Q = AV + BU \tag{24}$$

$$U = b_2 + AX - BY,\ V = AY + BX \tag{25}$$

By (15) it is also clear that

$$\mid \pi \mid^2, \mid \rho \mid^2, \mid \kappa \mid^2 \leq L^2 = (\mid \theta \mid +1)^2 < A^2 + B^2 + 1 + 2(A + B). \tag{26}$$

From (24) it follows that if $P = 0,\ Q < 0$ then $U < 0$, else if $P = 0,\ Q > 0$ then $V > 0$. Hence it follows that there does not exist periodic element in the first, the fourth and the third quadrant. Indeed, if $\mathrm{sgn}\,(U) = \mathrm{sgn}\,(V)$, $\mid U \mid, \mid V \mid > 0$ then $\mid Q \mid \geq A + B$, if $U > 0, V < 0$ then $P \geq b_1 + A + B$, $\mid Q \mid > 0$. It means that $\mid \pi \mid^2 = P^2 + Q^2 \geq (A+B)^2 + 1 = A^2 + B^2 + 2AB + 1$ which contradicts to (26) since if $A, B \geq 2$ then $A + B \leq AB$. We can also observe by (25) that if $X = 0$ and $Y < 0$ then $V < 0$, so the negative imaginary axis do not contain any periodic element. Let $U \leq 0,\ V > 0,\ Q > 0$. Suppose that $\mid U \mid \geq A + 1$. Then by (24) we have that $AV \geq 1 + B \mid U \mid \geq 1 + B(A + 1)$, therefore $V \geq B + 1$ but in this case $U^2 + V^2 > L^2$. It remains that $\mid U \mid \leq A$. Using (24) we can also observe that $-BU = AV - Q \geq A(V - 1)$, therefore $BA \geq A(V - 1)$. So we have that $V \leq B + 1$. Let $-A \leq P, U \leq 0,\ Q = B + 1$. In virtue of (24) we have that $AV - AB \leq Q$, therefore $AV \leq B(A + 1) + 1$. Hence, $V \leq B + 1$ and equality holds if and only if $A = B + 1$. Th. 1 shows that in this case the Gaussian integer $\eta = -A + Ai \in \mathcal{P}$ with period length one, and, since $\eta + 1 \equiv 0$ and $\Phi(\eta + 1) = i$ there does not exist any other periodic element with imaginary part $B + 1$. The case $Q = B$ was fully described by Th. 4 by which the only periodic element is $1 - \overline{\theta}$ with period length one. Suppose that $Q \leq B - 1$. Then by (24) we have that $AV + BU \leq B - 1$, hence $V \leq B - 1$. The proof of Theorem 6 is complete.$\diamondsuit$

Theorem 7. *Let* $\theta = A + Bi,\ A = -E,\ E, B \geq 2,\ (A, B) = 1$ *and let* $S_1^- = \{x + yi \in \mathbb{Z}[i],\ 0 \leq x \leq E,\ 0 < y \leq B - 1\}$, $S_2^- = \{0\}$. *Then* $\mathcal{P} \subseteq S_1^- \cup S_2^-$.

Proof. Th. 5 tells us that the only real periodic element is 0. Let $\pi = P + Qi,\ \rho = U + Vi$ and $\kappa = X + Yi$ be three arbitrary periodic elements, such that $\Phi(\pi) = \rho,\ \Phi(\rho) = \kappa$ as before. Then with suitable $b_1, b_2 \in \mathcal{A}$ we have

$$P = b_1 - EU - BV,\ Q = BU - EV \tag{27}$$

$$U = b_2 - EX - BY,\ V = BX - EY \tag{28}$$

Clearly, if $P = 0$ and $Q < 0$ then $V > 0$ else if $P = 0$ and $Q > 0$ then $U > 0$. Let us observe that if $E = 1$ then $L < B+1$. By (27) we get that if $\operatorname{sgn}(U) \neq \operatorname{sgn}(V)$ then $\mid Q \mid \geq E+B$. If $U, V < 0$ then $P \geq b_1+E+B$, therefore $P^2 + Q^2 > L^2$ which is a contradiction. Hence it follows that there does not exist any periodic element in the second, the third and in the fourth quadrant. Suppose that $U = 0$. Then $P, Q > 0$, $V < 0$ and by (27) $Q \geq E, P \geq B + b_1$. If $b_1 = 0$ then $B \mid E$ which is a contradiction. So $b_1 \geq 1$. If $\mid V \mid > 1$ then $P^2+Q^2 > L^2$ which is a contradiction again. If $V = -1$ then using (28) we can deduce that $X \leq E$ and $Y \leq B$. It is easy to check that if $P \leq E$ and $Q \leq B$ then $U \leq E$ and $V \leq B$. So there does not exist any periodic element in the imaginary axis (except 0). Suppose that $U \geq E+1$. The case $V = B$ was completely described. If $V > B$ then $E^2+B^2-1 \geq b_1 = P+EU+BV > P+E(E+1)+B^2$ which implies that $P < -E$ and this is a contradiction. If $B > V$ then $Q \geq E+B$ and $\mid P \mid \geq E(E+1)+B$ but in this case $P^2+Q^2 > L^2$ and it is a contradiction again. Therefore $U \leq E$. Suppose that $V \geq B+1$. Then $Q = BU - EV \leq BE - VE = E(B - V) \leq -E$ and this is a contradiction as well. It means that $V \leq B$. The proof is completed.$\Diamond$

Lemma 7. *Let $\theta = A + Bi$, $(A, B) = 1$, $B \geq 2$. Let furthermore $\alpha = U + Vi, \beta = \alpha + 1 \in \mathbb{Z}[i]$. If $A \geq 2, \alpha, \beta \in S_1^+$ or $A = -E, E \geq 1$, $\alpha, \beta \in S_1^-$ then $\Phi(\alpha) = \Phi(\beta)$.*

Proof. If $\beta \equiv c \pmod{\theta}$, $c \in \mathcal{A}$, $c \neq 0$ then $\alpha = \beta - 1 \equiv c-1 \pmod{\theta}$, hence $\Phi(\alpha) = \Phi(\beta)$. If $\beta \equiv 0 \pmod{\theta}$ then $\beta = \theta\gamma$ for some $\gamma \in \mathbb{Z}[i]$. It would mean that $\operatorname{Norm}(\beta) = \operatorname{Norm}(\theta)\operatorname{Norm}(\gamma)$, so $\operatorname{Norm}(\beta) \geq \operatorname{Norm}(\theta)$ which contradicts to $\beta \in (S_1^+ \cup S_1^-)$.$\Diamond$

Let $\theta = A+Bi$, $(A, B) = 1$, $B \geq 2$ and let us denote the number of periodic elements in a set S by $\#_p(S)$. Clearly, the number of all periodic elements $\#_p(\mathcal{P})$ is equal to $\#\mathcal{P}$. Lemma 7 shows that $\#_p(S_1^+) \leq B - 1$ and $\#_p(S_1^-) \leq B - 1$. On the other hand we have seen in section 5 that for each $\delta | B$ and for each $j = 0, 1, \ldots, l_\delta - 1$ there exists at least one period-cycle in $\mathcal{L}_\delta^{(j)}$. The length of a period in $\mathcal{L}_\delta^{(j)}$ is a multiple of t_δ, so it is at least t_δ. This means that $\#\mathcal{P} \geq \sum_{\delta|B} t_\delta l_\delta$. Since $t_\delta l_\delta = \varphi(B_\delta)$ therefore

$$\#\mathcal{P} \geq \sum_{\delta|B} \varphi(\frac{B}{\delta}) = B. \tag{29}$$

The Lemmas 1, 4 and Ths. 4, 5 show that if $A \geq 2$ then $\#_p(\mathcal{L}_B) = 2$, if $A = -E,\ E \geq 1$ then $\#_p(\mathcal{L}_B) = 1$. This fact together with (29) and with the theorems and lemmas proved in this paper provide our next result.

Theorem 8. *If $A \geq 2$ and $A = B + 1$ then*

$$\mathcal{G}(P) = \{(-1+i)j \to (-1+i)j,\ j = 0, 1, \ldots, A\}.$$

If $A \geq 1$ and $A \neq B + 1$ then $\#\mathcal{P} = |B| + 1$. If $A = -E,\ E \geq 1$ then $\#\mathcal{P} = |B|$.

Remarks. In $\mathcal{L}_\delta$ the length of all period-cycle is t_δ. Recall that if a and m are relative prime positive integers then the least positive integer x such that $a^x \equiv 1 \pmod{m}$ is called the *order of a modulo m*. The usual notation is $x = \operatorname{ord}_m a$. Let $a := A \pmod{B}$. Using this notation it is easy to see that $t_\delta = \operatorname{ord}_{B_\delta} a$. In other words for each $\delta,\ \delta|B,\ \delta < |B|$ there exist $\varphi(B_\delta)/\operatorname{ord}_{B_\delta} a$ cycles in $\mathcal{L}_\delta$ with period length $\operatorname{ord}_{B_\delta} a$.

If B_δ is prime then there is only one cycle in $\mathcal{L}_\delta$ with period length $B_\delta - 1$.

If $a \equiv 1 \pmod{B_\delta}$ then there are only loops in $\mathcal{L}_\delta$ and the number of them is $\varphi(B_\delta)$.

7. Example

If one wants to determine the imaginary parts of the elements of $\mathcal{L}_\delta$ one has to perform some greatest common divisor computations. Ths. 6 and 7 suggest a much faster way, namely using the sieve of Erathostenes. Then, one has to compute the orders to get the imaginary parts of the decomposition of $\mathcal{L}_\delta$. Finally, equation (18) gives also the real parts of the periodic elements.

Let $A = 5$ and $B = 12$. If $\delta = 1$ then $\varphi(B_1) = 4$, $\operatorname{ord}_{B_1} A = 2$. Therefore there are two cycles with period lenght 2. The periodic elements are $i \to -2+5i \to i$ and $-2+7i \to -4+11i \to -2+7i$.

If $\delta = 2$ then $\varphi(B_2) = 2$, $\operatorname{ord}_{B_2} A = 2$. Therefore there is one cycle with period length 2, namely $2i \to -4+10i \to 2i$.

If $\delta = 3$ then $\varphi(B_3) = 2$, $\operatorname{ord}_{B_3} A = 1$. There are two cycles with period length 1, namely $-1+3i \to -1+3i$ and $-3+9i \to -3+9i$.

If $\delta = 4$ then $\varphi(B_4) = 2$, $\operatorname{ord}_{B_4} A = 2$. Therefore there is one cycle with period length 2, namely $-1+4i \to -3+8i \to -1+4i$.

If $\delta = 6$ then $\varphi(B_6) = 1$, $\operatorname{ord}_{B_6} A = 1$. Therefore there is one cycle with period length 1, namely $-2+6i \to -2+6i$.

If $\delta = 12$ then there are two cycles with period length 1, namely $0 \to 0$ and $-4+12i \to -4+12i$.

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